Fractional-Order Observers Design for Fractional-Order Systems with Unknown Inputs

Ibrahima N'Doye, Mohamed Darouach and Michel Zasadzinski

Abstract— **This paper considers a method to design the fractional-order observers for continuous-time linear fractional-order systems with unknown inputs. The conditions for the existence of these observers are given. Sufficient conditions for the asymptotical stability of observers with the fractional order α satisfying $0 < \alpha < 2$ are derived in terms of linear matrix inequalities formulation. Two numerical examples are given to demonstrate the applicability of the proposed approach where the fractional order α belonging to $1 \leqslant \alpha < 2$ and $0 < \alpha \leqslant 1$ respectively.**

Index Terms— **Fractional-order systems, functional observer, existence condition, linear matrix inequality (LMI), unknown input, stability.**

I. INTRODUCTION

State estimation or observer design have been widely used in control and signal processing in the last few decades. They are of theoretical interest and also have some applications particularly in the failure detection and fault diagnosis problems, and chaotic synchronization and secure communications [1], [2], [3], [4], [5]. The problem of observing the state vector of a deterministic linear time-invariant multivariable system has been the object of numerous studies ever since the original work of Luenberger first appeared [6].

The problem of the functional observer design was related to the constrained or unconstrained Sylvester equations [7], [8]. Generally to solve this problem, many authors have proposed to transform the initial system to an equivalent one (by using some regular transformations) of reduced-order and to design an observer for this system. Necessary and sufficient conditions for the existence of these observers for linear systems were given in [9], [10]. The observers for systems with unknown inputs are of great interest in the failure detection and the control of systems in presence of disturbances [11]. This paper presents an extension of the approach developed in [11] to functional observers for general integer-order linear systems for which the unknown input affect also the measurements.

Recently, fractional-order systems have been studied by many authors in engineering science from an application point of view (see [12], [13], [14] and references therein). Many systems can be described with the help of fractional derivatives : electromagnetic systems [15], [16], dielectric polarization [17], viscoelastic systems [18], [19], chaotic synchronization and secure communications [20].

The question of stability is crucial in control theory. In the field of fractional-order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded input-bounded output stability, internal stability, etc. Stability results on fractional-order control systems have been presented in [21], [22], [23].

This paper is organized as follows.

In section II, we provide some background on the fractional derivative, the stability and the detectability of fractional-order systems with the fractional-order α belonging to $0 < \alpha < 2$.

In section III, we formulate the condition for the existence and the functional observers design problem for linear fractional-order systems. Sufficient conditions for the asymptotical stability of observers with fractional-order α belonging to $0 < \alpha \leqslant 1$ and $1 \leqslant \alpha < 2$ are presented in terms of linear matrix inequalities formulation. Finally, two illustrative examples are presented to illustrate of our proposed results.

Notations. M^T is the transpose of M, $\mathrm{Sym}\{X\}$ is used to denote $X^T + X$, Σ^+ is any generalized inverse of Σ satisfying $\Sigma\Sigma^+\Sigma = \Sigma$ and D^α represents initialized α^{th} order differintegration.

II. PRELIMINARY RESULTS

In this section, we present some preliminaries results on the fractional derivative systems which will be used in the sequel of this paper. Formulations of noninteger-order derivatives fall into two main classes: the Riemann-Liouville derivative and the Grûnward-Letnikov derivative, on one hand, defined as [12]

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\,\tau,\; n-1 \leqslant \alpha < n \tag{1}$$

or the Caputo derivative on the other, defined as [24],

$$D^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{\dfrac{\mathrm{d}^n f(\tau)}{\mathrm{d}\,t^n}}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\,\tau, \quad n-1 \leqslant \alpha < n \tag{2}$$

with $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, where $\Gamma(.)$ is the Gamma function and is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}\,t.$$

The physical interpretation of the fractional derivatives and the solution of fractional differential equations are given in [24]. Here and throughout the paper, only the Caputo definition is used since this Laplace transform allows using initial values of classical integer-order derivatives with clear

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physical interpretations. In the rest of this paper, D^α is used to denote the Caputo fractional derivative of order α.

