An Example of Output Regulation for Polynomially Stable SISO System with Infinite Dimensional Exosytem

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Abstract— **In this short paper we present an example of output regulation of periodic signals for a plant (perturbed wave equation) in the setting of polynomially stable *SISO* system with bounded control and observation operators. The periodic signals are assumed generated by a suitable infinite-dimensional exosystem. the solvability of this problem is strongly related to the solvability of a pair of linear regulator equations.**

Index Terms :Infinite-Dimensional Systems, Infinite-Dimensional Exosystems, Output Regulation, polynomial stability, perturbed wave equation.

I. INTRODUCTION

In order to situate this work, we first define the problem to be solved in the general setting. We are interested on the output regulation problem of the *SISO* system of the form described by the following equations:

$$\left\{\begin{array}{lcll} \dot z(t) &=& Az(t)+Bu(t) & t\geq 0\\ y(t) &=& Cz(t) & t\geq 0\\ z(0) &=& z_0\in Z & \end{array}\right. \tag{I.1}$$

Notice that there are no disturbance signals to the state of the plant. Here A generates a C_0-semigroup $T_A(t)$, $t\geq 0$, on a complex Banach space Z. The state of the plant (I.1) is denoted by $z(t)\in Z$. The continuous input $u:\mathbb{R}_+\longrightarrow\mathbb{C}$ and the continuous output $y:\mathbb{R}_+\longrightarrow\mathbb{C}$ take values in $\mathbb{C}$ (the space of complex numbers). The control operator $B\in\mathscr{L}(\mathbb{C},Z)$ and the observation operator $C\in\mathscr{L}(Z,\mathbb{C})$. In addition, we consider reference signals y_r which assumed generated by an infinite-dimensional exosystem

$$\left\{\begin{array}{lcll} \dot w(t) &=& Sw(t), & t\in\mathbb{R}\\ y_r(t) &=& Qw(t), & t\in\mathbb{R}\\ w(0) &=& w_0\in W. & \end{array}\right. \tag{I.2}$$

on a Hilbert space W. The generator S of the exosystem is assumed to generate an isometric C_0-group $T_S(t)$, $t\in\mathbb{R}$, on W with a suitable domain $D(S)$. We denote the error between the measured and reference outputs by

$$e(t):=y(t)-y_r(t)=Cz(t)-Qw(t). \tag{I.3}$$

The output regulation considers controlling a given plant such that its output asymptotically tracks a desired reference signals. The control law is constructed such that it stabilizes the closed-loop system consisting of the plant and the control appropriately. Many results presented in output regulation theory are based on a relationship between the closed loop system and a corresponding Sylvester type operator equation. In the case of finite or infinite-dimensional exosystem of the form (I.2), this equation is an infinite-dimensional Sylvester operator equation

$$A\Pi+P=\Pi S \tag{I.4}$$

Our procedure for regulate the output is to construct a suitable control law which depends on the state of the exosystem. Such a procedure is called state feedback regulation problem which is denoted by SFRP. More precisely, The task is to find a feedback control law of the form

$$u(t)=Lw(t)$$

such that $L\in\mathscr{L}(W,\mathbb{C})$ and for the extended closed loop system

$$\left\{\begin{array}{lclcl} \dot z(t) &=& Az(t)+BLw(t) &,& t\geq 0\\ \dot w(t) &=& Sw(t) &,& t\geq 0 \end{array}\right. \tag{I.5}$$

the tracking error $e(t)=Cz(t)-Qw(t)\longrightarrow 0$ as $t\longrightarrow\infty$ for any initial conditions $z_0\in Z$ and $w_0\in W$.
For finite-dimensional linear systems, the SFRP was studied by Davison, Francis, Wonham and others (see e.g. [6], [7], [8], [9] and the references therein). In [9], Francis was the first which presented a complete characterization for the SFRP in terms of solvability of the so called regulator equations:

$$\left\{\begin{array}{lclcl} A\Pi+B\Gamma &=& \Pi S & in & D(S)\\ C\Pi &=& Q & in & W \end{array}\right. \tag{I.6}$$

In [8], Davison has utilized the similar methods for the construction of the control law which will achieve asymptotic tracking of the reference signals. Many authors have also constructed such control law for infinite-dimensional linear systems with finite dimensional exosystems, e.g. Pohjolainen [4] and Byrnes et al.[5]. Under the exponential stabilizability assumption of the system (I.1), they proved a complete characterization for the existence (and construction) of a regulating control law in terms of solvability of the regulator equations (I.6). Subsequently, in [3], Immonen generalized the SFRP studied in [5] and [4] in the case that the plant (I.1) and the exosystem (I.2) are infinite dimensional systems. In his recent PhD-thesis, Immonen studied the output regulation problem under the strong stabilizability assumption of the system (I.1). He proved that if the pair (A,B) in the system (I.1) is strongly stabilizable using K and if the regulator equations (I.6) have a solution (Π,Γ), then the SFRP can be solved and the control law $u(t)=Kz(t)+(\Gamma-K\Pi)w(t)$ solves the SFRP. On the other hand, it is well known that this result has a converse provided that the exosystem is finite-dimentional and the system (I.1) is also exponentially stabilizable [5]. In [2], Phong establishes that for every $P\in\mathscr{L}(W,Z)$, if the semigroup $T_A(t)$ is exponentially stable then the Sylvester type operator equation (I.4) has a unique bounded solution given by

$$\Pi = \int_0^\infty T_A(t) P T_S(-t)\, dt \quad \text{(strong convergence).} \tag{I.7}$$

