Exponential stability criteria for systems with time-varying delays *

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Abstract— **This paper presents less conservative criteria for exponential stability of linear systems with a time-varying delay. Restrictions on the derivative of the time-varying delay is not required that allows to have a fast time-varying delay function. We propose a new Lyapunov-Krasovskii function and we present the result in the form of LMIs. By a known example, we show that our criteria give less conservative results than the previous ones.**

I. Introduction

Time delays are frequently encountered in various fields of science and engineering, especially in biology modeling, economics, physiology and many others [1-5]. Furthermore, the existence of this phenomenon is generally a source of instability and degradation of performance. Thus, stability analysis of linear systems with time varying delay has attracted a lot of researchers and many significant contributions have been reported in the literature, like [6-8]. The problem of exponential stability is always developing. For example, [9] gave a stability criterion based on the characteristic functions to guarantee α-stability. [10] derived an α-stability criterion, based on Lyapunov Krasovskii Functions, the descriptor model and the polytopic approach, by reducing the time-varying delay to a convex sum of its bounds. [11] studied the robust exponential stability and stabilization of linear parameter dependent system with both constant delay and interval time-varying delays and they proposed a new criterion based on the Lyapunov–Krasovskii functional and Newton–Leibniz formula. [12] were interested in the time-varying delays as a non-differentiable function in interval of time, and they used a combination of Lyapunov-Krasovskii Functions and Newton-Leibniz formula to present an improved exponential stability results. In this raison, we are interested in exponential stability of linear systems with interval time-varying delay.

The delay-dependant stability criterion that derived by using both of Lyapunov-Krasovskii and Newton-Leibniz formula to present an improved exponential stability results. In this raison, we are interested in exponential stability of linear systems with interval time-varying delay.

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The delay-dependant stability criterion that derived by using both of Lyapunov-Krasovskii and Newton-Leibniz formula is formulated in terms of LMIs and it can be solved by many algorithms such as the LMI Toolbox in Matlab.

This paper is organized as follows. In section 2, the problem is state and the required definition and some well-known technical propositions needed for the proof of the main results are formulated. In section 3, we present the main result, in terms of LMI, to solve the exponential stability problem with a convergence rate for the linear system with time-varying delays. In the purpose of showing the effectiveness of our proposed method, a numerical example is provided to show in section 4. Finally, section 5 is a conclusion.

II. Problem Statement and Preliminaries

A. The following notations will be used in this paper. R^+ denotes the set of all real non-negative numbers; for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). The notation A^T represents the transpose of matrix A. We use $\lambda_{\max}(.)$ and $\lambda_{\min x}(.)$ to denote, respectively, maximum and the minimum eigen-values of a real symmetric matrix. For $x \in R^n$, the norm of x, denoted by $\|x\|$, is defined by:

$$\|x\| = \left(\sum_{i=1}^{n} x_1^2 \right)^{1/2};$$

$$\|.\| = \sup_{-h \leq \theta \leq 0} \{\|x(t+\theta), \dot{x}(t+\theta)\|\}.$$

We consider the following linear parameter dependent system with delay

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - h(t)), & t \in R^+ \\ x(t) = \phi(t), & t \in [-h_2, 0] \end{cases} \quad (1)$$

where $x(t) \in R^n$ is the state vector and A, $B \in M^{n \times n}$. The time-varying delay $h(t)$ is a time-varying continuous function that satisfies:

$$0 \leq h_1 \leq h(t) \leq h_2 \quad (2)$$

The initial condition function $\phi(t)$ denotes a continuous vector-valuated initial function of $t \in [-h_2, 0]$.

Definition 2.1. Proposing the two times $h_a = 0.5(h_1 + h_2)$ and $h_r = 0.5(h_2 - h_1)$, $h(t)$ satisfying (2) can be expressed as:

$$h(t) = h_a - h_r \tau(t), \quad (3)$$

where,

$$\tau(t) = \begin{cases} \dfrac{2h(t) - (h_1 + h_2)}{h_2 - h_1}, & h_2 > h_1 \\ 0, & h_2 = h_1 \end{cases}$$

Obviously, $|\tau(t)| \le 1$. For this case, $h(t)$ is a function belonging to the interval $[h_a - h_r, h_a + h_r]$, where h_r can be taken as the range of variation of time-varying delay $h(t)$.