Now, consider the following linear fractional-order systems

$$\begin{cases} D^\alpha x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \\ x(0) = x_0 \end{cases} \qquad 0 < \alpha < 2 \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector and $y(t) \in \mathbb{R}^p$ is the measured output. A, B and C are known constant matrices.

It has been shown that system (3) is stable if the following condition is satisfied [21], [25] for $0 < \alpha \leqslant 1$, [26] for $1 < \alpha < 2$

$$|\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2} \tag{4}$$

where $\mathrm{spec}(A)$ represents the eigenvalues of matrix A.

The necessary and sufficient LMIs conditions to satisfy condition (4) when the fractional-order α belonging to $0 < \alpha < 2$ are given in the two following lemmas.

Lemma 1: [26], [27], [28] Let $A \in \mathbb{R}^{n\times n}$, then $|\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2}$, where $1 \leqslant \alpha < 2$, if and only if there exits a matrix $P_0 = P_0^T > 0$ such that

$$\begin{bmatrix} (AP_0 + P_0A^T)\sin\theta & (AP_0 - P_0A^T)\cos\theta \\ (P_0A^T - AP_0)\cos\theta & (AP_0 + P_0A^T)\sin\theta \end{bmatrix} < 0$$

where $\theta = \pi - \alpha\frac{\pi}{2}$. □

Lemma 2: [29] Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable $(i.e.\, |\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2})$ if and only if there exist two real symmetric matrices $P_{k1} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{n\times n}$, $k = 1, 2$, such that

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \mathrm{Sym}\{\Gamma_{ij} \otimes (AP_{ij})\} < 0 \tag{5}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \tag{6}$$

where

$$\Gamma_{11} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & -\cos(\alpha\frac{\pi}{2}) \\ \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}, \Gamma_{12} = \begin{bmatrix} \cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \end{bmatrix},$$
$$\Gamma_{21} = \begin{bmatrix} \sin(\alpha\frac{\pi}{2}) & \cos(\alpha\frac{\pi}{2}) \\ -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \end{bmatrix}, \Gamma_{22} = \begin{bmatrix} -\cos(\alpha\frac{\pi}{2}) & \sin(\alpha\frac{\pi}{2}) \\ -\sin(\alpha\frac{\pi}{2}) & -\cos(\alpha\frac{\pi}{2}) \end{bmatrix}. \tag{7}$$

□

Notice that the conditions given in lemma 2 are equivalent to those given in [27] and [30].

To prove the main results in the next section, we need the following lemmas.

Lemma 3: [31], [32] System (3) is detectable if and only if

$$\mathrm{rank}\left(\begin{bmatrix} \sigma I_n - A \\ C \end{bmatrix}\right) = n, \forall\, \sigma \in \mathbb{C} \quad \text{with} \quad |\arg(\sigma)| \leqslant \alpha\frac{\pi}{2}. \tag{8}$$

Proof: The proof of lemma 3 can be established as in the usual integer order case, since it involves algebraic properties of the pair (C, A) only. ■

Remark 1: [31], [32] In particular if (C, A) is observable, *i.e.*

$$\mathrm{rank}\left(\begin{bmatrix} \sigma I_n - A \\ C \end{bmatrix}\right) = n \quad \forall\, \sigma \in \mathbb{C} \quad \text{with} \quad |\arg(\sigma)| \leqslant \alpha\frac{\pi}{2}$$

the spectrum of $\mathbb{L} = A - LC$ can be assigned anywhere in the complex region of asymptotic stability $(i.e.\, |\arg(\mathrm{spec}(\mathbb{L}))| > \alpha\frac{\pi}{2})$.

Lemma 4: Let X represents an $m \times n$ matrix and Y an $n \times p$ matrix then $\mathrm{rank}\, XY = \mathrm{rank}\, Y$ if and only if

$$\mathrm{rank}\begin{bmatrix} X \\ I - YY^+ \end{bmatrix} = n.$$

III. UNKNOWN INPUT FUNCTIONAL OBSERVERS DESIGN

In this section, we give sufficient conditions for the existence and stability of the functional observer with unknown input. A constructive procedure for the design of this functional observer will be presented.

