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A new Saturation function to convert an output constraint into an input constraint

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Abstract— **The main result of our new method is to convert an output constraint into an input constraint for a nonlinear system. Then, standard saturated input design can be used as an anti-windup design for instance which is known to be efficient to enlarge a closed-loop system stability domain in presence of control saturations. We only address the single input single 'saturated' output problem in this paper.**

I. INTRODUCTION

Output and Input hard constraints problems arise in most of the control applications because achieving good performance often requires to reach the inherent physical and safety limitations of a concrete system. Even if one has to cope with possible future destabilizing effects, the application of a saturation placed just after an input is a simple and sure strategy to handle an input constraint whereas there does not exist an equivalent element for an output limitation. That's probably why the input constraint problem has received more attention in the past few decades (see for instance [4]).

Existing solutions to these problems can be divided into two groups : those who predict the future of the closed loop trajectory and check if the constraints will be violated and the other, which at each time 'do their best' to avoid the constraints. Concerning Linear systems, not only future prediction is easier but we can also apply some dedicated LMI-based methods [6], [2]. Moreover, state and input constraints problems are very close since it is possible to apply a relationship between a constrained output and an induced constrained input (whose constraints are state dependent) when the system is perfectly known and discretized [3], [1].

In this paper, we propose to generalize this idea to nonlinear systems by transforming an output constrained nonlinear system into an input constrained nonlinear system whose constraints are state dependent. As we will see our method can still be applied when some disturbances are applied to the nonlinear system and does not require to discretize the system but contrary to the linear case it can be slightly conservative.

By comparison to existing results based on nonlinear model predictive control [5], [9], [7], [10] (to cite a few), our method does not use prediction and can thus be easily implemented ; however, it is fair to say that we do not address several output constraints problems (where the number of inputs is inferior to the number of constrained outputs) in this preliminary work. Other possible and very useful methods [8], [11], [12] address output constraints problems for nonlinear systems of special form and/or try to escape the constraints while our design can be applied to a broad class of nonlinear system (for instance, thanks to our design, we can saturate any nonlinear function of the state if it is controllable) and enables to remain on the constraints limitations offering thus the possibility to use anti-windup loops as in [2], [1].

This paper is organized as follows : the basic notations and our two main results are given in Section 2 : the first one shows how one can transform an output constraint into an input constraint for a perfectly known general nonlinear system whereas the second one addresses the same problem in presence of disturbances. In Section 3, we illustrate our results on a motivating example and provides some numerical results. We finally give some future research directions and conclusions.

II. MAIN RESULTS

A. Notations

Let $\mathcal{R}$ (resp. $\mathcal{N}$) denote the set of real numbers (resp. natural integers). In this paper, we are interested in nonlinear systems of dimension $n \in \mathcal{N}$.

Given $\sigma \in \mathcal{C}^1(\mathcal{R}^n, \mathcal{R})$, $L_f\sigma := \sum_{i=1}^n f_i \partial_i \sigma$ denotes its Lie-derivative with respect to f.

Throughout this paper, we will use the following useful notation :

Given $r \in \mathcal{N}$ real numbers $K_1, \ldots, K_r$, we note

$$K_{j,r} := \prod_{i=j}^{r} K_i$$

and we also pose (for convenience) :

$$\forall j \in \mathcal{N}, \; K_{j+1,j} := 1$$

B. Problems formulation and main results

Problem 1*[From an output constraint to an input constraint]*

Let us consider the following non linear system

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= c(x) + d(x)u \end{cases}$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}^y$, $u \in \mathcal{R}$ and f, g, c, d are $\mathcal{C}^\infty$.

Let us suppose that a control law u (static or even dynamic) has been designed in order to achieve some desired objectives but without taking into account the fact that a given single output $z = \sigma(x) \in \mathcal{R}$ must remain inside a given (time-varying) interval.

