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Simple cascade observer for a class of nonlinear systems with long delay output

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***Abstract*— The present work proposes a state observer with a cascade structure for a class of nonlinear systems in the presence of delayed output measurements. The first system in the cascade allows to estimate the delayed state while each of the remaining ones is a predictor. Each predictor estimates the state of the preceding one with a prediction horizon equal to a fraction of the time delay in such a way that the state of the last predictor is an estimate of the system actual state. The design of the observer is achieved by assuming a set of conditions under which the exponential convergence of the estimation error to zero is established. It was in particular shown that the number of the systems in the cascade depends on the magnitude of the considered delay.**

I. Introduction

During the last two decades, an intensive research activity has been devoted to investigate the stability, control and state estimation for systems with time delays. A particular attention has been paid to the case of linear systems (See for instance [1], [2], [3], [4], [5], [6] and references therein) whereas only few results have been established in the nonlinear case (see for instance [7], [8], [9]). Moreover, in most works dealing with state estimation for delay systems, the output is assumed to be free-delay. In many real-time applications, some state variables may not be available instantaneously and corresponding measurements are systematically tainted with delay. One can cite the example of bioreactors where most of the component measurements are obtained with a more or less important time delay since they result from time consuming laboratory analyses. Another typical example is that of network connected systems where some output data are transmitted through low-rate communication systems. This generally introduces non negligible time-delays that has to be account for in order to ensure the viability of the control and monitoring system.

Works treating of systems observation with delayed output measurements are very few. A very interesting approach dealing with this problem was reported in ([10]) where the authors proposed a constant gain observer for a class of single output systems that are observable for any input with globally Lipschitz nonlinearities and where the output is available with a constant delay τ. The main characteristic of the proposed observer lies in its structure that consists in $m+1$ cascade subsystems where the first subsystem (the head of the cascade) is an observer of the delayed state. Each one of the m remaining subsystems is a predictor: the state of the predictor at rank j is an estimation of the delayed state with a time delay equal to $\tau - \frac{j}{m}\tau$ in such a way that the state of the predictor at rank m is an estimate of the actual state. The number m of the predictors in the cascade depends on the magnitude of the system nonlinearities Lipschitz constant and the time delay. More specifically, this dependence is explained under the form of an inequality that has to be satisfied when choosing m. The idea of designing an observer with a chain structure has been reconsidered in [11] where the formulation of the nonlinear observer design is based on the resolution of first order singular partial differential equations. The resulting observer is similar to that given in [10] but it involves more design parameters that allow to relax the condition between the number of predictors in the cascade versus the nonlinearities Lipschitz constant and the time delay magnitude. In the two chain observers referenced above, each predictor in the cascade involves a correction term the expression of which becomes complex when the predictor rank in the cascade is high. This may give rise to implementation problems when a great number of predictors in the cascade is required. Moreover, the time delay considered in the above two works is assumed constant. In a recent work ([12]), the case of the time-varying delay was considered: the authors proposed a constant gain observer where the dimension of the observer is equal to that of the system. The stability of the proposed observer is based on the existence of a symmetric positive definite matrix depending on the observer gain. As mentioned in ([12]), such a matrix may not exist when delays with relatively large magnitude are considered and thereby the observer cannot be designed.

In the present work, one shall firstly propose a cascade observer for a class of single output nonlinear systems that are observable for any input and where the output is available with a constant delay. The main characteristic of the proposed observer consists in the availability of a matrix design parameter that allows to assign the poles of the predictors and to get an identical simple structure for all the predictors gain.

This paper is organized as follows: in the next section the class of considered systems is introduced and some requisite preliminaries related to the observer design in the free delay output case are briefly presented. In section 3, the observer design is proposed with a full convergence analysis. Finally, some concluding remarks are given in section 4.

II. Problem formulation and preliminaries

One shall consider the following class of single input-single output (SISO) nonlinear systems

$$\left\{\begin{array}{lcl}\dot{x}(t) & = & f(x(t))+g(x(t))u(t)\\ y_\tau(t) & = & h(x(t-\tau))\end{array}\right. \quad (1)$$

where $\tau > 0$ is the measurement delay, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, the vector functions f, g and h are $\mathcal{C}^\infty$. The delayed available output $y_\tau(t) = y(t-\tau)$ where $y(t)$ is the undelayed output.

One assumes that system (1) is uniformly observable for any input ([13]). Such a system is then diffeomorphic to a system of the form

$$\left\{\begin{array}{lcl}\dot{z}(t) & = & Az(t)+\varphi\left(u(t),z(t)\right)\\ y_\tau(t) & = & z_1(t-\tau)=Cz(t-\tau)\end{array}\right. \quad (2)$$

where

$$A=\begin{pmatrix}0 & I_{n-1}\\ 0 & 0\end{pmatrix},\ C=\begin{pmatrix}1 & 0 & \dots & 0\end{pmatrix} \quad (3)$$

the state $z = \begin{pmatrix}z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix} \in \mathbb{R}^n$ with $z_i \in \mathbb{R}$ and I_{n-1} is the identity matrix of order $n-1$. The nonlinear vector function φ assumes a triangular structure with respect to z, i.e. $\varphi(u,z) = \begin{pmatrix}\varphi_1(u,z_1)\\ \varphi_2(u,z_1,z_2)\\ \vdots\\ \varphi_n(u,z)\end{pmatrix}$. In the free delay case ($y_\tau(t) = y(t)$), a high gain observer has been proposed for system (1). The design of the observer has been achieved under the following assumption:

(H1) The function φ is globally Lipschitz in z uniformly in u i.e. for all bounded inputs, there exists $L_\varphi > 0$ such that for all $z, \bar{z} \in \mathbb{R}^n$, one has :$\|\varphi(u,z)-\varphi(u,\bar{z})\| \leq L_\varphi\|z-\bar{z}\|$

The equations of the proposed observer can be written as follows in the z'coordiantes:

$$\dot{\hat{z}}(t) = A\hat{z}(t)+\varphi(u(t),\hat{z}(t))-\theta\Delta_\theta^{-1}K(C\hat{z}(t)-y(t)) \quad (4)$$

where K is a constant vector and is chosen such that the matrix $A-KC$ is Hurwitz, $\theta \geq 1$ is a design parameter and Δ_θ is the following diagonal matrix

$$\Delta_\theta = diag\left(1,\frac{1}{\theta},\dots,\frac{1}{\theta^{n-1}}\right) \quad (5)$$

The equations of the observer can be written in the original coordinates by considering the inverse of the transformation jacobian as follows:

$$\begin{array}{lcl}\dot{\hat{x}}(t) & = & f(\hat{x}(t))+g(\hat{x}(t))u(t)\\ & & -\left(\dfrac{\partial\Phi}{\partial x}(\hat{x}(t))\right)^{-1}\theta\Delta_\theta^{-1}K(h(\hat{x})-y(t))\end{array}$$

The exponential convergence to zero of the observation error for bounded inputs can be expressed as follows:

$$\|\hat{z}(t)-z(t)\| \leq \eta(\theta)e^{-\lambda(\theta)t}\|\hat{z}(0)-z(0)\| \quad (6)$$

where $\eta(\theta)$ is polynomial in θ and $\lim\limits_{\theta\to+\infty}\lambda(\theta)=+\infty$
Of course, since $z = \Phi(x)$ is a diffeomorphism, the observation error $\hat{x}(t)-x(t)$ also converges to zero, exponentially.

In the next sections, the observer design is detailed with a full convergence analysis . Then, the observer equations are given in the z'coordinates before being derived in the original ones in x.
The observer design requires the adoption of some assumptions that shall be stated in due courses. At this step, one assumes that (H1) holds throughout the paper.

III. Observer synthesis:

As in ([10], [11]), one adopts the following notations:

$$\begin{array}{lcl}z_j(t) & = & z\left(t-\tau+\dfrac{j}{m}\tau\right) \quad \text{and}\\ u_j(t) & = & u\left(t-\tau+\dfrac{j}{m}\tau\right)\end{array} \quad (7)$$

for $j = 0,\dots,m$ and $t \geq -\frac{j}{m}\tau$ where m in a positive integer that shall be specified more precisely later.
The observer we propose is composed by $m+1$ cascade subsystems where the first subsystem is an observer for the delayed state whereas each one of the remaining subsystems is a predictor. As in ([10], [11]), the predictor of rank j in the cascade predicts the state of the preceding subsystem with a prediction horizon equal to $\dfrac{\tau}{m}$ in such a way that the state of the $m'th$ predictor is an estimate of the system actual state.

According to the adopted notation, the state of the delayed system is governed by the following dynamical model

$$\begin{array}{lcl}\dot{z}_0(t) & = & A\hat{z}_0(t)+\varphi(u(t),z(t))\\ y_{z_0}(t) & = & y_\tau(t)=Cz_0(t)\end{array} \quad (8)$$

System (8) in under the form (2) where no delay is considered in the latter. As a result, a high gain observer under the form (4) can be used for the estimation of z_0. The equations of such an observer specialize as follows:

$$\begin{array}{lcl}\dot{\hat{z}}_0 & = & A\hat{z}_0(t)+\varphi(u(t),\hat{z}_0(t))\\ & & -\theta\Delta_\theta^{-1}K(C\hat{z}_0(t)-y_\tau(t))\end{array} \quad (9)$$

where, as in (4), K is a vector such that $(A-KC)$ is Hurwitz and $\theta \geq 1$ is a design parameter.

One now shall focus on the structure of the remaining m predictors. As it has been mentioned in the introduction, a main characteristic of these predictors is that their respective gains have the same structure and this is true whatever is the rank of the predictor in the cascade. The design of these predictors is described below.