We notice that the operator equation (I.4) does not necessarily always have a solution in the case that $T_A(t)$ is only strongly stable; but if we hope to have a solution, we may need to be some regularity for P. In this way, Immonen, in [3], assumed that the operator P is regular for the semigroup $T_A(t)$ (see section:necessary conditions for the solvability of the SFRP in [3]). He established that the operator equation (I.4) has a solution $\Pi \in \mathscr{L}(W,Z)$ if and only if $P \in \mathscr{L}(W,Z)$ is regular for $T_A(t)$. Hence, if $T_A(t)$ is exponentially stable, then for every operator $P \in \mathscr{L}(W,Z)$ is regular for $T_A(t)$. Now we can arise the question: what about if $T_A(t)$ is polynomially stable and not exponentially stable? To our knowledge the output regulation problem of polynomially stable C_0-semigroup under perturbations of its generator has not been studied previously in the literature. The concept of polynomial stability is interesting because it allows for a uniform polynomial decay rate for all sufficiently smooth initial conditions. This type of stability can occur for e.g. if the spectrum of A is contained in open left half plane, but approaches the imaginary axis at $\pm\infty$. In this paper, our objective is to apply the output regulation problem for a system generating by a polynomially stable but not exponentially stable C_0-semigroup. In order to achieve this, we will give a characterization of the solution of the operator equation (I.4) in the case that $T_A(t)$ is polynomially stable. We will show that if the operator equation (I.4) has a solution $\Pi \in \mathscr{L}(W,Z)$ then necessarily we have

$$\Pi w = \int_0^\infty T_A(t) P T_S(-t) w\, dt \tag{I.8}$$

for all $w \in D(S)$. This allows us to employ the concept of operator conform (see Section (IV)) and characterize this property by the solvability of the operator equation (I.4). Thus a complete characterization for the solvability of the regulator equations and the SFRP to be solve will be given. The plan of the paper is as follows. In Section (II), we formulate the plant consisting of perturbed wave equation with bounded control and observation operators. We define also the exosytem which is responsible for generating the reference signals. In Section (III), we formulate the output regulation problem to be solved. In Section (IV), we present, in the general setting, sufficient and necessary conditions for the solvability of the SFRP. In the last Section, we apply the results presented in Section (IV) to our example.

II. PLANT AND EXOSYSTEM

Consider the following perturbed one-dimensional wave equation on (0.1)

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t) + b_0(x)[h_1(x)v(x,t) + h_2(x)\frac{\partial v}{\partial t}(x,t)]$$
$$v(0,t) = v(1,t) = 0$$
$$v(x,0) = v_0 \quad \frac{\partial v}{\partial t}(x,0) = v_1$$

where $b_0(x) = \sqrt{3}(1-x)$, h_1 and h_2 are defined in [11, Theorem 13] Define the operator $A_0 : D(A_0) \subset L^2(0,1) \longrightarrow L^2(0,1)$ by

$$A_0 = -\frac{d^2}{dx^2}$$

with domain $D(A_0) = \{v \in L^2(0,1) | v, v' abs.cont. v'' \in L^2(0,1), v(0) = v(1) = 0\}$. The operator A_0 has a positive self-adjoint square root $A_0^{1/2}$ and the space $Z = D(A_0^{1/2}) \times L^2(0,1)$ equipped with an inner product

$$\langle v,w\rangle_Z = \langle A_0^{1/2} v_1, A_0^{1/2} w_1\rangle_{L^2} + \langle v_2, w_2\rangle_{L^2}$$

is a Hilbert space. Next we introduce

$$z = \begin{bmatrix} v \\ \frac{dv}{dt} \end{bmatrix} \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \quad \Delta = \begin{bmatrix} 0 & 0 \\ b_0 h_1 & b_0 h_2 \end{bmatrix}$$

$$D(A) = D(A_0) \times D(A_0^{1/2}).$$

Notice that $\Delta \in \mathscr{L}(Z)$. We introduce the control operator $B\xi = b\xi = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \xi$ where $b_1(x) = x(1-x)$ and the observation operator $C = \sum_{k\neq 0} < \cdot, \phi_k > \overline{< b, \phi_k >} \in \mathscr{L}(Z, \mathbb{C})$ where $\phi_k(x) = \frac{1}{\lambda_k}\begin{bmatrix} sin(k\pi x) \\ \lambda_k sin(k\pi x) \end{bmatrix}$ and $\lambda_k = ik\pi$ for $k \in \mathbb{Z}\backslash\{0\}$. Thus the above perturbed wave equation can be written as

$$\begin{cases} \dot{z}(t) &= (A+\Delta)z(t) + Bu(t), \quad z(0) = z_0 = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in Z \\ y(t) &= Cz(t),\ t \geq 0 \end{cases} \tag{II.1}$$