Consider the following linear fractional-order systems

$$\begin{cases} D^\alpha x(t) = Ax(t) + Fd(t) + Bu(t) \\ y(t) = Cx(t) + Gd(t) \\ z(t) = Lx(t) \\ x(0) = x_0 \end{cases} \qquad 0 < \alpha < 2 \tag{9}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the measured output, $d(t) \in \mathbb{R}^q$ is the unknown input vector and $z(t) \in \mathbb{R}^r$ is the vector to be estimated where $r \leqslant n$. A, B, C, F, G and L are known constant real matrices of compatible dimensions, which must be determined such that $\widehat{z}(t)$ asymptotically converges to $z(t)$.

In order to reconstruct the state function we require a functional observer of the form

$$\begin{cases} D^\alpha \eta(t) = N\eta(t) + Jy(t) + Hu(t) \\ \widehat{z}(t) = \eta(t) + Ey(t) \\ \eta(0) = \eta_0 \end{cases} \qquad 0 < \alpha < 2 \tag{10}$$

where $\eta(t) \in \mathbb{R}^r$ is the state vector of observer and $\widehat{z}(t) \in \mathbb{R}^r$ is the estimate of $z(t)$. Matrices N, J, H and E are unknown matrices of appropriate dimensions to be designed.

The following proposition gives the conditions for the existence and stability of functional observer (10).

Proposition 1: System (10) is an asymptotic functional observer where $0 < \alpha < 2$, *i.e.* $\lim\limits_{t\to\infty} \widehat{z}(t) - z(t) = 0$, for any $x(0)$, $\widehat{z}(0)$ and $u(t)$ if

i) $D^\alpha e(t) = Ne(t)$ is asymptotically stable,
ii) $PA - NP - JC = 0$,
iii) $PF + NEG - JG = 0$,
iv) $H = PB$,
v) $EG = 0$, □

where $P = L - EC$.

Proof: Define $e(t) = z(t) - \widehat{z}(t)$, the error between $z(t)$ and $\widehat{z}(t)$, then its fractional-order dynamic is given by

$$D^\alpha e(t) = D^\alpha z(t) - D^\alpha \widehat{z}(t) \tag{11}$$

or equivalently

$$\begin{aligned} D^{\alpha}e(t) = {} & Ne(t)+(PA-NP-JC)x(t)+(PB-H)u(t) \\ & +(PF+NEG-JG)d(t)-EGD^{\alpha}d(t) \end{aligned} \tag{12}$$

where $P = L - EC$.

If conditions i), ii), iii), iv) and v) are satisfied, then $\lim\limits_{t\to\infty} e(t) = 0$ for any $x(0)$, $\widehat{z}(0)$, $d(t)$ and $u(t)$. Then, there must be matrices N, P, J and H such that

$$PA - NP - JC = 0 \tag{13a}$$

$$PF + NEG - JG = 0 \tag{13b}$$

$$PB = H \tag{13c}$$

$$EG = 0. \tag{13d}$$

From equation (12), one can see that under conditions (13a), (13b), (13c) and (13d) the fractional-order dynamic of this observer error is given by

$$D^{\alpha}e(t) = Ne(t) \quad \text{with} \quad 0 < \alpha < 2. \tag{14}$$

in this case $\lim\limits_{t\to\infty} e(t) = 0$ if (14) is asymptotically stable. ■

Now the design of the functional observer is reduced to finding the matrices N, P, J, H such that proposition 1 is satisfied. By using the definition of P, equations (13a) and (13b) can be written

$$NL + ECA + KC = LA \tag{15a}$$

$$KG + ECF = LF. \tag{15b}$$

where $K = J - NE$.