A. The predictors' structure

Recall that the variable z_j denotes the delayed state with a time delay equal to $\tau - \dfrac{j}{m}\tau$. Since observer (9) allows the estimation of the state z_0, one shall focus on the case where $j = 1, \ldots, m$: the aim is to design a predictor that estimates the state z_j. The latter is governed by the following dynamical equation

$$\dot{z}_j(t) = Az_j(t) + \varphi(u_j(t), z_j(t)) \tag{10}$$

System (10) can be rewritten as follows

$$\dot{z}_j(t) = \bar{A}z_j(t) + \varphi(u_j(t), z_j(t)) + (A - \bar{A})z_j(t) \tag{11}$$

where $\bar{A}$ is a Hurwitz matrix. As it shall be detailed later, the consideration of this matrix allows to derive a simple gain structure for the predictor.
According to (11), the state z_j can be explained as follows:

$$\begin{aligned} z_j(t) &= e^{\bar{A}\frac{\tau}{m}} z_j\left(t - \frac{\tau}{m}\right) \\ &+ \int_{t-\frac{\tau}{m}}^{t} e^{\bar{A}(t-s)} \left(\varphi(u_j(s), z_j(s)) + (A - \bar{A})z_j(s)\right) ds \\ &= e^{\bar{A}\frac{\tau}{m}} z_{j-1}(t) \\ &+ e^{\bar{A}t} \int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s} \left(\varphi(u_j(s), z_j(s)) + (A - \bar{A})z_j(s)\right) ds \end{aligned} \tag{12}$$

Let us denote by $\hat{z}_j$ the state of the predictor that shall provide an estimate of the state z_j. Motivated by implementation simplicity and miming the structure of the delayed state observer (9), one assigns the dynamics of $\hat{z}_j$ as follows:

$$\dot{\hat{z}}_j(t) = A\hat{z}_j(t) + \varphi(u_j(t), \hat{z}_j(t)) - G_j(t) \tag{13}$$

where $G_j(t)$, $j = 1, \ldots, m$ is a corrective term that has to be chosen in order to guarantee the exponential convergence to zero of the prediction error, e.g. $\tilde{z}_j(t) = \hat{z}_j(t) - z_j(t)$. Notice that the structure of the state delayed observer (9) is similar to that of predictor (13) with

$$\begin{aligned} G_0(t) &= -\theta\Delta_\theta^{-1}K(C\hat{z}_0(t) - y_\tau(t)) \\ &= -\theta\Delta_\theta^{-1}KC(\hat{z}_0(t) - z_0(t)) \\ &\triangleq -\theta\Delta_\theta^{-1}KC\tilde{z}_0(t) \end{aligned}$$

where $\tilde{z}_0(t)$ is the observation error corresponding to the delayed state and it converges to zero exponentially according to the above developments.

Now, miming (11), the predictor equation (13) can be rewritten as follows:

$$\begin{aligned} \dot{\hat{z}}_j(t) &= \bar{A}\hat{z}_j(t) + \varphi(u_j(t), \hat{z}_j(t)) \\ &+ (A - \bar{A})\hat{z}_j(t) - G_j(t) \end{aligned} \tag{14}$$

Again and according to (14), the prediction $\hat{z}_j$ can be explained as follows:

$$\begin{aligned} \hat{z}_j(t) &= e^{\bar{A}\frac{\tau}{m}} \hat{z}_j\left(t - \frac{\tau}{m}\right) + e^{\bar{A}t} \int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s} \times \\ &\left(\varphi(u(s), \hat{z}_j(s)) + (A - \bar{A})\hat{z}_j(s) - G_j(s)\right) ds \end{aligned} \tag{15}$$

Now, one imposes the following relationship between the states corresponding to two successive predictors, e.g. $\hat{z}_j$ and $\hat{z}_{j-1}$, $j = 1, \ldots, m$

$$\begin{aligned} \hat{z}_j(t) &= e^{\bar{A}\frac{\tau}{m}} \left(\hat{z}_{j-1}(t) - r_j(t)\right) \\ &+ e^{\bar{A}t} \int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s} \left(\varphi(u_j(s), \hat{z}_j(s)) + (A - \bar{A})\hat{z}_j(s)\right) ds \end{aligned} \tag{16}$$

where the r_j's, $j = 1, \ldots, m$, are functions that are differentiable with respect to time and shall be determined simultaneously with the G_j's in the next subsection. One notices that, the introduction of the functions r_j is motivated by the aim to derive simple expressions for the predictors gain. Indeed, one shall show later that unlike the predictors proposed in ([10], [11]) where the gain expression complexity grows with the predictor rank in the cascade, the structure of the predictors gain proposed in this paper is simple and is same for all the predictors. Moreover, one shall show that the functions r_j only intervene in the analysis of the observer convergence and they do not appear in the observer equations.

B. Determination of the predictors' gain

Subtracting equation (16) from equation (15) gives

$$\begin{aligned} &e^{\bar{A}\frac{\tau}{m}} \left(\hat{z}_j\left(t - \frac{\tau}{m}\right) - \hat{z}_{j-1}(t) + r_j(t)\right) = \\ &e^{\bar{A}t} \int_{t-\frac{\tau}{m}}^{t} e^{-\bar{A}s} G_j(s) ds \end{aligned}$$

Differentiating with respect to time each side of the above equation yields to

$$\begin{aligned} &e^{\bar{A}\frac{\tau}{m}} \left(\dot{\hat{z}}_j\left(t - \frac{\tau}{m}\right) - \dot{\hat{z}}_{j-1}(t) + \dot{r}_j(t)\right) = \\ &\bar{A}e^{\bar{A}\frac{\tau}{m}} \left(\hat{z}_j\left(t - \frac{\tau}{m}\right) - \hat{z}_{j-1}(t) + r_j(t)\right) + \\ &G_j(t) - e^{\bar{A}\frac{\tau}{m}} G_j\left(t - \frac{\tau}{m}\right) \end{aligned} \tag{17}$$