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Our reformulation of the problem consists in finding some nonlinear transfer functions $h_1(y)$ and $h_2(y)$ such that when u belongs to $[h_1, h_2]$ and $x_0 := x(t=0)$ to a suitable set $\mathcal{C}_0$, z will remain in the interval $[z_{\min}(t), z_{\max}(t)]$ (where $\forall t, z_{\max}(t) > z_{\min}(t)$).
To sum up, we want to find $h_1, h_2, \mathcal{C}_0$ such that:

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x) + g(x)u \\ y & = & c(x) + d(x)u \\ z & := & \sigma(x) \\ u & \in & [h_1(y, z_{\min}, z_{\max}), h_2(y, z_{\min}, z_{\max})] \\ x_0 & \in & \mathcal{C}_0 \end{array}\right.$$

$$\Downarrow$$

$$\forall t, \quad z(t) \in [z_{\min}(t), z_{\max}(t)]$$

Let us now stem our first result which is a possible solution to this problem :

Theorem 1 Suppose $y = x$ and $z = \sigma(x)$ is of relative degree $r \in \mathcal{N}_{>0}$ i.e

$$z^{(r)} = L_f^r \sigma + (L_g L_f^{r-1} \sigma) u$$

where $\forall x, L_g L_f^{r-1} \sigma(x) > 0$
Let $K_1, \dots, K_r > 0$
If one chooses :

$$h_1(x) = \frac{K_{1,r} z_{\min} - \sum_{i=0}^{r} K_{i+1,r} L_f^i \sigma(x)}{L_g L_f^{r-1} \sigma(x)} \tag{1}$$

$$h_2(x) = \frac{K_{1,r} z_{\max} - \sum_{i=0}^{r} K_{i+1,r} L_f^i \sigma(x)}{L_g L_f^{r-1} \sigma(x)} \tag{2}$$

and if x_0 satisfies the following inequalities which thus define the set $\mathcal{C}_0$

$$z_{\min}(0) \le \sigma(x(0)) \le z_{\max}(0)$$

$\forall j \in [1, r-1]$,

$$K_{1,j} z_{\min}(0) \le \sum_{i=0}^{j} K_{i+1,j} L_f^i \sigma(x(0)) \le K_{1,j} z_{\max}(0)$$

then z will remain in the set $[z_{\min}, z_{\max}]$ if u is forced to be inside $[h_1, h_2]$

Remark 1: it is obvious that if $\forall x, L_g L_f^{r-1} \sigma(x) < 0$, the same result is obtained when $u \in [h_2, h_1]$.
Remark 2: the choice of the gains K_i is based on a trade off : theoretically, the bigger the gains K_i are, the bigger our initial admissible set is and so the less conservative is our input equivalent constraint conversion. However, as it can be observed in numerical simulations, high values of the K_i lead to high value of the control law when we are on the output constraint.
Remark 3: the relative degree 0 case is not covered by theorem 1 but it is easy to see that if $z = \sigma_1(x) + \sigma_2(x)u$ (with $\forall x, \sigma_2(x) \neq 0$) and if $z(0) \in [z_{\min}(0), z_{\max}(0)]$, $z(t)$ will respect its constraints if u satisfies

$$\forall x, \quad u(x) \in \left[\min\left\{ \frac{z_{\min} - \sigma_1(x)}{\sigma_2(x)} , \frac{z_{\max} - \sigma_1(x)}{\sigma_2(x)} \right\} , \max\left\{ \frac{z_{\min} - \sigma_1(x)}{\sigma_2(x)} , \frac{z_{\max} - \sigma_1(x)}{\sigma_2(x)} \right\} \right] \tag{3}$$

(remark : this formulation is independent of the sign of σ_2)

Proof: Let us prove the result by recurrence on the relative degree $r \in \mathcal{N}_{>0}$

- first step : suppose $r = 1$ it is easy to see that given $K_1 > 0$, if $\forall t \ge 0, \dot{z} \in [-K_1(z - z_{\min}), -K_1(z - z_{\max})]$ then z remains inside $[z_{\min}, z_{\max}]$ if z_0 is inside this interval. Indeed, since

$$\dot{z} \ge -K_1(z - z_{\min}) \ge 0 \quad \text{if } z \le z_{\min}$$

and

$$\dot{z} \le -K_1(z - z_{\max}) \le 0 \quad \text{if } z \ge z_{\max}$$

z cannot violate its constraints provided $z(0) \in [z_{\min}, z_{\max}]$.
Moreover, since $r = 1$, we have $\dot{z} = L_f \sigma + L_g \sigma u$ where $L_g \sigma > 0$ and so writing $\dot{z} \in [-K_1(z - z_{\min}), -K_1(z - z_{\max})]$ is equivalent to write