Equations (15a), (15b) and (13d) can be written as

$$\begin{bmatrix} N & K & E \end{bmatrix} \Sigma_1 = \Sigma_2 \tag{16}$$

where

$$\Sigma_1 = \begin{bmatrix} L & 0 & 0 \\ C & G & 0 \\ CA & CF & G \end{bmatrix} \quad \text{and} \quad \Sigma_2 = \begin{bmatrix} LA & LF & 0 \end{bmatrix}. \tag{17}$$

The following lemma gives the necessary and sufficient conditions for the existence of the solution of (16).

Lemma 5: There exists a solution to (16) if and only if

$$\operatorname{rank} \begin{bmatrix} L & 0 & 0 \\ C & G & 0 \\ CA & CF & G \\ LA & LF & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} L & 0 & 0 \\ C & G & 0 \\ CA & CF & G \end{bmatrix}. \tag{18}$$

□

Proof: From the general solution of linear matrix equations [33], there exists a solution of (16) if and only if

$$\Sigma_2 \Sigma_1^{+} \Sigma_1 = \Sigma_2 \tag{19}$$

where Σ_1^{+} is the generalized inverse matrix of Σ_1. Equation (19) is equivalent

$$\operatorname{rank} \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} = \operatorname{rank} \Sigma_1 \tag{20}$$

which is the condition (18). ■

In this case the general solution of (16) is given by

$$\begin{bmatrix} N & K & E \end{bmatrix} = \Sigma_2 \Sigma_1^{+} - Z(I - \Sigma_1 \Sigma_1^{+}) \tag{21}$$

where Z is an arbitrary matrix of appropriate dimension.

From (21), we obtain

$$N = \mathbb{A} - Z\mathbb{B} \tag{22}$$

where

$$\mathbb{A} = \Sigma_2 \Sigma_1^{+} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbb{B} = (I - \Sigma_1 \Sigma_1^{+}) \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Matrices J and H are obtained according to

$$J = K + NE \tag{24}$$

$$H = (L - EC)B \tag{25}$$

By using this algorithm we can compute all the observer parameters which provide a fractional-order functional observer of the form (10).

Under condition (18) and by using (22), the observer error dynamics can be written as

$$D^{\alpha}e(t) = Ne(t) = (\mathbb{A} - Z\mathbb{B})e(t) \quad \text{with} \quad 0 < \alpha < 2. \tag{26}$$

Now, the problem of the functional observer (10) design is reduced to the determination of the free matrix parameter Z such that condition i) of proposition 1 is satisfied.

The following lemma gives the necessary and sufficient conditions for the existence of the matrix parameter Z.

Lemma 6: There exists a matrix parameter Z such that (26) is asymptotically stable if and only if

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} \sigma L - LA & -LF & 0 \\ C & G & 0 \\ CA & CF & G \end{bmatrix} & = \operatorname{rank} \Sigma_1 \quad \forall\, \sigma \in \mathbb{C} \\ \forall\, \sigma \in \mathbb{C} \quad \text{with} & \quad |\arg(\sigma)| \leqslant \alpha \frac{\pi}{2} \end{aligned} \tag{27}$$

□

Proof: From (26), the matrix $N = \mathbb{A} - Z\mathbb{B}$ is asymptotically stable if and only if the pair $(\mathbb{B}, \mathbb{A})$ is detectable or equivalently

$$\operatorname{rank} \left(\begin{bmatrix} \sigma I - \mathbb{A} \\ \mathbb{B} \end{bmatrix} \right) = r, \ \forall\, \sigma \in \mathbb{C} \text{ with } |\arg(\sigma)| \leqslant \alpha \frac{\pi}{2}. \tag{28}$$

The left hand side of (27), can be written as

$$\begin{aligned} & \operatorname{rank} \begin{bmatrix} \sigma L - LA & -LF & 0 \\ C & G & 0 \\ CA & CF & G \end{bmatrix} \\ & = \operatorname{rank} \left[\begin{bmatrix} \sigma I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Sigma_1 - \begin{bmatrix} \Sigma_2 \\ 0 \end{bmatrix} \right] \\ & = \operatorname{rank} \left[\begin{bmatrix} \sigma I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} \Sigma_2 \Sigma_1^{+} \\ 0 \end{bmatrix} \right] \Sigma_1. \end{aligned}$$