Using the Newton-Leibniz formula:

$$x(t - h(t)) = x(t - h_a) - \int_{t-h(t)}^{t-h_a} \dot{x}(s)ds \quad (4)$$

we have

$$\dot{x}(t) = Ax(t) + Bx(t - h_a) - B \int_{t-h(t)}^{t-h_a} \dot{x}(s)ds \quad (5)$$

Note that (5) requires initial function $\psi(t)$ in $[-2h_2, 0]$:

$$\psi(s) = \phi(s + h(0)), \quad -h_2 - h(0) \le s \le -h(0),$$
$$\psi(s) = x(t + s), \quad -h(0) \le s \le 0,$$

and as shown in [13], it is a special case of the system (1) such that the stability property of the system (5) will establish the stability property of the system (1). Consequently, we will consider the stability of the system (5) in the interest of confirming the stability of (1).

I. MAIN RESULT

Now, we consider state and ensure the following result for the stability of (1).

Theorem 3.1. *For given non-negative scalars* h_1, h_2 *and* $\alpha > 0$, *system (1) with a time-varying satisfying (2) and constant matrices* A *and* B *is exponentially stable with decay rate* α *if there exist symmetric positive definite* P, U, S, W *and the matrices* M_i *and* T_i, $i = (1,2,3)$ *appropriate dimensions such that the following LMI holds:*

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & 2h_a M_1^T & h_r T_1^T B \\ * & \Omega_{22} & \Omega_{23} & 2h_a M_2^T & h_r T_2^T B \\ * & * & \Omega_{33} & 2h_a M_3^T & h_r T_3^T B \\ * & * & * & -2h_a S & 0 \\ * & * & * & * & -h_r W \end{pmatrix} \le 0 \quad (6)$$

where

$$\Omega_{11} = 2\alpha P + U + M_1^T + M_1 + T_1^T A + AT_1,$$
$$\Omega_{12} = -M_2 + M_1^T + T_1^T B + AT_2,$$
$$\Omega_{13} = P + M_3 + M_1^T - T_1^T + A^T T_3,$$
$$\Omega_{22} = -e^{-2\alpha h_a} U - M_2 - M_2^T + T_2^T B + B^T T_2,$$
$$\Omega_{23} = -M_3 - T_2^T + B^T T_3,$$
$$\Omega_{33} = h_a S + h_r W - T_3^T - T_3.$$

Moreover, the solution of the system satisfies:

$$\|x(t)\| \le \sqrt{\frac{a}{b}} e^{-\alpha t} \|\phi\|_{cl}, \quad t \ge 0,,$$

where

$$a = \lambda_{\min}(P),$$
$$b = \lambda_{\max}(P) + [\lambda_{\max}(U) + \lambda_{\max}(S) + \lambda_{\max}(W)] \left(\frac{1 - e^{-2\alpha h_2}}{2\alpha} \right).$$

Proof. We consider the following Lyapunov-Krasovskii functional for the system (1)

$$V(x(t)) = \sum_{i=1}^{4} V_i(t), \quad (7)$$

where

$$V_1(t) = e^{2\alpha t} x^T(t) P x(t),$$
$$V_2(t) = \int_{t-h_a}^{t} e^{2\alpha s} x^T(s) U x(s) ds,$$
$$V_3(t) = \int_{-h_a}^{0} \int_{t+s}^{t} e^{2\alpha s} \dot{x}^T(\theta) S \dot{x}(\theta) d\theta ds,$$
$$V_4(t) = \int_{-h_a}^{-h_1} \int_{t+s}^{t} e^{2\alpha s} \dot{x}^T(\theta) W \dot{x}(\theta) d\theta ds.$$

and matrices $P > 0$, $U > 0$, $S > 0$, $W > 0$ need to be determined.

The derivative of $V(x(t))$ along the trajectory of system (5) is given by:

$$\dot{V}(x(t)) = \sum_{i=1}^{4} \dot{V}_i(t), \quad (8)$$

Where

$$\dot{V}_1(t) = 2\alpha e^{2\alpha t} x^T(t)Px(t) + 2x^T(t)P\dot{x}(t),$$
$$\dot{V}_2(t) = e^{2\alpha t}[x^T(t)Ux(t) - e^{-2\alpha h_a} x^T(t-h_a)Ux(t-h_a)],$$
$$\dot{V}_3(t) = e^{2\alpha t}[\dot{x}^T(t)h_a S\dot{x}(t) - \int_{t-h_a}^{t} e^{-2\alpha(s-t)}\dot{x}^T(s)S\dot{x}(s)ds],$$
$$\dot{V}_4(t) = e^{2\alpha t}[\dot{x}^T(t)h_r W\dot{x}(t) - \int_{t-h_a e}^{t-h_1} e^{-2\alpha(s-t)}\dot{x}^T(s)W\dot{x}(s)ds].$$

Certainly, for any scalar $s \in [t-h_a, t]$, we have:

$$e^{-2\alpha h_a} \le e^{-2\alpha(s-t)} \le 1,$$

and

$$-\int_{t-h_a}^{t} e^{-2\alpha(s-t)}\dot{x}^T(s)S\dot{x}(s)ds \le e^{-2\alpha h_a}\int_{t-h_a}^{t}\dot{x}^T(s)S\dot{x}(s)ds \quad (9)$$