Substituting in (17) $\dot{\hat{z}}_{j-1}(t)$ and $\dot{\hat{z}}_j\left(t - \frac{\tau}{m}\right)$ by their expressions given by (13) leads to

$$\begin{aligned} &e^{\bar{A}\frac{\tau}{m}} \left(A\left(\hat{z}_j\left(t - \frac{\tau}{m}\right) - \hat{z}_{j-1}(t)\right)\right. \\ &+\varphi(u_{j-1}, \hat{z}_j\left(t - \frac{\tau}{m}\right)) - \varphi(u_{j-1}, z_{j-1}(t)) + \\ &\left.G_{j-1}(t) - G_j\left(t - \frac{\tau}{m}\right) + \dot{r}_j(t)\right) = \\ &\bar{A}e^{\bar{A}\frac{\tau}{m}} \left(\hat{z}_j\left(t - \frac{\tau}{m}\right) - \hat{z}_{j-1}(t) + r_j(t)\right) \\ &+G_j(t) - e^{\bar{A}\frac{\tau}{m}} G_j\left(t - \frac{\tau}{m}\right) \end{aligned} \tag{18}$$

Notice that one has used in the last equation the identity $u_j\left(t - \frac{\tau}{m}\right) = u_{j-1}(t)$. Now, using the fact that the matrices $\bar{A}$ and $e^{\bar{A}\frac{\tau}{m}}$ commute, equation (18) leads to

$$\begin{aligned} G_j(t) &= e^{\bar{A}\frac{\tau}{m}} \left((A - \bar{A})\left(\hat{z}_j\left(t - \frac{\tau}{m}\right) - \hat{z}_{j-1}(t)\right)\right. \\ &\left.+\varphi(u_{j-1}, \hat{z}_j\left(t - \frac{\tau}{m}\right)) - \varphi(u_{j-1}, z_{j-1}(t))\right) \\ &+e^{\bar{A}\frac{\tau}{m}} \left(\dot{r}_j(t) - \bar{A}r_j(t) + G_{j-1}(t)\right) \end{aligned} \tag{19}$$

Now, if one chooses $r_j(t)$ such that

$$\begin{aligned} \dot{r}_j(t) &= \bar{A}r_j(t) - G_{j-1}(t) \\ \text{with } r_j(0) &= 0,\ j = 1, \ldots, m \end{aligned} \quad (20)$$

the expression of $G_j(t)$ becomes

$$G_j(t) = e^{\bar{A}\frac{\tau}{m}} \Big((A - \bar{A}) \left(\hat{z}_j \left(t - \frac{\tau}{m} \right) - \hat{z}_{j-1}(t) \right) + \varphi(u_{j-1}, \hat{z}_j \left(t - \frac{\tau}{m} \right)) - \varphi(u_{j-1}, z_{j-1}(t)) \Big) \quad (21)$$

Before giving the main theorem that summarizes the results obtained through the above developments, one recalls that since the matrix $\bar{A}$ is Hurwitz, there exist positive numbers β and $\bar{a}$ such that ([14], [11])

$$\forall t \geq 0 : \ \|e^{\bar{A}t}\| \leq \beta e^{-\bar{a}t} \quad (22)$$

The equations of the candidate observer for system (2) are then given by

$$\begin{aligned} &\text{for } j = 0, \ldots, m : \\ &\dot{\hat{z}}_j(t) = A\hat{z}_j(t) + \varphi(u_j(t), \hat{z}_j(t)) - G_j(t) \\ &G_0(t) = \theta\Delta_\theta^{-1}KC(\hat{z}_0(t) - y_\tau(t)) \\ &\text{and for } j = 1, \ldots, m : \\ &G_j(t) = e^{\bar{A}\frac{\tau}{m}} \Big((A - \bar{A}) \left(\hat{z}_j \left(t - \frac{\tau}{m} \right) - \hat{z}_{j-1}(t) \right) \\ &+\varphi \left(u_{j-1}(t), \hat{z}_j \left(t - \frac{\tau}{m} \right) \right) - \varphi(u_{j-1}(t), \hat{z}_{j-1}(t)) \Big) \end{aligned} \quad (23)$$

where the matrix $\bar{A}$ is Hurwitz, the vector K is such that $A - KC$ is Hurwitz, Δ_θ is the diagonal matrix given by (5) and $\theta \geq 1$ is a design parameter.

One states the following

Theorem *3.1:* For system (2) assume that Hypothesis (H1) holds true. If

$$\beta \left(L_\varphi + \|\bar{A} - A\| \right) \frac{\tau}{m} < 1 \quad (24)$$

then, there exist positive constants μ and α such that for all $t \geq -\tau$

$$\|\hat{z}_m(t) - z(t)\| \leq \mu e^{-\alpha t} \quad (25)$$

with μ being dependent on the observer initialization error. The proof of the theorem is given below.