The eigenvalues of A are λ_k for $k \in \mathbb{Z}\backslash\{0\}$ and the corresponding eigenvectors are ϕ_k form an orthonormal basis in Z. Due to Paunonen [11], the perturbed operator $\tilde{A} := A + \Delta$ is a Riesz-spectral operator and that $\sigma(\tilde{A}) = \{\frac{-\nu\pi}{k^2} + ik\pi\}_{k\neq 0}$ for some $\nu > 0$. Further, the eigenvectors Φ_k of $\tilde{A}$ form a Riesz basis of Z. Since $\tilde{A}$ is similar to a normal operator then, due to [1, Theorem 4.1], $\tilde{A}$ generates a polynomially stable C_0-semigroup $T_{\tilde{A}}(t)$ and we have

$$\|T_{\tilde{A}}(t)\tilde{A}^{-1}\| \leq \frac{N}{\sqrt{t}} \tag{II.2}$$

for all $t > 0$ and some $N > 0$. Since $T_{\tilde{A}}(t)$ is bounded then the estimate (II.2) is equivalent to

$$\|T_{\tilde{A}}(t)(-\tilde{A})^{-2r}\| \leq \frac{N}{t^r} \tag{II.3}$$

for all $r > 0, t > 0$. Since $\sigma(\tilde{A}) \subset \mathbb{C}^-$ and it has no finite accumulation points on $i\mathbb{R}$, the operator $-\tilde{A}$ is an invertible sectorial operator. Using the fractional domains of the operator $-\tilde{A}$, we have, for $\beta \geq 0$

$$Z_\beta = D((-\tilde{A})^\beta) = \{z \in Z | \sum_{k\neq 0} |\mu_k|^{2\beta} | < z, \phi_k > |^2 < \infty\}$$

The space $(Z_\beta, \|\cdot\|_\beta)$ is a Hilbert space with norm defined by

$$\|z\|_{2\beta}^2 = \sum_{k\neq 0} |\mu_k|^{2\beta} | < z, \phi_k > |^2$$

where $\mu_k = \frac{-\nu\pi}{k^2} + ik\pi$ for $k \neq 0$.

Now we introduce an interesting functions space whose

elements can be generated by the exosystem (I.2) if its parameters are suitably chosen. Let $p > 0$, let $\omega_k = \frac{2\pi k}{p}$ for $k \in \mathbb{Z}$, let $(f_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ such that $f_k \geq 1$, $k \in \mathbb{Z}$ and $(f_k^{-1})_{k\in\mathbb{Z}} \in \ell^2$ (the space of square summable complex sequences). We define the state space of the exosystem as follows:
$W = \{y : \mathbb{R} \longrightarrow \mathbb{C} \,|\, y(t) = \sum_{k\in\mathbb{Z}} y_k e^{i\omega_k t}\; t \in \mathbb{R}\,, \sum_{k\in\mathbb{Z}} |y_k|^2 f_k^2 < \infty\,, (y_k)_{k\in I} \subset \mathbb{C}\}$. Put $\theta_k(t) = e^{i\omega_k t}$ for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}$. It is clear that the set $\{\theta_k\}_{k\in\mathbb{Z}}$ form an orthonormal basis in W with the L^2-inner product which is denoted by $<,>_{L^2}$. But it shall be interesting to use the inner product in W defined by

$$< u, y >_f = \sum_{k\in\mathbb{Z}} u_k \overline{y_k} f_k$$

where $u_k =< u, \theta_k >_{L^2}$ and $y_k =< y, \theta_k >_{L^2}$ for all $k \in \mathbb{Z}$. It is easy to see that W is a Hilbert space with the inner product $<,>_f$; the corresponding norm is

$$\|y\|_f = \sqrt{\sum_{k\in\mathbb{Z}} | < y, \theta_k > |^2 f_k^2}$$

thus the set $\{\theta_k\}_{k\in\mathbb{Z}}$ with this inner product form an orthogonal basis in W, with $\|\theta_k\|_f = f_k$ for all $k \in \mathbb{Z}$. We define the operator

$$S = \sum_{k\in\mathbb{Z}} i\omega_k < \cdot, \theta_k >_{L2} \theta_k$$

$$D(S) = \{y \in W \mid \sum_{k\in\mathbb{Z}} \omega_k^2 | < y, \theta_k >_{L^2} |^2 f_k^2 < \infty\}.$$