For any scalar $s \in [t-h_a, t-h_1]$, we have:

$$e^{-2\alpha h_r} \le e^{-2\alpha(s-t)} \le 1,$$

and

$$\begin{aligned} -\int_{t-h_a}^{t-h_1} e^{-2\alpha(s-t)}\dot{x}^T(s)W\dot{x}(s)ds &\le e^{-2\alpha h_r}\int_{t-h_a}^{t-h_1}\dot{x}^T(s)W\dot{x}(s)ds \\ &\le e^{-2\alpha h_r}\int_{t-h_a}^{t-h(t)}\dot{x}^T(s)W\dot{x}(s)ds \end{aligned} \quad (10)$$

Using (9) and (10), we obtain:

$$\begin{aligned} \dot{V}(x(t)) &\le 2\alpha e^{2\alpha t}x^T(t)Px(t) + 2x^T(t)P\dot{x}(t) \\ &+ e^{2\alpha t}\left[x^T(t)Ux(t) - e^{-2\alpha h_a}x^T(t-h_a)Ux(t-h_a)\right] \\ &+ e^{2\alpha t}\left[\dot{x}^T(t)h_a S\dot{x}(t) - e^{-2\alpha h_a}\int_{t-h_a}^{t}\dot{x}^T(s)S\dot{x}(s)ds\right] \\ &+ e^{2\alpha t}\left[\dot{x}^T(t)h_r W\dot{x}(t) - e^{-2\alpha h_r}\int_{t-h_a}^{t-h(t)}\dot{x}^T(s)W\dot{x}(s)ds\right] \end{aligned}$$

$$\begin{aligned} &= e^{2\alpha t}\big[x^T(t)(2\alpha P + U(t)x(t) + 2x^T(t)P\dot{x}(t) \quad (11) \\ &+ \dot{x}(t)(h_a S + h_r W)\dot{x}(t) \\ &- e^{-2\alpha h_a}x^T(t-h_a)Ux(t-h_a) \\ &- e^{-2\alpha h_a}\int_{t-h_a}^{t}\dot{x}^T(s)S\dot{x}(s)ds \\ &- e^{-2\alpha h_r}\int_{t-h_a}^{t-h(t)}\dot{x}^T(s)W\dot{x}(s)ds \ \big] \end{aligned}$$

Expending the Newton-Leibniz, we have:

$$x(t) - x(t-h_a) - \int_{t-h_a}^{t}\dot{x}(s)ds = 0 \quad (12)$$

Thus, by introducing the matrices M_i and T_i, $i = (1,2,3)$ *we obtain the following inequalities:*

$$\begin{aligned} &2e^{2\alpha t}[x^T(t)M_1^T + x^T(t-h_a)M_2^T + \dot{x}^T(t)M_3^T]\times \\ &[x(t) - x(t-h_a) - \int_{t-h_a}^{t}\dot{x}(s)ds] = 0 \end{aligned} \quad (13)$$

and

$$\begin{aligned} &2e^{2\alpha t}[x^T(t)T_1^T + x^T(t-h_a)T_2^T + \dot{x}^T(t)T_3^T]\times \\ &[Ax(t) - Bx(t-h_a) - B\int_{t-h(t)}^{t-h_a}\dot{x}(s)ds - \dot{x}(t)] = 0 \end{aligned} \quad (14)$$

Adding all the zero items of (14) into (11), we get:

$$\begin{aligned} \dot{V}(x(t)) \le= &e^{2\alpha t}[x^T(t)(2\alpha P + U)x(t) + 2x^T(t)P\dot{x}(t) \\ &+ \dot{x}^T(t)(h_a S + h_r W)\dot{x}(t) \\ &- e^{-2\alpha h_a}x^T(t-h_a)Ux(t-h_a) \\ &- e^{-2\alpha h_a}\int_{t-h_a}^{t}\dot{x}^T(s)S\dot{x}(s)ds \\ &- e^{-2\alpha h_a}\int_{t-h_a}^{t-h(t)}\dot{x}^T(s)W\dot{x}(s)ds \\ &+ 2e^{2\alpha t}[x^T(t)M_1^T + x^T(t-h_a)M_2^T + \dot{x}^T(t)M_3^T]\times \\ &[x(t) - x(t-h_a) - \int_{t-h_a}^{t}\dot{x}(s)ds] \end{aligned} \quad (15)$$

$$\begin{aligned} &2e^{2\alpha t}[x^T(t)T_1^T + x^T(t-h_a)T_2^T + \dot{x}^T(t)T_3^T]\times \\ &[Ax(t) - Bx(t-h_a) - B\int_{t-h(t)}^{t-h_a}\dot{x}(s)ds - \dot{x}(t)] \end{aligned}$$

Then $\dot{V}(x(t)) \le \xi^t(t)\Omega\xi(t)$

Where $\xi(t) = [x^T(t) \quad x^T(t-h_a) \quad \dot{x}^T(t)]^T$ and Ω is verifying (6).