C. Proof of Theorem 3.1

Let $\tilde{z}_j = \hat{z}_j - z_j$ denotes the estimation error for $j = 0, \ldots, m$. It is clear from the developments detailed above that the high gain observer (9) that constitutes the head of the cascade in the observer equations (23) provides an estimate of the delayed state z_0 and the associated estimation error satisfies an inequality similar to that given by (6). One thus has

$$\|\tilde{z}_0(t)\| \leq \eta(\theta)e^{-\lambda(\theta)t}\tilde{z}_0(0) \quad (26)$$

According to the expression of G_0 given by (23), one also has

$$\begin{aligned} \|G_0(t)\| &\leq \theta\|\Delta_\theta^{-1}KC\|\eta(\theta)e^{-\lambda(\theta)t}\tilde{z}_0(0) \\ &\leq \theta^n\|KC\|\eta(\theta)e^{-\lambda(\theta)t}\tilde{z}_0(0) \end{aligned}$$

Set

$$\mu_0 = \theta^n\|KC\|\eta(\theta)e^{-\lambda(\theta)t}\tilde{z}_0(0), \ \alpha_0 = \lambda(\theta) \quad (27)$$

Since the design parameter θ can be chosen high enough, one can assume without loss of generality that θ is chosen such that $\alpha_0 > \bar{a}$ where $\bar{a}$ is given by (22) and $\|\theta\Delta_\theta^{-1}KC\| \geq 1$. With the so adopted notations and conventions, one has

$$\|\tilde{z}_0(t)\| \leq \mu_0 e^{-\alpha_0 t} \leq \mu_0 e^{-\bar{a}t} \text{ and } \|G_0(t)\| \leq \mu_0 e^{-\alpha_0 t} \quad (28)$$

In order to prove inequality (25) of the theorem, one shall proceed by a mathematical induction on the value of m. One has first to prove this inequality for $m = 1$. Indeed, according to equation (20) and since $\bar{A}$ is Hurwitz and G_0 converges exponentially to zero, the function $r_1(t)$ also converges exponentially to zero. More precisely, from (20), one obtains: $r_1(t) = -\int_0^t e^{\bar{A}(t-s)}G_0(s)ds$ and according to (22) and (28), this leads to

$$\begin{aligned} \|r_1(t)\| &\leq \beta \int_0^t e^{-\bar{a}(t-s)}\|G_0(s)\|ds \\ &\leq \beta\mu_0 e^{-\bar{a}t} \int_0^t e^{(\bar{a}-\alpha_0)s}ds \\ &= \beta\mu_0 e^{-\bar{a}t} \frac{1 - e^{(\bar{a}-\alpha_0)t}}{\alpha_0 - \bar{a}} \\ &\leq \frac{\beta}{\alpha_0 - \bar{a}}\mu_0 e^{-\bar{a}t} \text{ since } \alpha_0 > \bar{a} \end{aligned} \quad (29)$$

Now, from (12) and (16), one obtains

$$\begin{aligned} &\tilde{z}_j(t) = e^{\bar{A}\frac{\tau}{m}} \left(\tilde{z}_{j-1}(t) - r_j(t) \right) \\ &+ \int_{t-\frac{\tau}{m}}^t e^{\bar{A}(t-s)} \big(\varphi(u_j(s), \hat{z}_j(s)) - \varphi(u_j(s), z_j(s)) \\ &+(A - \bar{A})\tilde{z}_j(s) \big) ds \implies \end{aligned}$$

$$\begin{aligned} &\|\tilde{z}_j(t)\| \leq \|e^{\bar{A}\frac{\tau}{m}}\| \left(\|\tilde{z}_{j-1}(t)\| + \|r_j(t)\| \right) \\ &+ \int_{t-\frac{\tau}{m}}^t \|e^{\bar{A}(t-s)}\| \\ &\big(\|\varphi(u_j(s), \hat{z}_j(s)) - \varphi(u_j(s), z_j(s))\| \\ &+\|(A - \bar{A})\|\|\tilde{z}_j(s)\| \big) ds \\ &\leq \|e^{\bar{A}\frac{\tau}{m}}\| \left(\|\tilde{z}_{j-1}(t)\| + \|r_j(t)\| \right) \\ &+(L_\varphi + \|A - \bar{A}\|) \int_{t-\frac{\tau}{m}}^t \|e^{\bar{A}(t-s)}\|\|\tilde{z}_j(s)\|ds \\ &\leq \|e^{\bar{A}\frac{\tau}{m}}\| \left(\|\tilde{z}_{j-1}(t)\| + \|r_j(t)\| \right) \\ &+\beta(L_\varphi + \|A - \bar{A}\|) \int_{t-\frac{\tau}{m}}^t \|\tilde{z}_j(s)\|ds \end{aligned} \quad (30)$$

Setting $j = 1$ in the above inequality, one gets:

$$\begin{aligned} \tilde{z}_1(t) \leq\ & \|e^{\bar{A}\frac{\tau}{m}}\| \left(\|\tilde{z}_0(t)\| + \|r_1(t)\| \right) \\ &+\beta(L_\varphi + \|A - \bar{A}\|) \int_{t-\frac{\tau}{m}}^t \|\tilde{z}_1(s)\|ds \end{aligned} \quad (31)$$