Notice that the operator S generates an isometric C_0-group on W given by:

$$T_S(t) = \sum_{k\in\mathbb{Z}} e^{i\omega_k t} < \cdot, \theta_k > \theta_k.$$

We fix $Q = \delta_0$ (point evaluation at 0). Due to [3], the exogenous system (I.2) with $W, S, Q = \delta_0$ and $w_0 \in W$, can generate all reference signals in W and only those. Moreover, every reference function $y_r \in W$ is generated by the choice $w_0 = y_r \in W$. In fact

$$\delta_0 T_S(t) h = h(x+t)|_{x=0} = h(t)$$

for every $h \in W$ and $t \in \mathbb{R}$. (For more details see [3, Chapter II]).
Notice that $Q = \delta_0 \in \mathscr{L}(W, Z)$ because if $y = \sum_{k\in\mathbb{Z}} < y, \theta_k >_{L^2} \theta_k$, we have

$$\begin{aligned} |\delta_0 y| &= |\textstyle\sum_{k\in\mathbb{Z}} < y, \theta_k >_{L^2} | \\ &\leq \sqrt{\textstyle\sum_{k\in\mathbb{Z}} f_k^{-2}} \sqrt{\textstyle\sum_{k\in\mathbb{Z}} | < y, \theta_k >_{L^2} |^2 f_k^2} \\ &= c\|y\|_f \end{aligned}$$

III. FORMULATION OF THE MAIN PROBLEM

Our goal is to regulate the plant so that its output tracks a reference signal generated by the exosystem. To this end, we follow the approach given in [3] and seek the control as a feedback of the state of exosystem

$$u(t) = Lw(t) \tag{III.1}$$

where $L \in \mathscr{L}(W, \mathbb{C})$. Hence the plant becomes

$$\begin{cases} \dot{z}(t) &= \tilde{A}z(t) + BLu(t), \\ z(0) &= z_0 = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \end{cases} \tag{III.2}$$

and the extended system consisting of the plant and the exosystem is given by

$$\begin{cases} \dot{z}(t) = \tilde{A}z(t) + BLw(t) \quad , \quad t \geq 0 \\ \dot{w}(t) = Sw(t) \quad , \quad t \geq 0 \end{cases} \tag{III.3}$$

Our tracking problem is formulated in terms of the error $e(t)$ defined by

$$e(t) = y(t) - y_r(t) \tag{III.4}$$

for $t \geq 0$, where y is the output of the extended system (III.3) and y_r is the reference signal to be tracked. Since

$$y(t) = Cz(t) \quad \text{and} \quad y_r(t) = Qw(t) \tag{III.5}$$

then the error (III.4) can be written as

$$e(t) = Cz(t) - Qw(t) = [C \;\; -Q] \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \tag{III.6}$$

We are now ready to define our main problem.

Problem 3.1: (The Main Problem). The task is to find a feedback control of the form

$$u(t) = Lw(t)$$

such that for every initial conditions of the plant and the exosystem, the error $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

IV. SUFFICIENT AND NECESSARY CONDITIONS FOR THE SOLVABILITY OF THE SFRP

As mentioned in the introduction, Immonen, in [3], establishes that if the operator $\tilde{A}$ in the plant (I.1) is strongly stable and if the regulator equations (I.6) have a solution (Π, Γ), then the SFRP can be solved and the control law $u(t) = \Gamma w(t)$ solves the SFRP. In our case, we have $\tilde{A}$ generates a bounded polynomially stable C_0-semigroup $T_{\tilde{A}}(t)$. Hence, by density argument, $T_{\tilde{A}}(t)$ is strongly stable. consequently, it suffices to solve the regulator equations (I.6) in order to guarantees that the error $e(t)$ converges to 0 as $t \longrightarrow \infty$. Thus we can state the theorem which gives the sufficient conditions for the solvability of the SFRP.

Theorem 4.1: If the regulator equations (I.6) have a solution (Π, Γ), then the control law $u(t) = \Gamma w(t)$ solves the SFRP.
Before study the necessary conditions for the solvability of the SFRP, we will look at the characterization of the Sylvester type operator equation

$$\tilde{A}\Pi + P = \Pi S \tag{IV.1}$$

where $P \in \mathscr{L}(W, Z)$.
Using the polynomial stability of the C_0-semigroup $T_{\tilde{A}}(t)$, we can give a characterization of the solvability of (IV.1)