From this equality one can see that (27) is satisfied if and only if

$$\operatorname{rank}\left[\left[\begin{matrix}\sigma I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{matrix}\right]-\begin{bmatrix}\Sigma_2\Sigma_1^{+}\\ 0\end{bmatrix}\right]\Sigma_1=\operatorname{rank}\Sigma_1$$

$$\forall\,\sigma\in\mathbb{C}\quad\text{with}\quad|\arg(\sigma)|\leqslant\alpha\frac{\pi}{2}.$$

Using lemma 4, this is equivalent to

$$\operatorname{rank}\left[\begin{matrix}\left[\begin{matrix}\sigma I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{matrix}\right]-\begin{bmatrix}\Sigma_2\Sigma_1^{+}\\ 0\end{bmatrix}\\ I-\Sigma_1\Sigma_1^{+}\end{matrix}\right]\ \text{is a full column rank,}$$

$$\forall\,\sigma\in\mathbb{C}\quad\text{and}\quad|\arg(\sigma)|\leqslant\alpha\frac{\pi}{2}.$$

Or equivalently, matrix

$$\left[\begin{matrix}\sigma I-\mathbb{A} & -\Sigma_2\Sigma_1^{+}\begin{bmatrix}0\\ I\\ 0\end{bmatrix} & -\Sigma_2\Sigma_1^{+}\begin{bmatrix}0\\ 0\\ I\end{bmatrix}\\ \mathbb{B} & (I-\Sigma_1\Sigma_1^{+})\begin{bmatrix}0\\ I\\ 0\end{bmatrix} & (I-\Sigma_1\Sigma_1^{+})\begin{bmatrix}0\\ 0\\ I\end{bmatrix}\\ 0 & I & 0\\ 0 & 0 & I\end{matrix}\right]$$

must be of full column row, $\forall\,\sigma\in\mathbb{C}$ and $|\arg(\sigma)|\leqslant\alpha\frac{\pi}{2}$. This is equivalent to

$$\operatorname{rank}\begin{bmatrix}\sigma I-\mathbb{A}\\ \mathbb{B}\end{bmatrix}=r\quad\forall\,\sigma\in\mathbb{C}\quad\text{with}\quad|\arg(\sigma)|\leqslant\alpha\frac{\pi}{2},$$

this ends the proof. ■

The asymptotical stability for the fractional-order observer error system (26) where $0<\alpha\leqslant 1$ and $1\leqslant\alpha<2$ is given in the two following theorems.

Theorem 1: Under conditions (18) and (27), there exists an asymptotically stable observer of the form (10) where $1\leqslant\alpha<2$, if there are matrices $X\in\mathbb{R}^{m\times n}$ and $P_0=P_0^T>0\in\mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12}^T & \Omega_{22}\end{bmatrix}<0 \tag{29}$$

where

$$\Omega_{11}=\Omega_{22}=(P_0\mathbb{A}+\mathbb{A}^TP_0-X\mathbb{B}-\mathbb{B}^TX^T)\sin\theta$$

$$\Omega_{12}=(\mathbb{A}^TP_0-P_0\mathbb{A}+X\mathbb{B}-\mathbb{B}^TX^T)\cos\theta$$

Moreover, a stabilizing gain matrix Z is given by

$$Z=XP_0^{-1}$$

□

Proof: From lemmas 3 and 27, one can see that the necessary condition that condition i) of proposition 1 is satisfied, by using (22), that the pair $(\mathbb{B},\mathbb{A})$ is detectable. Now, suppose that there exist matrices $X\in\mathbb{R}^{m\times n}$ and $P_0=P_0^T>0\in\mathbb{R}^{n\times n}$ such that (29) holds. It follows from lemma 1 that $|\arg(\operatorname{spec}(N))|>\alpha\frac{\pi}{2}$ is equivalent to