Using (28) and (29), inequality (31) becomes

$$\begin{aligned} \tilde{z}_1(t) \leq\ & \|e^{\bar{A}\frac{\tau}{m}}\|\mu_0 \left(1 + \frac{\beta}{\alpha_0 - \bar{a}} \right) e^{-\bar{a}t} \\ &+\beta(L_\varphi + \|A - \bar{A}\|) \int_{t-\frac{\tau}{m}}^t \|\tilde{z}_1(s)\|ds \end{aligned}$$

Now, one needs the following technical Lemma:

Lemma *3.1:* ([10]) Consider a function $s(t) \geq 0$, $t \in [\delta, +\infty[$, with $\delta > 0$ such that

$$\int_{-\delta}^{0} s(\nu)d\nu < +\infty, \quad s(t) \leq \mu e^{-\bar{\alpha}t} + \gamma \int_{t-\delta}^{t} s(\nu)d\nu \qquad (32)$$

where $\mu, \bar{\alpha}$ and γ are positive reals. If $\gamma\delta < 1$ then there exists a positive real $\alpha \leq \bar{\alpha}$ such that

$$s(t) \quad \leq \quad \bar{\mu}e^{-\alpha t} \ , \ t \geq 0 \qquad (33)$$

where $\bar{\mu} = \frac{e^{\alpha\delta}}{1-c}\left(\mu + \gamma \int_{-\delta}^{0} s(\nu)d\nu\right)$ and $c = \frac{\gamma}{\alpha}(e^{\alpha\delta} - 1) < 1$.

Due to Lemma 3.1 and the assumptions of Theorem 3.1 and by setting $\delta = \frac{\tau}{m}$, $\mu = \mu_0$, $\bar{\alpha} = \bar{a}$ and $\gamma = \beta(L_\varphi + \|A - \bar{A}\|)$, it can be easily concluded that there exists a pair of positive numbers $(\mu_1, \bar{\alpha}_1)$ with $0 < \bar{\alpha}_1 \leq \bar{a}$ such that

$$\|\tilde{z}_1(t)\| \leq \mu_1 e^{-\bar{\alpha}_1 t} \qquad (34)$$

where

$$\begin{aligned} \mu_1 &= \frac{e^{\bar{\alpha}_1 \frac{\tau}{m}}}{1 - c_1}\left(\|e^{\bar{A}\frac{\tau}{m}}\|\left(1 + \frac{\beta}{\alpha_0 - \bar{a}}\right)\mu_0 \right. \\ &\left. +\gamma \int_{-\frac{\tau}{m}}^{0} \|\tilde{z}_1(\nu)\|d\nu\right) \text{ and} \\ c_1 &= \frac{\gamma}{\bar{\alpha}_1}(e^{\bar{\alpha}_1 \frac{\tau}{m}} - 1) < 1 \qquad (35) \end{aligned}$$

Now, let $\alpha_1 > 0$ be a positive number such that $\alpha_1 < \bar{\alpha}_1 (\leq \bar{a})$. It is clear that one has

$$\|\tilde{z}_1(t)\| \leq \mu_1 e^{-\alpha_1 t} \qquad (36)$$

where μ_1 is given by (35).

Let us continue with the mathematical induction and suppose that $m > 1$. Let $2 \leq j \leq m$; for the induction hypothesis, assume that for every $1 \leq i \leq j-1$, there exists a pair of positive numbers (μ_i, α_i) with $0 < \mu_1 \leq \ldots \leq \mu_{j-1}$ and $0 < \alpha_{j-1} \leq \ldots \leq \alpha_1 < \bar{a}$ such that

$$\|\tilde{z}_i(t)\| \leq \mu_i e^{-\alpha_i t} \qquad (37)$$

One needs to show that there also exist positive numbers $\mu_j \geq \mu_{j-1}$ and $0 < \alpha_j \leq \alpha_{j-1}$ such that

$$\|\tilde{z}_j(t)\| \leq \mu_j e^{-\alpha_j t} \qquad (38)$$

Indeed, from (21), one has

$$\begin{aligned} & G_{j-1}(t) = e^{\bar{A}\frac{\tau}{m}}\left((A - \bar{A})\left(\hat{z}_{j-1}(t - \frac{\tau}{m}) - \hat{z}_{j-2}(t)\right)\right. \\ & \left. +\varphi(u_{j-2}, \hat{z}_{j-1}(t - \frac{\tau}{m})) - \varphi(u_{j-2}, z_{j-2}(t))\right) \\ & = e^{\bar{A}\frac{\tau}{m}}\left((A - \bar{A})\left(\tilde{z}_{j-1}(t - \frac{\tau}{m}) - \tilde{z}_{j-2}(t)\right)\right) \\ + \quad & e^{\bar{A}\frac{\tau}{m}}\left(\varphi(u_{j-2}, \hat{z}_{j-1}(t - \frac{\tau}{m})) - \varphi(u_{j-2}, z_{j-1}(t - \frac{\tau}{m}))\right. \\ & +\varphi(u_{j-2}, z_{j-2}(t) - \varphi(u_{j-2}, z_{j-2}(t)))) \end{aligned}$$