Proposition 4.2: The Sylvester type operator equation (IV.1) has a bounded solution $\Pi \in \mathscr{L}(W,Z)$ if and only if the operator

$$\Pi\omega := \int_0^\infty T_{\tilde{A}}(t)PT_S(-t)wdt$$

define a linear bounded operator from $D(S)$ to Z, where $D(S)$ is endowed with the induced norm of W.
subsequently, such operator Π is said conform for the semigroup $T_{\tilde{A}}(t)$.
Proof: Let $\Pi \in \mathscr{L}(W,Z)$ such that $\Pi(D(S)) \subset D(\tilde{A})$ and

$$\tilde{A}\Pi + P = \Pi S \tag{IV.2}$$

From [3, lemma 3.1], we have

$$\Pi w = T_{\tilde{A}}(\sigma)\Pi T_S(-\sigma)w + \int_0^\sigma T_{\tilde{A}}(t)PT_S(-t)wdt \tag{IV.3}$$

for all $w \in W$ and $\sigma \geq 0$.
Since $\Pi(D(S)) \subset D(\tilde{A})$ and $D(S)$ is $T_S(t)$-invariant subspace then $\Pi T_S(-\sigma)w \in D(\tilde{A})$ for all $w \in D(S)$ and $\sigma \geq 0$. Hence, using (II.2), we conclude that the first term in the right hand side of the equality (IV.3) converges to 0 as $\sigma \to \infty$ for all $w \in D(S)$. Consequently, we have

$$\Pi w = \int_0^\infty T_{\tilde{A}}(t)PT_S(-t)wdt. \tag{IV.4}$$

for all $w \in D(S)$. Conversely,
Suppose that there exists $\Pi \in \mathscr{L}(D(S),Z)$ such that

$$\Pi w = \int_0^\infty T_{\tilde{A}}(\sigma)PT_S(-\sigma)wd\sigma$$

for every $w \in D(S)$. By density argument, Π can be extended to a linear bounded operator from W to Z noted also by Π.
Let $w \in D(S)$ and $t > 0$ be arbitrary. We have

$$T_{\tilde{A}}(t)\Pi w - \Pi w$$
$$= \int_0^\infty T_{\tilde{A}}(t+\sigma)PT_S(-\sigma)wd\sigma - \int_0^\infty T_{\tilde{A}}(\sigma)PT_S(-\sigma)wd\sigma$$
$$= \int_t^\infty T_{\tilde{A}}(\sigma)PT_S(t-\sigma)wd\sigma - \int_0^\infty T_{\tilde{A}}(\sigma)PT_S(-\sigma)wd\sigma$$
$$= \int_0^\infty T_{\tilde{A}}(\sigma)PT_S(-\sigma)(T_S(t)w - w)d\sigma - \int_0^t T_{\tilde{A}}(\sigma)PT_S(t - \sigma)wd\sigma$$

hence, for all $t \in (0,1)$, we have

$$\frac{T_{\tilde{A}}(t)\Pi w - \Pi w}{t}$$
$$= \int_0^\infty T_{\tilde{A}}(\sigma)PT_S(-\sigma)\left(\frac{T_S(t)w-w}{t}\right)d\sigma - \frac{1}{t}\int_0^t T_{\tilde{A}}(\sigma)PT_S(t - \sigma)wd\sigma = \Pi\left(\frac{T_S(t)w-w}{t}\right) - \frac{1}{t}\int_0^t T_{\tilde{A}}(\sigma)PT_S(t-\sigma)wd\sigma$$

In the first term on the right hand side, we use the fact that $\frac{T_S(t)w-w}{t} \in D(S)$ for every $w \in D(S)$.
Since $\lim_{t \longrightarrow 0^+} \frac{T_S(t)w-w}{t} = Sw$ and $\Pi \in \mathscr{L}(W,Z)$, then this first term converges to ΠSw as $t \longrightarrow 0^+$. For another hand, since the map $\sigma \longrightarrow T_{\tilde{A}}(\sigma)PT_S(t-\sigma)w$ is continuous on $\mathbb{R}^+$ for every $t \in (0,1)$ and $T_S(t)$ is an isometric group on W then the second term converges to $-Pw$. consequently,

$$\lim_{t \longrightarrow 0^+} \frac{T_{\tilde{A}}(t)\Pi w - \Pi w}{t} \text{ exists in Z.}$$

Therefore, $\Pi w \in D(\tilde{A})$ and $\tilde{A}\Pi w = \Pi Sw - Pw$. ■
In the following proposition, we will give a sufficient condition for which an operator is conform (see Proposition 4.2) for a polynomially stable C_0-semigroup.

Proposition 4.3: If there exists $\varepsilon > 0$ such that $P \in \mathscr{L}(W, Z_{\alpha+\varepsilon})$, then the operator P is conform for the semigroup $T_{\tilde{A}}(t)$.

