A New Approach to Optimal Energy Management with Discrete Control

P. Balaguer, J.C. Alfonso, J. Zhang and X. Xia

Abstract— **In this article, we present a new approach to the solution of optimization problems with discrete control variables modeled by binary integer programs (BIP). The solution of BIP is computationally demanding when the number of the BIP variables increase. Two instances that increase the number BIP variables in practical applications are the reduction of the discretization sampling time and the increase of the optimization time period. The proposed approach transforms a single BIP optimization into a linear program (LP) and N feasibility BIP's, with less number of variables. The reduction of the number of variables increases the algorithm speed in providing a solution. The approach permits to solve optimization problems with longer time intervals and with a higher number of control variables, while being computationally tractable. A case study on the electricity cost minimization in a pumping station shows the applicability of the method.**

I. INTRODUCTION

The presence of discrete control variables transforms standard continuous optimization problems [1] into integer optimization problems, which are computationally more demanding and shown to be NP-hard [2]. Integer optimization applications are prevalent in electric power systems such as the minimization of electricity cost by means of loads management [3] [4], optimization of hybrid systems operation [5] [6], reactive power and voltage control [7], and on optimal energy management in liberalized markets [8], among others.

The previously introduced minimization problems are normally cast on time periods of 24 hours because of the periodic nature of the problem to be solved. The common approach to the solution of integer minimization problem is to discretize the continuous objective function and constraints using a one hour discretization sampling time [3] [4] [5] [6] [7].

Although useful practical results may be accomplished using the discretization process, the procedure has fundamental limitations that prevent their utilization as a general approach. The main problem relies on the curse of dimensionality problem of integer programs [9]. From a practical point of view, the curse of dimensionality imposes a trade-off between the discretization time, equivalently the number of program variables, and the optimality of the achieved solution. In fact optimality of the solution is compromised by the discretization [10] because not only the minimum on-off time is determined by the sampling time but also the on-off time instants are fixed by the discretization procedure. In general the chosen discretization pattern is unable to provide the optimal solution.

Moreover, the discretization approach also shows fundamental limitations in other settings, for instance, when incorporating predictions on the optimization [6]. In this case the problem time period can be enlarged beyond the 24 hours if longer forecast periods are available, thus increasing the number of the program variables. On the other hand, time period of 24 hours may not be large enough in certain applications. Even if a feasible solution to the minimization problem is found, it has been shown that the feasible solution, when periodically repeated, may become infeasible [11]. Prediction time periods and feasibility issues of repeated optimal solutions motivate the necessity of optimization procedures that can enlarge the optimization time horizon.

In this article we present a new approach for solving binary integer programs (BIP) that increases the time horizon without compromising problem solvability due to computational burden. In this way the solution is given in a two steps procedure. In the first step, the BIP is transformed into a linear program (LP) by integration of the binary variables. The solution of the LP provides the optimal energy consumption integral per interval. In the second step, the optimal energy value calculated is supplied in a binary way, by solving a BIP feasibility problem. In this way, a unique BIP is transformed into a LP plus N feasibility BIP's related to shorter time intervals with fewer variables. The reduction of the number of variables presents an advantage because algorithms work faster with fewer variables. Moreover, in the second stage, the algorithm only requires a feasible solution, not an optimal one. Finally, the approach permits to keep the complexity of the feasible BIP bounded while the optimization period can be made arbitrarily large, increasing its practical utility.

II. PROBLEM FORMULATION

A. Problem Statement

We consider the optimal energy cost management of a number of M loads during a time period T by minimizing the following cost function

$$J = \sum_{m=1}^{M} \int_0^T C(t) P_m(t) dt \qquad (1)$$

where $C(t)$ is the energy cost, and $P_m(t)$ is the power consumption of load m.

P. Balaguer and J.C. Alfonso are with Departament d'Enginyeria de Sistemes Industrials i Disseny, Universitat Jaume I de Castelló, Castelló, Spain. pbalague@uji.es, calfonso@uji.es

J. Zhang and X. Xia are with the Centre of New Energy Systems, Department of Electrical, Electronic and Computer Engineering, University of Pretoria, South Africa. zhang@up.ac.za, xxia@up.ac.za

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Assumption 1: Function $C(t)$ is piecewise constant with N intervals per period.