Using the Lipschitz assumption on φ, one gets

$$\begin{aligned} \|G_{j-1}(t)\| \quad \leq \quad & \|e^{\bar{A}\frac{\tau}{m}}\|(\|A - \bar{A}\| + L_\varphi) \\ & \left(\|\tilde{z}_{j-1}(t - \frac{\tau}{m})\| + \|\tilde{z}_{j-2}(t)\|\right) \end{aligned}$$

and according to the induction hypothesis, the last inequality leads to

$$\begin{aligned} \|G_{j-1}(t)\| \quad \leq \quad & \|e^{\bar{A}\frac{\tau}{m}}\|(\|A - \bar{A}\| + L_\varphi) \\ & \left(\mu_{j-1}e^{-\alpha_{j-1}(t - \frac{\tau}{m})} + \mu_{j-2}e^{-\alpha_{j-2}t}\right) \\ \leq \quad & \beta e^{-\bar{a}\frac{\tau}{m}}(\|A - \bar{A}\| + L_\varphi) \\ & \left(\mu_{j-1}e^{-\alpha_{j-1}(t - \frac{\tau}{m})} + \mu_{j-2}e^{-\alpha_{j-2}t}\right) \\ \leq \quad & \beta(\|A - \bar{A}\| + L_\varphi) \\ & \left(\mu_{j-1}e^{-\alpha_{j-1}t} + \mu_{j-2}e^{-\alpha_{j-2}t}\right) \\ \leq \quad & \beta(\|A - \bar{A}\| + L_\varphi)\left(\mu_{j-1} + \mu_{j-2}\right)e^{-\alpha_{j-1}t} \\ \leq \quad & \frac{m}{\tau}\left(\mu_{j-1} + \mu_{j-2}\right)e^{-\alpha_{j-1}t} \\ \leq \quad & 2\frac{m}{\tau}\mu_{j-1}e^{-\alpha_{j-1}t} \qquad (39) \end{aligned}$$

From (20) and (39), one gets:
$r_j(t) = -\int_0^t e^{\bar{A}(t-s)}G_{j-1}(s)ds \Longrightarrow$

$$\begin{aligned} \|r_j(t)\| \quad \leq \quad & \beta \int_0^t e^{-a(t-s)}\|G_{j-1}(s)\|ds \\ \leq \quad & 2\frac{m}{\tau}\mu_{j-1}e^{-\bar{a}t}\int_0^t e^{(\bar{a}-\alpha_{j-1})s}ds \\ = \quad & 2\frac{m}{\tau}\mu_{j-1}e^{-\bar{a}t}\frac{e^{(\bar{a}-\alpha_{j-1})t} - 1}{\bar{a} - \alpha_{j-1}} \\ \leq \quad & 2\frac{m\mu_{j-1}}{\tau(\bar{a} - \alpha_{j-1})}e^{-\alpha_{j-1}t} \\ \leq \quad & 2\frac{m\mu_{j-1}}{\tau(\bar{a} - \alpha_1)}e^{-\alpha_{j-1}t} \end{aligned}$$

Now, using inequality (30), one gets

$$\begin{aligned} \|\tilde{z}_j(t)\| \quad \leq \quad & \|e^{\bar{A}\frac{\tau}{m}}\|\left(\|\tilde{z}_{j-1}(t)\| + \|r_j(t)\|\right) \\ & +\beta(L_\varphi + \|A - \bar{A}\|)\int_{t-\frac{\tau}{m}}^{t}\|\tilde{z}_j(s)\|ds \\ \leq \quad & \|e^{\bar{A}\frac{\tau}{m}}\|\mu_{j-1}\left(1 + 2\frac{m}{\tau(\bar{a} - \alpha_1)})\right)e^{-\alpha_{j-1}t} \\ & +\beta(L_\varphi + \|A - \bar{A}\|)\int_{t-\frac{\tau}{m}}^{t}\|\tilde{z}_j(s)\|ds \end{aligned}$$

Again, due to Lemma 3.1 and the assumptions of Theorem 3.1, it can be easily concluded that there exists a pair of positive numbers (μ_j, α_j) such that

$$\|\tilde{z}_j(t)\| \leq \mu_j e^{-\alpha_j t} \qquad (40)$$

where

$$\begin{aligned} \mu_j \quad = \quad & \frac{e^{\alpha_j \frac{\tau}{m}}}{1 - c_j}\left(\|e^{\bar{A}\frac{\tau}{m}}\|\mu_{j-1}\left(1 + 2\frac{m}{\tau(\bar{a} - \alpha_1)})\right)\right. \\ & \left. +\gamma\int_{-\frac{\tau}{m}}^{0}\|\tilde{z}_j(\nu)\|d\nu\right) \text{ and} \\ c_j \quad = \quad & \frac{\gamma}{\alpha_j}(e^{\alpha_j \frac{\tau}{m}} - 1) < 1 \end{aligned}$$