Proof: Since $T_{\tilde{A}}(t)$ is a bounded polynomially stable C_0-semigroup then, by inequality (II.3), with $r = 1 + \frac{\varepsilon}{2}$, we have

$$\|T_{\tilde{A}}(t)(-\tilde{A})^{-2-\varepsilon}\| \leq N_\varepsilon t^{-1-\frac{\varepsilon}{2}},$$

or equivalently

$$\|T_{\tilde{A}}(t)z\| \leq N_\varepsilon t^{-1-\frac{\varepsilon}{2}}\|z\|_{2+\varepsilon}. \tag{IV.5}$$

for all $z \in Z_{2+\varepsilon}$. Let $a > 0$ such that $\|Pw\|_{2+\varepsilon} \leq a\|w\|$ for all $w \in W$. Then

$$\begin{array}{rcl} \|T_{\tilde{A}}(t)PT_S(-t)w\| & \leq & N_\varepsilon t^{-1-\frac{\varepsilon}{2}}\|PT_S(-t)w\|_{2+\varepsilon} \\ & \leq & aN_\varepsilon t^{-1-\frac{\varepsilon}{2}}\|T_S(-t)w\| \\ & = & aN_\varepsilon t^{-1-\frac{\varepsilon}{2}}\|w\| \end{array}$$

consequently, the function $t \longrightarrow T_{\tilde{A}}(t)PT_S(-t)w$ is integrable on $[0,\infty)$ and

$$\Pi w = \int_0^\infty T_{\tilde{A}}(t)PT_S(-t)wdt$$

define a bounded operator from W to Z; i.e. the operator P is conform for the semigroup $T_{\tilde{A}}(t)$. ■
The following theorem presents some conditions under which the solvability of the regulator equations (I.6) is necessary for the solvability of the SFRP.

Theorem 4.4: If the SFRP is solvable for some control law $u(t) = Lw(t)$ such that the operator $BL \in \mathscr{L}(W,Z)$ is conform for the semigroup $T_{\tilde{A}}(t)$, then there exits $\Pi \in \mathscr{L}(W,Z)$ and $\Gamma \in \mathscr{L}(W,\mathbb{C})$ such that $\Pi(D(S)) \subset D(\tilde{A})$, $L = \Gamma$ and the regulator equations (I.6) are satisfied.

Proof: Since BL is conform for $T_{\tilde{A}}(t)$, by Proposition 4.2, there exists $\Pi \in \mathscr{L}(W,Z)$ such that $\Pi(D(S)) \subset D(\tilde{A})$ and

$$\Pi S = \tilde{A}\Pi + BL$$

Let $\Gamma = L \in \mathscr{L}(W,\mathbb{C})$. Then Π and Γ solve the first regulator equation of (I.6).
Next, we show that also the second regulator equation is satisfied.
First we compute the tracking error $e(t)$ in (I.3). By the boundedness of BL, the composite operator $\mathscr{A}$ on the extended state space $Z \times W$ defined as

$$\mathscr{A} = \begin{pmatrix} \tilde{A} & BL \\ 0 & S \end{pmatrix}$$

generates a C_0-semigroup $T_{\mathscr{A}}(t)$ on $Z \times W$. An easy calculation reveals that this semigroup is given by

$$T_{\mathscr{A}}(t) = \begin{pmatrix} T_{\tilde{A}}(t) & \int_0^t T_{\tilde{A}}(t-\sigma)BLT_S(\sigma)d\sigma \\ 0 & T_S(t) \end{pmatrix}.$$

By using (IV.3), we have

$$T_{\mathscr{A}}(t)\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} T_{\tilde{A}}(t)(z_0 - \Pi w_0) + \Pi T_S(t)w_0 \\ T_S(t)w_0 \end{pmatrix}$$

The explicit expression for the tracking error $e(t)$ is as follows:

$$\begin{aligned} e(t) &= [C \ \ -Q]\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = [C \ \ -Q]\, T_{\mathscr{A}}(t)\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} \\ &= CT_{\tilde{A}}(t)(z_0 - \Pi w_0) + C\Pi T_S(t)w_0 - QT_S(t)w_0 \end{aligned}$$

then

$$e(t) = CT_{\tilde{A}}(t)(z_0 - \Pi w_0) + (C\Pi - Q)T_S(t)w_0 \qquad \text{(IV.6)}$$

Now let $w_0 \in W$ be arbitrary and take $z_0 = \Pi w_0 \in Z$. Then the corresponding tracking error $e(t)$ is given by

$$= (C\Pi - Q)T_S(t)w_0$$

Since the exosystem generates periodic signals then we must have

$$C\Pi w_0 - Qw_0 = 0.$$

(fore more details see [3, Chapter 3]). ■

Due to the above discussion we obtain the following complete characterization for the solvability of the regulator equations I.6 and the SFRP, which summaries in the following theorem

Theorem 4.5: The SFRP is solvable using the control law $u(t) = Lw(t)$, where $L \in \mathscr{L}(W, \mathbb{C})$ and BL is conform for $T_{\tilde{A}}(t)$, if and only if there exists $\Pi \in \mathscr{L}(W, Z)$ and $\Gamma \in \mathscr{L}(W, \mathbb{C})$ such that $\Pi(D(S)) \subset D(\tilde{A})$, $L = \Gamma$ and the regulator equations (I.6) are satisfied.