The power consumption is controlled by discrete variables $u_m(t), \quad m = 1, \ldots, M$, where

$$u_m(t) = \begin{cases} 1, & \text{load m is on} \\ 0, & \text{load m is off} \end{cases}$$

Assumption 2: The power consumption of load m, $P_m(t)$ is given by $P_m(t) = P_m^{ss} u_m(t)$, where P_m^{ss} is the steady state power consumption, and $u_m(t)$ is the discrete control action.

The objective function (1) is to be minimized subject to the following constraints

$$s.t. \qquad \dot{x}(t) = Ax(t) + B_w w(t) + Bu(t) \tag{2}$$

$$x_{min} \leq x(t) \leq x_{max} \tag{3}$$

$$u(t) \in \{0, 1\} \tag{4}$$

where equation (2) is a continuous time dynamic model represented in state space form [12] with state vector $x(t) \in \mathbb{R}^n$, disturbance vector $w(t) \in \mathbb{R}^D$, input vector $u(t) \in \mathbb{R}^M$, and dynamic matrix $A \in \mathbb{R}^{n \times n}$, disturbance matrix $B_w \in \mathbb{R}^{n \times D}$ and input matrix $B \in \mathbb{R}^{n \times M}$. Equation (3) are bounds on the state $x(t)$ and equation (4) refers to the discrete value of control actions.

The BIP mathematical program set by cost function (1) and constrains (2)-(4) captures the energy cost minimization of a continuous time dynamical system with bounds on the state values. The mathematical formulation captures significant practical problems such as the energy management of a colliery [4], peak load management in steel plants [13], demand side management of a mine winder system [14], optimization of wind-hydro plant operation [6], and energy cost minimization on pumping stations [15], among others. Moreover further constraints commonly used in industrial settings such production constrains, storage constraints, sequential constraints, and down time of machines, among others can be cast in the framework proposed [3].

III. PROBLEM TRANSFORMATION

The complexity of solving the BIP is circumvented by a two step approach. First the BIP is transformed into a LP that can be solved more easily. Secondly N simpler feasibility BIP are solved to recover the discrete control actions.

A. From BIP to LP

The minimization of cost function (1), subject to (2)-(4), is a linear BIP because the on-off nature of the control actions. However if the binary decision variable $u_m(t)$ is integrated, the integral $\int_{\Delta t} u_m(t)dt$ is no longer binary but real, thus yielding a linear program. Following assumptions 1 and 2, the cost function (1) can be rewritten as

$$J = \sum_{m=1}^{M} \sum_{i=1}^{N} C_i P_m^{ss} \int_{t_{i-1}}^{t_i} u_m(t)dt \tag{5}$$

where $U_{mi} = \int_{t_{i-1}}^{t_i} u_m(t)dt$ is the operation time of load m scheduled during time period $\Delta t_i = t_i - t_{i-1}$, at constant price C_i. Figure 1 presents the time interval T divided into intervals Δt_i with constant electricity price, together with the variables associated to each interval. The objective function (5) is rewritten as

$$J = \sum_{m=1}^{M} P_m^{ss} \sum_{i=1}^{N} C_i U_{mi} \tag{6}$$

which is a linear function with MN decision variables U_{mi}.

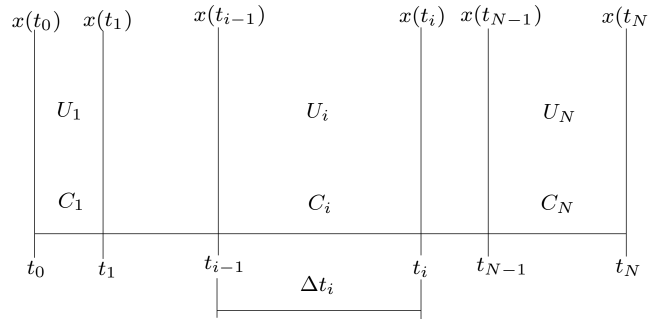

Fig. 1. Problem Variables.