$$u \in \left[\frac{K_1(z_{\min} - \sigma) - L_f \sigma}{L_g \sigma} , \frac{K_1(z_{\max} - \sigma) - L_f \sigma}{L_g \sigma} \right]$$

which is the same that writing $u \in [h_1, h_2]$ for $r = 1$.
- second step : suppose the theorem is proved for any output of relative degree $j \le r - 1$.
In particular, according to the theorem, for any $K_2, \dots, K_r > 0$, $\dot{z}$ which is of relative degree $r - 1$ will remain inside $[\dot{z}_{\min}, \dot{z}_{\max}]$ if $u \in [\bar{h}_1, \bar{h}_2]$ where :

$$\bar{h}_1 = \frac{K_{2,r} \dot{z}_{\min} - \sum_{i=0}^{r-1} K_{i+2,r} L_f^i \dot{z}}{L_g L_f^{r-2} \dot{z}}$$

$$\bar{h}_2 = \frac{K_{2,r} \dot{z}_{\max} - \sum_{i=0}^{r-1} K_{i+2,r} L_f^i \dot{z}}{L_g L_f^{r-2} \dot{z}}$$

if $x(0)$ satisfies the following inequalities :

$$\dot{z}_{\min} \le \dot{z}(x(0)) \le \dot{z}_{\max}$$

$\forall j \in [2, r-1]$,

$$K_{2,j} \dot{z}_{\min} \le \sum_{i=0}^{j} K_{i+2,j} L_f^i \dot{z}(x(0)) \le K_{1,j} \dot{z}_{\max}$$

Let us consider $z := \sigma(x)$ which is of relative degree r.
Let $K_1 > 0$, it is easy to prove that if $z(0) \in [z_{\min}, z_{\max}]$ and if $\forall t$

$$-K_1(z - z_{\min}) \le \dot{z} \le -K_1(z - z_{\max})$$

$z(t)$ will remain in the set $[z_{\min}, z_{\max}]$. Indeed, since

$$\dot{z} \geq -K_1(z - z_{\min}) \geq 0 \quad \text{if } z \leq z_{\min}$$

and

$$\dot{z} \leq -K_1(z - z_{\max}) \leq 0 \quad \text{if } z \geq z_{\max}$$

z cannot violate its constraints provided $z(0) \in [z_{\min}, z_{\max}]$.
Thus, this boils down to saying that if $\dot{z} \in [-K_1(z - z_{\min}), -K_1(z - z_{\max})]$ and if $z(0) \in [z_{\min}, z_{\max}]$, z will remain in the desired set.
Keeping $\dot{z}$ in this set amounts to apply the recurrence hypothesis by choosing $\dot{z}_{\min} := -K_1(z - z_{\min})$ and $\dot{z}_{\max} := -K_1(z - z_{\max})$.
Replacing $\dot{z}_{\min}$ (resp. $\dot{z}_{\max}$) by $-K_1(z - z_{\min})$ (resp. $-K_1(z - z_{\max})$) and $\dot{z}$ by $\frac{dz}{dt} := L_f\sigma(x)$ in the recurrence hypothesis and adding the initial condition constraint on $x(0)$ according to $z(0) := \sigma(x(0))$ must be inside $[z_{\min}, z_{\max}]$ gives our result.

C. Robustness issues

Problem 2*[Robust output constraint problem]* Let us consider the following non linear system

$$\left\{\begin{array}{lcl} \dot{x} &=& f(x) + g_1(x)u + g_2(x)d \\ y &=& c(x) + d_1(x)u + d_2(x)d \end{array}\right.$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, $u \in \mathcal{R}$ and $f, g_1, g_2, d_1, d_2, c, d$ are $\mathcal{C}^\infty$ and where the disturbance $d \in \mathcal{R}$ is bounded by a time-varying limit

$$\forall t, |d(t)| \leq M(t)$$

Let us suppose that a control law u (static or even dynamic) has been designed in order to achieve some desired objectives and that we have found some nonlinear transfer functions $h_1(y)$ and $h_2(y)$ such that when u belongs to $[h_1, h_2]$, z remains in the interval $[z_{\min}, z_{\max}]$ when no disturbance is applied to the nonlinear system. Then, given a bounded disturbance $|d(t)| \leq M(t)$, the robust output constraint problem consists in changing these nonlinear transfer functions which now depend on M such that when u belongs to $[h_1(M, y, z_{\min}, z_{\max}), h_2(M, y, z_{\min}, z_{\max}]$ and x_0 to a suitable set $\mathcal{C}_0$, z will remain in the interval $[z_{\min}, z_{\max}]$ (where $\forall t, z_{\max}(t) > z_{\min}(t)$)
To sum up, we want to find $h_1, h_2, \mathcal{C}_0$ such that :