As an immediate consequence of the explicit expression for the tracking error $e(t)$ in (IV.6), if the regulator equations (I.6) have a solution (Π, Γ), we can give the rate of decay of $\|e(t)\|$ if more regularity of initial conditions is given:

Corollary 4.6: Assume that $z_0 \in D(\tilde{A})$ and $w_0 \in D(S)$. If the regulator equations (I.6) have a solution (Π, Γ), then there is a positive constant m depending only on $\|C\|, \|z_0\|_{\tilde{A}}$ and $\|w_0\|_S$ such that

$$\|e(t)\| \le m t^{-\frac{1}{2}}, \qquad \forall t > 0. \qquad \text{(IV.7)}$$

where the Banach spaces $D(\tilde{A})$ and $D(S)$ are endowed with the graph norms $\|\cdot\|_{D(\tilde{A})}$ and $\|\cdot\|_{D(S)}$.

Proof: Due to (IV.6), since the regulator equations (I.6) have a solution (Π, Γ), then

$$e(t) = CT_{\tilde{A}}(t)(z_0 - \Pi w_0).$$

Since $z_0 \in D(\tilde{A})$ and $w_0 \in D(S)$ then $z_0 - \Pi w_0 \in D(\tilde{A})$ and due to (II.2) we have

$$\|e(t)\| \le m t^{-\frac{1}{2}}, \qquad \forall t > 0.$$

where $m = N'(\|z_0\|_{\tilde{A}} + \|w_0\|_S)$, for some $N' > 0$. ■

Remark 4.7: In the expression of m, we utilized the fact that $\Pi \in \mathscr{L}(D(S), D(\tilde{A}))$ since

$$\tilde{A}\Pi + P = \Pi S \quad \text{in D(S)}$$

V. ASYMPTOTIC TRACKING OF PERIODIC REFERENCE SIGNALS FOR THE PERTURBED WAVE EQUATION

In this Section, our objective is to construct a control law $u(t) = Lw(t)$ for the asymptotic tracking of periodic reference signals. Consider the regulator equations

$$\begin{cases} \tilde{A}\Pi + B\Gamma &= \Pi S \\ C\Pi &= Q \end{cases} \qquad \text{(V.1)}$$

Assume the pair(Π, Γ) solves (V.1). Then, for each $k \in \mathbb{Z}$, the first regulator equation applied to θ_k gives

$$\tilde{A}\Pi\theta_k + B\Gamma\theta_k = \Pi S\theta_k. \qquad \text{(V.2)}$$

Since $i\omega_k$ is an eigenvalue of S with θ_k as eigenvector this equation simplify to

$$(i\omega_k - \tilde{A})\Pi\theta_k = B\Gamma\theta_k. \qquad \text{(V.3)}$$

Since $\sigma(\tilde{A}) \cap \sigma(S) = \emptyset$ then $i\omega_k - \tilde{A}$ is invertible for each $k \in \mathbb{Z}$. Hence we can solve (V.3) for $\Pi\theta_k$

$$\Pi\theta_k = R(i\omega_k, \tilde{A})B\Gamma\theta_k. \qquad \text{(V.4)}$$

Applying C to both sides and using the second regulator equation of V.1, we have

$$Q\theta_k = CR(i\omega_k, \tilde{A})B\Gamma\theta_k. \qquad \text{(V.5)}$$

where in (V.5), $CR(\lambda, \tilde{A})B$ is the transfer function of the plant which we denote by

$$H(\lambda) = CR(\lambda, \tilde{A})B \qquad \text{(V.6)}$$

for all $\lambda \in \rho(\tilde{A})$. Thus we can write (V.5) as

$$Q\theta_k = H(i\omega_k)\Gamma\theta_k. \qquad \text{(V.7)}$$

In order to solve this equation explicitly for $\Gamma\theta_k$, we need to invert $H(i\omega_k)$. In this case we obtain

$$\Gamma\theta_k = H(i\omega_k)^{-1}Q\theta_k. \qquad \text{(V.8)}$$

Since $Q = \delta_0$ then the equality (V.8) becomes

$$\Gamma\theta_k = H(i\omega_k)^{-1}. \qquad \text{(V.9)}$$

for each $k \in \mathbb{Z}$. Consequently, if $H(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$ and if the sequence $(H(i\omega_k)^{-1} f_k^{-1})_{k \in \mathbb{Z}} \in \ell^2$ then, using the expansion $y = \sum_{k \in \mathbb{Z}} < y, \theta_k > \theta_k$ for every $y \in W$, the expression of Γ is

$$\Gamma y = \sum_{k \in \mathbb{Z}} < y, \theta_k > \theta_k H(i\omega_k)^{-1} \qquad \text{(V.10)}$$

It is clear that the operator Γ is bounded from W to $\mathbb{C}$ if and only if $(H(i\omega_k)^{-1} f_k^{-1})_{k \in \mathbb{Z}} \in \ell^2$. Hence it is not in general bounded. In order to avoid this case, we can act on the degree of smoothness of the reference signals by a suitable choice of the sequence $f_{k k \in \mathbb{Z}}$.