II. NUMERICAL EXAMPLE

In this section, we will prove the efficiency of our new criterion concerning time varying-delay system. So, we propose this numerical example used by [14] and [10]. Yet, we introduce the little following modification:

[14] and [10] considered that the time varying delay verifies:

$$0 \le h(t) \le \bar{h} < \infty \qquad (16)$$

For that, we suppose that $h_1 = 0$ and $h_2 = \bar{h}$.

We consider the following system with an interval time-varying delay $[h_1, h_2]$:

$$\dot{x}(t) = \begin{pmatrix} -1 & 1 \\ 2 & -0.5 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ 0 & -0.5 \end{pmatrix} x(t - h(t)) \quad (17)$$

Our aim is to find the maximum admissible delay h_2 with the convergence rate α. So, the system (17) under study is -α-stable. We summarize the results given by [14, 10], and us in Table 1.

Table 1. Comparative results of h_{max} for various α

α	0	0.2	0.4	0.6
[14]	0.783	0.563	0.385	0.224
[10]	0.999	0.840	0.722	0.629
Theorem 3.1	1.200	1.140	1.080	0.990
α	0.8	1	2	2.3
[14]	0.081	0	0	0
[10]	0.558	0.503	0.329	0
Theorem 3.1	0.860	0.590	0.590	0.590

The table 1 demonstrates clearly, that our method gives less conservative results than those proposed by [14] and [10]. In fact, [14] found an exponential stability until $\alpha = 1$. The method proposed by [10] gave a good solution until $\alpha = 2.3$. Besides, our result is better than these and available until $\alpha = 3$. Thus, the theorem 3.1 gives an upper bound of the interval time-varying delay.

The result of the simulation of this example is depicted in Figures 1, 2 and 3. The evolution of states x_1 and x_2 is given. It is shown that the time-delay dynamical system (16) is exponential stable for different values of delays and convergence rates.

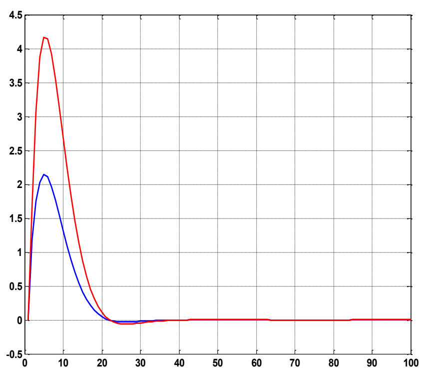

Figure 1: Evolution of states x_1 and x_2 for $\alpha = 0$ and $h_{\max} = 1.2$

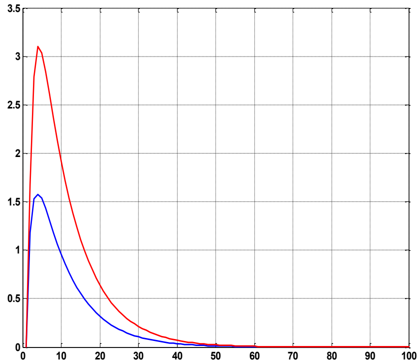

Figure 2: Evolution of states x_1 and x_2 for $\alpha = 0.6$ and $h_{\max} = 0.99$

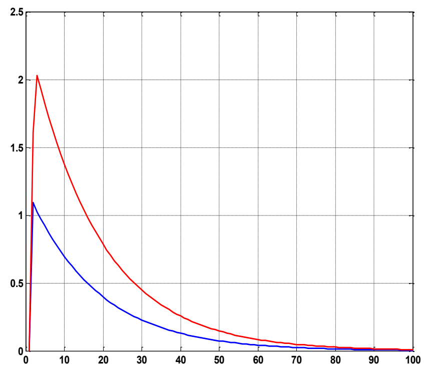

Figure 3: Evolution of states x_1 and x_2 for $\alpha = 2$ and $h_{\max} = 0.59$

III. CONCLUSION

We considered the stability of linear systems with time-varying delay. In this paper, the constraint on the time derivative of the interval time-varying delay has been removed to have a fast time-varying function. Moreover, our delay-dependent sufficient conditions for exponential stability which are based on Lyapunov-Krasovskii theory have been derived in the form of LMIs. Finally, by a numerical example, we showed the usefulness and effectiveness of our result.

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