$$\begin{bmatrix}(P_0N+N^TP_0)\sin\theta & -(P_0N-N^TP_0)\cos\theta\\ (P_0N-N^TP_0)\cos\theta & (P_0N+N^TP_0)\sin\theta\end{bmatrix}$$

$$=\operatorname{Sym}\left\{\begin{bmatrix}P_0\mathbb{A}\sin\theta & -P_0\mathbb{A}\cos\theta\\ P_0\mathbb{A}\cos\theta & P_0\mathbb{A}\sin\theta\end{bmatrix}\right\}$$

$$+\operatorname{Sym}\left\{\begin{bmatrix}-X\mathbb{B}\sin\theta & X\mathbb{B}\cos\theta\\ -X\mathbb{B}\cos\theta & -X\mathbb{B}\sin\theta\end{bmatrix}\right\}<0 \tag{30}$$

where $Z=XP_0^{-1}$ and $\theta=\pi-\alpha\frac{\pi}{2}$. Inequality (30) is equivalent to (29). This ends the proof. ■

Theorem 2: Under conditions (18) and (27), there exists an asymptotically stable observer of the form (10) where $0<\alpha\leqslant 1$, if and only if there are matrices $Q\in\mathbb{R}^{m\times n}$ and $P_0=P_0^T>0\in\mathbb{R}^{n\times n}$ such that

$$\sum_{i=1}^{2}\left(\operatorname{Sym}\{\Gamma_{i1}\otimes(\mathbb{A}^TP_0)\}-\operatorname{Sym}\{\Gamma_{i1}\otimes(\mathbb{B}^TQ)\}\right)<0 \tag{31}$$

where $\Gamma_{i1}(i=1,2)$ satisfy (7) and the stabilizing gain matrix Z is given by $Z=P_0^{-1}Q^T$. □

Proof: From lemmas 3 and 27, one can see that the necessary condition that condition i) of proposition 1 is satisfied, by using (22), that the pair $(\mathbb{B},\mathbb{A})$ is detectable.

Suppose that there exist matrices $Q\in\mathbb{R}^{m\times n}$ and $P_0=P_0^T>0\in\mathbb{R}^{n\times n}$ such that (31) holds. It follows from lemma 2 that $|\arg(\operatorname{spec}(N))|>\alpha\frac{\pi}{2}$ is equivalent to

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\operatorname{Sym}\{\Gamma_{ij}\otimes(N^TP_{ij})\}<0 \tag{32}$$

where $N=\mathbb{A}-Z\mathbb{B}$ and $\Gamma_{ij}(i,j=1,2)$ satisfy (7). By setting $P_{11}=P_{21}=P_0$, $P_{12}=P_{22}=0$ in (32), one can conclude that if

$$\operatorname{Sym}\{\Gamma_{11}\otimes(N^TP_0)\}+\operatorname{Sym}\{\Gamma_{21}\otimes(N^TP_0)\}<0 \tag{33}$$

the fractional-order system $D^\alpha e(t)=Ne(t)$ where $0<\alpha\leqslant 1$ is asymptotically stable.

Substitute $N=\mathbb{A}-Z\mathbb{B}$ into (33) and set $Z=P_0^{-1}Q^T$, we obtain

$$\sum_{i=1}^{2}\left(\operatorname{Sym}\{\Gamma_{i1}\otimes(\mathbb{A}^TP_0)\}-\operatorname{Sym}\{\Gamma_{i1}\otimes(\mathbb{B}^TQ)\}\right)<0 \tag{34}$$

Inequality (34) is equivalent to (31). This completes the proof. ■

IV. NUMERICAL EXAMPLES

In this section, we provide two numerical examples to illustrate the applicability of the proposed method.

A. Example 1 : $\alpha=1.76$

Consider the linear fractional-order system (9) with the following matrices

$$A=\begin{bmatrix}-1 & 0 & 0 & 0\\ 1 & -2 & 1 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & -4\end{bmatrix},\quad F=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One can see that conditions (18) and (27) are satisfied.