$$\left\{\begin{array}{lcl} \dot{x} &=& f(x) + g_1(x)u + g_2(x)d \\ y &=& c(x) + d_1(x)u + d_2(x)d \\ z &:=& \sigma(x) \\ u &\in& [h_1(M, y), h_2(M, y)] \\ x_0 &\in& \mathcal{C}_0 \end{array}\right.$$

$$\Downarrow$$

$$\forall t, \quad z(t) \in [z_{\min}(t), z_{\max}(t)]$$

Theorem 2 Suppose $y = x$ and $z = \sigma(x)$ is of relative degree $r \in \mathcal{N}_{>0}$(resp. $s \leq r$) with respect to u (resp. d) i.e

$$\left\{\begin{array}{l} \forall x, \ \forall i < r-1, \ L_{g_1}L_f^i\sigma(x) = 0 \ \& \ L_{g_1}L_f^{r-1}\sigma(x) \neq 0 \\ \forall x, \ \forall i < s-1, \ L_{g_2}L_f^i\sigma(x) = 0 \ \& \ L_{g_2}L_f^{s-1}\sigma(x) \neq 0 \end{array}\right.$$

Suppose also that $\forall x, L_gL_f^{r-1}\sigma(x) > 0$
Let $K_1, \ldots, K_{s-1}, K_{s_0}, K_{s+1}, \ldots, K_r > 0$ be fixed and let us define the time-varying positive gain K_s by

$$K_s(t) = \frac{K_{s_0} + 2M(t)\varepsilon_r(x(t))}{K_{1,s-1}K_{s+1,r}(z_{\max} - z_{\min})}$$

(where ε_r is defined by (6)).
Let us note $K_{j,r} := \prod_{i=j:r} K_i$
If one chooses :

$$h_1(x) = \frac{K_{1,r}z_{\min} - \sum_{i=0}^{r} K_{i+1,r}L_f^i\sigma(x) + M\varepsilon_r(x)}{L_gL_f^{r-1}\sigma(x)} \quad (4)$$

$$h_2(x) = \frac{K_{1,r}z_{\max} - \sum_{i=0}^{r} K_{i+1,r}L_f^i\sigma(x) - M\varepsilon_r(x)}{L_gL_f^{r-1}\sigma(x)} \quad (5)$$

with $\forall j \in [s, r]$

$$\varepsilon_j(x) := \sum_{i=s}^{j} K_{i+1,j}|L_{g_2}L_f^{i-1}\sigma(x)| \quad (6)$$

and if x_0 satisfies the following conditions:

$$z_{\min}(0) \leq \sigma(x(0)) \leq z_{\max}(0)$$

$\forall j \in [1, s-1]$,

$$K_{1,j}z_{\min}(0) \leq \sum_{i=0}^{j} K_{i+1,j}L_f^i\sigma(x(0)) \leq K_{1,j}z_{\max}(0)$$

$\forall j \in [s, r]$,

$$K_{1,j}z_{\min}(0) + M(0)\varepsilon_j(x_0) \leq \sum_{i=0}^{j} K_{i+1,j}L_f^i\sigma(x(0))$$
$$\&$$
$$\sum_{i=0}^{j} K_{i+1,j}L_f^i\sigma(x(0)) \leq K_{1,j}z_{\max}(0) - M(0)\varepsilon_j(x_0)$$

then z will remain in the set $[z_{\min}, z_{\max}]$ when $u \in [h_1, h_2]$

Remark 1: it is possible with painstaking computations to extend our result to the case $d \in \mathcal{R}^k$ with $k > 1$.
Remark 2: it is obvious that if $\forall x, L_gL_f^{r-1}\sigma(x) < 0$, the same result is obtained when $u \in [h_2, h_1]$.