Now we are ready to solve our problem.
We have

$$R(i\omega_k,\tilde{A}) = \sum_{n\neq 0} \frac{< \cdot, \phi_n > \Phi_n}{i\omega_k - \mu_n} \qquad \text{(V.11)}$$

for all $k \in \mathbb{Z}$. Hence the transfer function evaluated at the frequencies $i\omega_k$ is

$$H(i\omega_k) = CR(i\omega_k,\tilde{A})B = \sum_{n\neq 0} \frac{| < b, \phi_n > |^2}{i\omega_k - \mu_n}. \qquad \text{(V.12)}$$

For a suitable choice of $b \in \mathbb{Z}$, it is easy to see that $H(i\omega_k) \neq 0$ for all $k \in \mathbb{Z}$. On the other hand, put

$$Ly := \sum_{k\in\mathbb{Z}} H(i\omega_k)^{-1} < y, \theta_k >_{L^2} . \qquad \text{(V.13)}$$

for $y \in W$ such that $L \in \mathscr{L}(W,\mathbb{C})$. We saw that the operator L is bounded if and only if the condition

$$\sum_{k\in\mathbb{Z}} |H(i\omega_k|^{-2}|f_k|^{-2} < \infty. \qquad \text{(V.14)}$$

holds. Assume that the condition (V.14) is verified. We verify that $b \in Z_{2+\varepsilon} = D((-\tilde{A})^{2+\varepsilon})$ for some $\varepsilon > 0$.
For all $k \neq 0$, we have
$< b, \phi_k >_Z = < b_1, \sin k\pi\cdot >_{L^2} = \int_0^1 b_1(x) \sin k\pi x dx$
Hence

$$< b, \phi_k >_Z = \frac{2(1-(-1)^k)}{k^3\pi^3}.$$

Since $| \mu_k | = O(k^2)$ as $k \longrightarrow \infty$ then

$$| \mu_k |^{2(2+\varepsilon)} | < b, \phi_k >_Z |^2 = O(k^{2\varepsilon-2})$$

Hence the series $\sum_{k\neq 0} |\mu_k|^{2(2+\varepsilon)} | < z, \phi_k > |^2 < \infty$ for $0 < \varepsilon < \frac{1}{2}$. Thus if we choose $0 < \varepsilon < \frac{1}{2}$, the operator BL is bounded from W to $Z_{2+\varepsilon}$ and due to the Proposition (4.3), we conclude that BL is a conform operator for the semigroup $T_{\tilde{A}}(t)$ and the first regulator equation of (I.6) has a bounded solution given by

$$\Pi y := \int_0^\infty T_{\tilde{A}}(t) BLT_S(-t) y dt$$

for all $y \in D(S)$. According to the discussion above, it is clear that the operator Π satisfied the equality (I.7). By applying C to both sides of this equality, we deduce that

$$C\Pi\theta_k = H(i\omega_k)\Gamma\theta_k. \qquad \text{(V.15)}$$

Since $\Gamma\theta_k = H(i\omega_k)^{-1}$ for all $\in \mathbb{Z}$, we conclude that

$$C\Pi\theta_k = 1 = Q\theta_k. \qquad \text{(V.16)}$$

Thus, using the expansion $y = \sum_{k\in\mathbb{Z}} < y, \theta_k > \theta_k$ for every $y \in W$, we have

$$C\Pi y = \sum_{k\in\mathbb{Z}} < y, \theta_k > C\Pi\theta_k \sum_{k\in\mathbb{Z}} < y, \theta_k > = y(0) = Qy. \qquad \text{(V.17)}$$

Consequently, the second regulator equation is also verified. Due to the theorem (4.5), the SFRP is solvable using $u(t) = Lw(t)$. Moreover, for every $y_r \in W$ the corresponding control law u_{y_r} which achieves the asymptotic tracking of y_r is given, for all $t \geq 0$, by

$$u_{y_r}(t) = \sum_{k\in\mathbb{Z}} H(i\omega_k)^{-1} < y_r, \theta_k >_{L^2} e^{i\omega_k t}.$$

VI. CONCLUSION

In this paper we studied an example of output regulation of periodic signals for a plant (perturbed wave equation) in the setting of polynomially stable *SISO* system with bounded control and observation operators. We assumed that the reference signals to be tracked are generated by an infinite dimensional exosystem. We showed that the solvability of the state feedback regulation problem SFRP is equivalent to the solvability of the so called regulator equations under some assumptions. In contrast to the case of strong stability, the polynomial stability of the plant allows us to give a rate of decay of the tracking error if more regularity of initial conditions is given.

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