From the results of section III, we obtain

$$\mathbb{A} = -2.2857$$

and

$$\mathbb{B} = \begin{bmatrix} 0.1429 & 0.1429 & -0.2857 & 0 & -0.1429 \end{bmatrix}^T.$$

A feasible solution of LMI (29) where $\alpha = 1.76$ is as follows

$$P_0 = 4.9402 \times 10^8,$$

$$X = 10^7 \times \begin{bmatrix} 3.0876 & 3.0876 & -6.1752 & 0 & -3.0876 \end{bmatrix}.$$

Then, the asymptotically stabilizing state-feedback gain is obtained as

$$Z = XP_0^{-1} = \begin{bmatrix} 0.0625 & 0.0625 & -0.1250 & 0 & -0.0625 \end{bmatrix}.$$

Using the algorithm in section III, we obtain all the following functional observer parameters

$$N = -2.3482, \quad K = \begin{bmatrix} 0.6518 & -0.3036 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & -0.6518 \end{bmatrix}, \quad J = \begin{bmatrix} 0.6518 & 1.2270 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0.6518 & 1 & 0 \end{bmatrix} \quad \text{and} \quad H = 0.6518.$$

Finally, the estimate $\widehat{z}(t)$ is given by the following observer

$$\begin{cases} D^{\alpha}\eta(t) = -2.3482\eta(t) + \begin{bmatrix} 0.6518 & 1.2270 \end{bmatrix} y(t) + 0.6518u(t) \\ \widehat{z}(t) = \eta(t) - 0.6518 y_2(t) \end{cases}$$

with $\alpha = 1.76$.

Figures 1 and 2 show the performances of the functional observer presented in this paper for $\alpha = 1.76$ with the unknown input vector $d(t) = 0.5\sin(60\pi t)$.

B. Example 2 : $\alpha = 0.77$

Consider the linear fractional-order system (9) with the following matrices

$$A = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 3 & -8 & 1 & 0 \\ 0 & -1 & -10 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}, \; F = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It can easy to see that conditions (18) and (27) are satisfied.

From the results of section III, we obtain

$$\mathbb{A} = -10.8381$$

and

$$\mathbb{B} = \begin{bmatrix} 0.0381 & -0.1143 & -0.1524 & 0 & -0.0190 \end{bmatrix}^T.$$

A feasible solution of LMI (31) where $\alpha = 0.77$ is as follows

$$P_0 = 2.4398 \times 10^7,$$

$$Q = 10^5 \times \begin{bmatrix} 0.8576 & -2.5738 & -3.4303 & 0 & -0.4288 \end{bmatrix}^T.$$

Then, the asymptotically stabilizing state-feedback gain is obtained as

$$Z = P_0^{-1} Q^T = \begin{bmatrix} 0.0035 & -0.0105 & -0.0141 & 0 & -0.0018 \end{bmatrix}.$$

Using the algorithm in section III, we obtain all the following functional observer parameters

$$N = -10.8416, \quad K = \begin{bmatrix} -7.4752 & 2.8664 \end{bmatrix},$$

$$E = \begin{bmatrix} -2 & 0.4208 \end{bmatrix}, \quad J = \begin{bmatrix} 14.2080 & -1.6957 \end{bmatrix},$$

$$P = \begin{bmatrix} -2 & -0.8416 & 1 & 0 \end{bmatrix} \quad \text{and} \quad H = -2.8416.$$

Finally, the estimate $\widehat{z}(t)$ is given by the following observer

$$\begin{cases} D^{\alpha}\eta(t) = -10.8416\eta(t) + \begin{bmatrix} 14.2080 & -1.6957 \end{bmatrix} y(t) - 2.8416u(t) \\ \widehat{z}(t) = \eta(t) + \begin{bmatrix} -2 & 0.4208 \end{bmatrix} y(t) \end{cases}$$

where $\alpha = 0.77$.

Figures 3 and 4 show the performances of the functional observer presented in this paper for $\alpha = 0.77$ with the unknown input vector $d(t) = 0.5\sin(60\pi t)$.