Sketch of the Proof: Due to space restrictions, we only give here the guidelines of the proof.
First, replacing f by $f + g_2d$, we carry on the computations of the proof of theorem 1 putting carefully at each stage of the recursive design the part containing d in the right and left limit (in order to be able to derive the remaining term). We also keep in mind that σ is of relative degree s with

respect to d. Finally, one can prove that if x_0 satisfies the following conditions :

$$z_{\min} \leq \sigma(x(0)) \leq z_{\max}$$

$\forall j \in [1, s-1]$,

$$K_{1,j} z_{\min}(0) \leq \sum_{i=0}^{j} K_{i+1,j} L_f^i \sigma(x(0)) \leq K_{1,j} z_{\max}(0)$$

$\forall j \in [s, r]$,

$$\begin{array}{rcl} K_{1,j} z_{\min}(0) - \lambda_j(x_0) d(0) & \leq & \sum_{i=0}^{j} K_{i+1,j} L_f^i \sigma(x(0)) \\ & \& & \\ \sum_{i=0}^{j} K_{i+1,j} L_f^i \sigma(x(0)) \leq & K_{1,j} & z_{\max}(0) - \lambda_j(x_0) d(0) \end{array}$$

where $\lambda_j(x) := \sum_{i=s}^{j} K_{i+1,j} L_{g_2} L_f^{i-1} \sigma(x)$ and if one chooses :

$$\begin{array}{rcl} h_1^{\#}(x) & = & \dfrac{K_{1,r} z_{\min} - \sum_{i=0}^{r} K_{i+1,r} L_f^i \sigma(x) + \lambda_j(x) d}{L_g L_f^{r-1} \sigma(x)} \\ h_2^{\#}(x) & = & \dfrac{K_{1,r} z_{\max} - \sum_{i=0}^{r} K_{i+1,r} L_f^i \sigma(x) + \lambda_j(x) d}{L_g L_f^{r-1} \sigma(x)} \end{array}$$

Then z will remain in $[z_{\min}, z_{\max}]$ if $u \in [h_1^{\#}, h_2^{\#}]$. However, this result is useless since d is unknown and it is necessary to use the bound M(t). We obtain our result by using

$$\forall x,\ |\lambda_j(x) d| \leq M \varepsilon_j(x) \ \&\ h_1^{\#}(x) \leq h_1(x) \ \&\ h_2(x) \leq h_2^{\#}(x)$$

Looking at the expressions of h_1 and h_2 of theorems 1 and 2, one problem can appear in theorem 2 ! Indeed,

- in theorem 1 u must remain inside $[h_1, h_2]$ which is a non zero measured interval since

$$\Delta_1 := h_2 - h_1 = \frac{K_{1,r}(z_{\max} - z_{\min})}{L_g L_f^{r-1}} > 0$$

- however in theorem 2, u must remain inside $[h_1, h_2]$ where :

$$\Delta_2 := h_2 - h_1 = \frac{K_{1,r}(z_{\max} - z_{\min}) - 2M\varepsilon_r}{L_{g_1} L_f^{r-1}}$$

  and so we need to prove that this quantity is positive $\forall x$ in order to get some admissible values for u.

That's why we have used a time varying gain K_s in theorem 2. Replacing K_s by its value solves our problem since:

$$\forall x,\ \Delta_2(x) = \frac{K_{s_0}}{L_{g_1} L_f^{r-1}(x)} > 0$$

III. Illustrative example and numerical results

A. A basic flight combat aircraft example

To illustrate our idea, let us consider the Longitudinal flight of a combat aircraft where the N_z load factor must remain inside $[N_{\min}, N_{\max}]$:

$$\left\{ \begin{array}{rcl} \dot{\alpha} & = & q + k(M_a) N_z(M_a, \alpha) \cos(\alpha) \\ \dot{q} & = & W(M_a, \alpha, q) + B(M_a, \alpha, q) u \end{array} \right.$$

where $\forall M_a, \alpha, q,\quad \frac{\partial N_z(M_a,\alpha)}{\partial \alpha} B(M_a, \alpha, q) > 0$.
This model is nonlinear because the aerodynamic coefficients nonlinearly depend on M_a the Mach Number (so on the flight conditions) and on α the angle of attack and q (which can take large values). In this example, M_a is constant (frozen velocity case). To solve this problem without using prediction and then many computations is difficult since we must make α track its reference while keeping N_z inside a given set.