By setting $j = m$, the above developments proved the existence of a pair of positive numbers α_m and μ_m such that

$$\|\tilde{z}_m(t)\| \leq \mu_m e^{-\alpha_m t} \qquad (41)$$

The parameter α required by the theorem is equal to α_m. In the rest of the proof, some technical developments are given and they allow to derive a simple expression of the second parameter μ required by the theorem. Indeed, one can check that the function $\alpha \mapsto \frac{e^{\alpha\delta}-1}{\alpha\delta}$ where $\delta = \frac{\tau}{m}$ is monotone and increases from 1 to ∞ as α goes from 0 to ∞. As a result, the following inequalities hold:

$$
\begin{aligned}
c_j = \frac{\gamma}{\alpha_j}(e^{\alpha_j\delta}-1) = \gamma\delta\frac{e^{\alpha_j\delta}-1}{\alpha_j\delta} &\leq \gamma\delta\frac{e^{\alpha_1\delta}-1}{\alpha_1\delta} \\
&\leq \gamma\delta\frac{e^{\bar{\alpha}_1\delta}-1}{\bar{\alpha}_1\delta} \\
&= c_1 < 1 \qquad (42)
\end{aligned}
$$

where c_1 is given by (35).

Similarly, since $\alpha_j < \bar{a}$ and according to (42) and (22), one can show that

$$
\begin{aligned}
\mu_j &= \frac{e^{\alpha_j\frac{\tau}{m}}}{1-c_j}\left(\|e^{\bar{A}\frac{\tau}{m}}\|\mu_{j-1}\left(1+2\frac{m}{\tau(\bar{a}-\alpha_1)})\right)\right. \\
&\quad \left. +\gamma\int_{-\frac{\tau}{m}}^{0}\|\tilde{z}_j(\nu)\|d\nu\right) \\
&\leq \frac{\beta}{1-c_1}\left(1+2\frac{m}{\tau(\bar{a}-\alpha_1)})\right)\mu_{j-1} \\
&\quad +\gamma\frac{e^{\bar{a}\frac{\tau}{m}}}{1-c_1}\int_{-\frac{\tau}{m}}^{0}\|\tilde{z}_j(\nu)\|d\nu \qquad (43)
\end{aligned}
$$

Set $\rho = \frac{\beta}{1-c_1}\left(1+2\frac{m}{\tau(\bar{a}-\alpha_1)}\right)$; using (43), one can deduce that for $j = m$:

$$
\mu_m \leq \rho^{m-1}\mu_1 + \sum_{j=0}^{m-2}\rho^j\gamma\frac{e^{\bar{a}\frac{\tau}{m}}}{1-c_1}\int_{-\frac{\tau}{m}}^{0}\|\tilde{z}_{m-j}(\nu)\|d\nu \quad (44)
$$

Substituting successively μ_1 and μ_0 by their respective expressions (35) and (27), one gets the expression of the parameter μ required by the theorem

$$
\begin{aligned}
\mu_m &\leq \mu \triangleq \theta^n\eta(\theta)\|KC\|\frac{\beta\rho^{m-1}}{1-c_1}\left(1+\frac{\beta}{\lambda(\theta)-\bar{a}}\right) \\
&\quad e^{-\lambda(\theta)t}\tilde{z}_0(0) + \sum_{j=0}^{m-1}\rho^j\gamma\frac{e^{\bar{a}\frac{\tau}{m}}}{1-c_1}\int_{-\frac{\tau}{m}}^{0}\|\tilde{z}_{m-j}(\nu)\|d\nu
\end{aligned}
$$

This ends the proof of the theorem.

D. Equations of the observer in the original coordinates

Let $\Psi : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto z \triangleq \Psi(x)$ be a diffeomorphism that puts system (1) under the form (2). It is easy to deduce that observer (23) can be written in the original coordinates as follows

$$
\begin{aligned}
&\dot{\hat{x}}_j(t) = f(\hat{x}(t)) + g(\hat{x}(t))u(t) \\
&-\left(\frac{\partial\Psi(\hat{x}_j}{\partial\hat{x}_j}(t)\right)^{-1}G_j(t),\ j=0,\ldots,m \\
&G_0(t) = \theta\Delta_\theta^{-1}K\left(h(\hat{x}_0(t)) - y_\tau(t)\right) \qquad (45) \\
&\text{and for } j = 1,\ldots,m: \\
&G_j(t) = e^{\bar{A}\frac{\tau}{m}}(\\
&(A-\bar{A})\left(\Psi(\hat{x}_j\left(t-\frac{\tau}{m}\right)) - \Psi(\hat{x}_{j-1}(t))\right) \\
&+\varphi(u_{j-1}(t), \Psi(\hat{x}_j\left(t-\frac{\tau}{m}\right))) \\
&-\varphi(u_{j-1}(t), \Psi(\hat{x}_{j-1}(t))))
\end{aligned}
$$

IV. Conclusion

The design of a cascade observer to estimate the state of a class of nonlinear systems where the output is available with a long time delay, is presented. A main characteristic of the proposed observer lies in the ease of its implementation since the gain of all the predictors in the cascade have the same structure. For clarity purposes, only the single output case has been treated. The extension of the observer design to the multi-output case as well as its use to design state feedback controllers for systems with delayed measurements is under study.

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