V. CONCLUSION

In this paper, we have presented a simple method to design a functional observer for linear fractional-order systems. This method reduces the design procedure to one of full-order system. The conditions for the existence of these observers are given, sufficient conditions for their stability are derived in terms of linear matrix inequalities formulation with fractional-order α belonging to $0 < \alpha < 2$. Two illustrative examples have shown the effectiveness of our results.

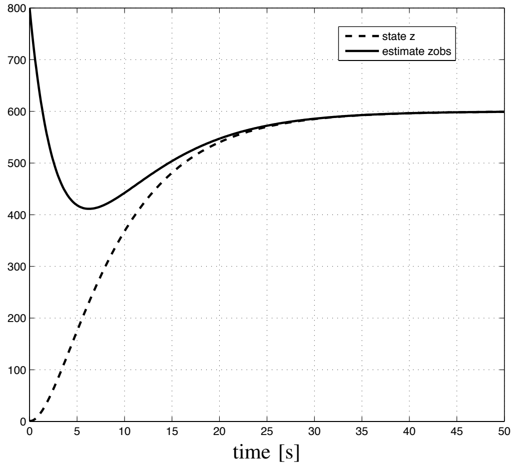

Fig. 1. State response z and this estimate $\widehat{z}$ in example IV-A with fractional order $\alpha = 1.76$.

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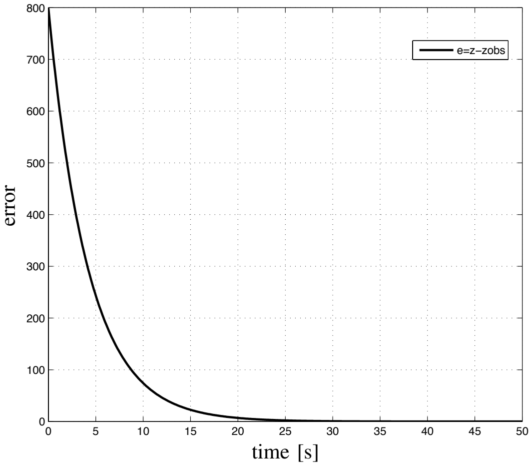

Fig. 2. Error function of functional observer in example IV-A with fractional order $\alpha = 1.76$.

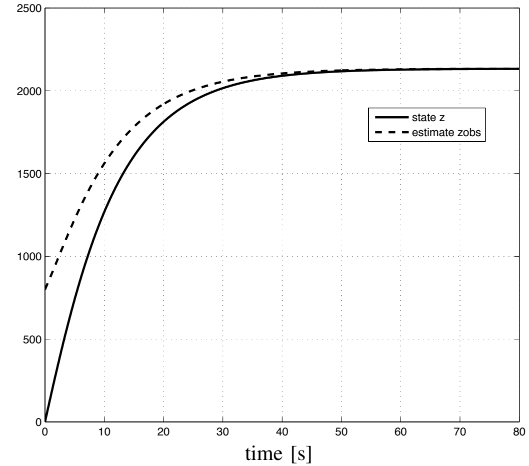

Fig. 3. State response z and this estimate $\widehat{z}$ in example IV-B with fractional order $\alpha = 0.77$.

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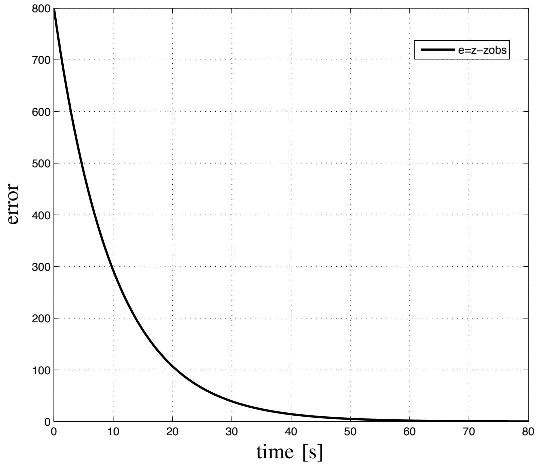

Fig. 4. Error function of functional observer in example IV-B with fractional order $\alpha = 0.77$.

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