B. Full state feedback case

Let us suppose that :

- the full state is available (i.e $y = x$)
- a dynamic control law u has been designed in order to make α track a given reference signal α_c

For instance, using a classical Backstepping control law design, we choose :

$$\left\{ \begin{array}{rcl} \dot{x}_c & = & \alpha - \alpha_c \\ u & = & \frac{-W(M_a,\alpha,q) - k_d(q - q_c) + \dot{q}_c - (\alpha - \alpha_c)}{B(M_a,\alpha,q)} \end{array} \right.$$

where

$$q_c := -k(M_a) N_z(M_a, \alpha) \cos(\alpha) - k_p(\alpha - \alpha_c) - k_i x_c$$

We select $k_i, k_p, k_d > 0$ in order to get a stable closed loop system :

$$\left\{ \begin{array}{rcl} \dot{x}_c & = & \alpha - \alpha_c \\ \dot{\alpha} - \dot{\alpha}_c & = & -k_i x_c - k_p(\alpha - \alpha_c) + (q - q_c) \\ \dot{q} - \dot{q}_c & = & -k_d(q - q_c) - (\alpha - \alpha_c) \end{array} \right.$$

The constrained output is $\sigma = N_z(M_a, \alpha)$ and our problem is to find two nonlinear functions $h_1(x)$ and $h_2(x)$ such that when u belongs to $[h_1(x), h_2(x)]$, N_z remains in its interval $[N_{\min}, N_{\max}]$.
Let us remark that our method consists in shifting the constraints step by step towards the control input u.

- First, we note that given $K_1 > 0$, the application of the following constraint :

$$-K_1(N_z - N_{\min}) \leq \dot{N}_z \leq -K_1(N_z - N_{\max})$$

  enables to keep N_z inside its interval provided $N_z(0)$ is inside this interval.
- Secondly, considering $K_2 > 0$ and noting $\dot{N}_{z\min} := -K_1(N_z - N_{\min})$ and $\dot{N}_{z\max} := -K_1(N_z - N_{\max})$, the application of the following constraint :

$$-K_2(\dot{N}_z - \dot{N}_{\min}) \leq \ddot{N}_z \leq -K_2(\dot{N}_z - \dot{N}_{\max})$$

  enables to keep $\dot{N}_z$ inside its interval $[\dot{N}_{\min}, \dot{N}_{\max}]$ provided $\dot{N}_z(0)$ starts inside this interval. Thanks to the preceding point, this fact also implies that N_z will stay inside $[N_{\min}, N_{\max}]$ provided $N_z(0)$ starts inside this interval.

- finally, since computing u appears in $\ddot{N}_z$

$$\begin{aligned} \ddot{N}_z &= \frac{\partial N_z}{\partial \alpha}\ddot{\alpha} + \frac{\partial^2 N_z}{\partial \alpha^2}\dot{\alpha}^2 \\ &= \frac{\partial N_z}{\partial \alpha}Bu + \frac{\partial N_z}{\partial \alpha}W + \frac{\partial^2 N_z}{\partial \alpha^2}\dot{\alpha}^2 \end{aligned}$$

and so replacing this and using the last two sets of inequalities gives exactly the functions h_1 and h_2 of theorem 1 when we compute the various Lie derivatives for this example of relative degree two.

$$h_1 = \frac{K_{1,2}(N_{\min} - N_z) - K_2\dot{N}_z - \frac{\partial N_z}{\partial \alpha}W - \frac{\partial^2 N_z}{\partial \alpha^2}\dot{\alpha}^2}{\frac{\partial N_z}{\partial \alpha}B}$$

$$h_2 = \frac{K_{1,2}(N_{\max} - N_z) - K_2\dot{N}_z - \frac{\partial N_z}{\partial \alpha}W - \frac{\partial^2 N_z}{\partial \alpha^2}\dot{\alpha}^2}{\frac{\partial N_z}{\partial \alpha}B}$$

Since our method consists in saturating the input we can also add a 'basic' MRAW (Model Recovery Anti-Windup [13]) based anti-windup loop which acts only when the input saturates. This consists in using the following control law:

$$\begin{cases} \dot{x}_c &= \alpha - \alpha_c + \alpha_{aw} \\ u &= \frac{-W(M_a,\alpha,q) - k_d(q-q_c) + \dot{q}_c - (\alpha - \alpha_c)}{B(M_a,\alpha,q)} \end{cases}$$

where

$$\begin{cases} \dot{\alpha}_{aw} = -k_p\alpha_{aw} + q_{aw} \\ \dot{q}_{aw} = -k_d q_{aw} - \alpha_{aw} + B(M_a,\alpha,q)\left(u - Sat_{h_1,h_2}(u)\right) \end{cases}$$

To illustrate this idea numerically, we have studied three possible laws (see figure 1):

- the first control law is the nominal control law which enables α to follow a reference α_c (in green). The outputs are drawn in magenta : α follows its reference but the load factor N_z can exceed its limits (drawn in red)
- the second control law uses a new saturation function '$Sat_{[h_1,h_2]}$'. The outputs are drawn in blue : the saturation function enables the constraint output N_z to remain inside the specified interval. However, due to a wind up effect, α does not follow exactly its reference and there is a bias.
- the third control law uses the saturation function and a anti-windup loop. The outputs are drawn in black : this time all the specifications are achieved : α follows its reference and N_z respects its specified interval.

Remark : When the constraints of a given r-relative degree output z are constant, another and possible interpretation of our problem formulation is to keep the trajectory $t \mapsto (z(t), \dot{z}(t), \ldots, z^{(r)}(t))$ between two hyperplanes of a $\mathcal{R}^{r+1}$ vectorial space. In this example, keeping u inside $[h_1, h_2]$ is equivalent to keep $\ddot{N}_z$ inside $[-K_2\dot{N}_z - K_{1,2}(N_z - N_{\min}), -K_2\dot{N}_z - K_{1,2}(N_z - N_{\max})]$. This fact is illustrated by figure 2 where the two last control laws (which succeed to keep N_z inside $[N_{\min}, N_{\max}]$) also keeps the trajectory $t \mapsto (N_z(t), \dot{N}_z(t), \ddot{N}_z(t))$ between two hyperplanes defined by $\ddot{N}_z = -K_2\dot{N}_z - K_{1,2}(N_z - N_{\min})$ and $\ddot{N}_z = -K_2\dot{N}_z - K_{1,2}(N_z - N_{\max})$.

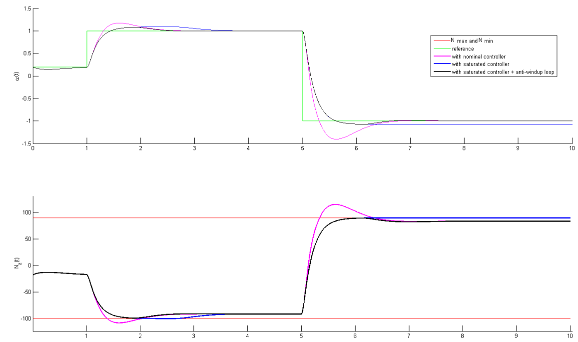

Fig. 1. Comparison of three possible control laws (from top to bottom : α and N_z)

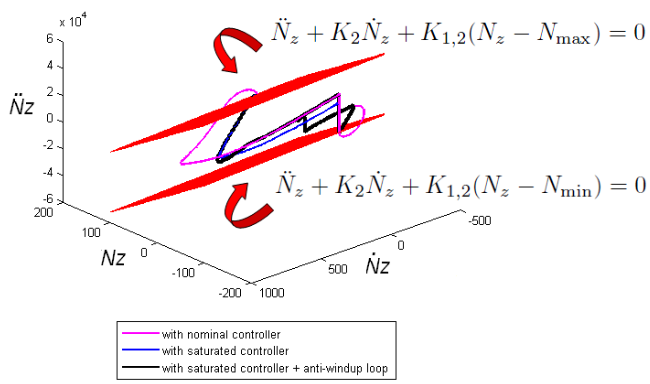

Fig. 2. Illustration of the equivalent 'hyperplanes' condition

C. Output feedback case

Let us illustrate this robustness problem by using an output feedback on our motivating example. This time the controller can use $y = \alpha$ and we estimate the remaining state components by using a nonlinear observer. We remark that this problem boils down to studying 'our robust output problem'. Indeed, we have :

$$\begin{cases} \dot{\alpha} &= q + k(M_a)N_z(M_a,\alpha)\cos(\alpha) \\ \dot{q} &= W(M_a,\alpha,q) + B(M_a,\alpha,q)u(\alpha,\hat{q}) \\ & \quad +B(M_a,\alpha,q)(u(\alpha,q) - u(\alpha,\hat{q})) \\ N_z(M_a,\alpha) &\in [N_{\min}, N_{\max}] \end{cases}$$

where $\hat{q}$ is given by a nonlinear observer. Therefore, noting $d = u(\alpha,q) - u(\alpha,\hat{q})$, on can simply write :

$$\begin{cases} \dot{\alpha} &= q + k(M_a)N_z(M_a,\alpha)\cos(\alpha) \\ \dot{q} &= W(M_a,\alpha,q) + B(M_a,\alpha,q)u \\ & \quad +B(M_a,\alpha,q)d \\ N_z(M_a,\alpha) &\in [N_{\min}, N_{\max}] \end{cases}$$

From now on, we use what we called the "saturated+AW" control law presented in the preceding subsection (and which was the only law achieving both perfect tracking and good constraints satisfaction). For different initial conditions and so different initial observer error conditions, we plot the outputs obtained by this law

- on figure 3, we see that the output constraints are violated because of the disturbance d
- on figure 4, we change a little bit the bounds of our saturation function (theoretically, this amounts to designing some ε_1 and ε_2 functions) and we obtain the expected results (good tracking performance for the output α and specified bounds on the output N_z).

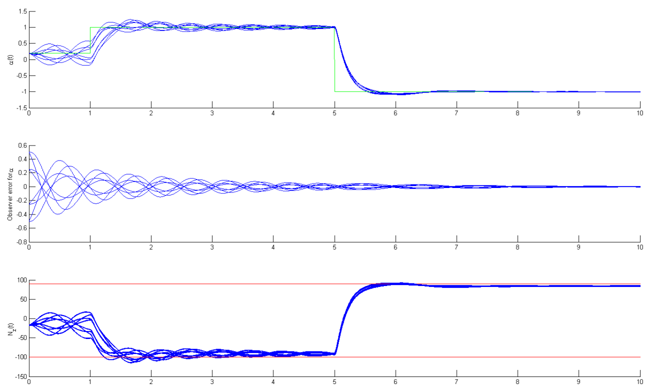

Fig. 3. Application of the saturated +AW law for several initial observer errors (from top to bottom : α, $q - \hat{q}$ and N_z)

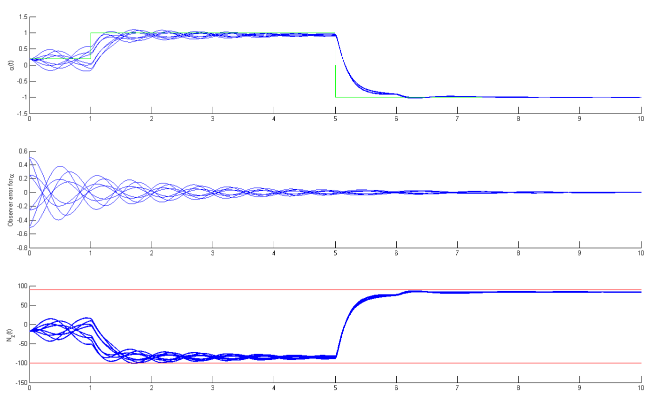

Fig. 4. Application of the robustified saturated + AW law for several initial observer errors $\hookrightarrow$ this time the constraints are not violated. (from top to bottom : α, $q - \hat{q}$ and N_z)

Let us also add a comment about the architecture of our law : the use of an observer amounts to using dynamic functions h_1 and h_2 ; indeed,

$$h_i(y) := h_i(\hat{x}, M) \quad \text{where} \quad \dot{\hat{x}} = f_{\text{observer}}(\hat{x}, y, u)$$

IV. Conclusions and future work

This paper has addressed the difficult problem of controlling nonlinear systems submitted to an output hard constraint. Thanks to a novel approach which has shifted 'r-times' the constraint towards the input (where r is the relative degree of the constrained output with respect to the control), we have changed this output constraint into an input constraint which enables us to use more classical saturated control methodologies. Moreover, we have shown that our methodology can still be applied when the system is submitted to some disturbances.

Future research will be dedicated to the multi-input multi-output problem and to the addition of anti-windup loops.

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