The constraints (2)-(4) are also integrated. First integrating the system dynamics (2) we have

$$\begin{aligned} x(t_i) \quad - \quad x(t_{i-1}) &= A \int_{t_{i-1}}^{t_i} x(t)dt + \\ + \quad B_w \int_{t_{i-1}}^{t_i} w(t)dt &+ B \int_{t_{i-1}}^{t_i} u(t)dt. \end{aligned} \tag{7}$$

Performing the change of variables $X_i = \int_{t_{i-1}}^{t_i} x(t)dt$ and $W_i = \int_{t_{i-1}}^{t_i} w(t)dt$, and reordering expression (7), the state value at time instant t_i is

$$x(t_i) = x(t_{i-1}) + AX_i + B_w W_i + BU_i. \tag{8}$$

The disturbance W_i is known, whereas X_i and U_i are the decision variables, hence by the variables grouping

$$\bar{A} = \begin{bmatrix} A & B \end{bmatrix} \tag{9}$$

$$\bar{V}_i = \begin{bmatrix} X_i \\ U_i \end{bmatrix} \tag{10}$$

we have that, finally, equation (8) can be expressed as

$$x(t_i) = x(t_{i-1}) + B_w W_i + \bar{A}\bar{V}_i \tag{11}$$

for $i = 1, \ldots, N$.

Applying the state constraint (3) to equation (11) yields

$$x_{min} - x(0) - B_w \sum_{i=1}^{j} W_i \quad \leq \quad \bar{A} \sum_{i=1}^{j} \bar{V}_i \tag{12}$$

$$x_{max} - x(0) - B_w \sum_{i=1}^{j} W_i \quad \geq \quad \bar{A} \sum_{i=1}^{j} \bar{V}_i \tag{13}$$

for $j \in [1, \ldots, N]$, which can be finally written as

$$\begin{bmatrix} -A_T \\ A_T \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \vdots \\ \bar{V}_N \end{bmatrix} \leq \begin{bmatrix} -\mathbb{1}_{1\times n} \otimes (x_{min} - x(0)) + B_w \otimes W \\ \mathbb{1}_{1\times n} \otimes (x_{max} - x(0)) - B_w \otimes W \end{bmatrix} \tag{14}$$

where $\otimes$ is the matrix Kronecker product [16, p. 399], and

$$A_T = \begin{bmatrix} \bar{A} & 0 & \ldots & 0 \\ \bar{A} & \bar{A} & \ldots & 0 \\ \bar{A} & \bar{A} & \ldots & \bar{A} \end{bmatrix}, \quad W = \begin{bmatrix} W_1 \\ W_1 + W_2 \\ \vdots \\ \sum_i^N W_i \end{bmatrix} \tag{15}$$

In this way the inequality (14) is written as the standard inequality $Ax \leq b$ of linear programs.

The integral of the control action is also limited by

$$0_{M\times 1} \leq U_i \leq \mathbb{1}_{M\times 1}\Delta t_i \tag{16}$$

which sets the constraints on the control action. In fact constraint (16) sets that the minimum value of the control action is 0, that is the load is off during all the time interval. On the other hand, the bound Δt_i, sets that the load can be, at most, on during all the time interval.

It should be remarked that the proposed transformation is only a necessary condition. In fact, if the optimal solution of the problem defined by cost function (1) and constraints (2)-(4) is given by $u^*(t) \in \{0, 1\}^M$, it follows that

$$\int_{\Delta t_k} u^*(t)dt \geq U_k \tag{17}$$

where $U_k \in \mathbb{R}^p$ is the solution provided by the first stage LP. However, there are problems were in (17) we have an equality relation, as will be shown in the example presented in the article, thus motivating the approach.

B. Discrete Control Actions

Once the LP defined by cost function (6) and constraints (14)-(16) is solved, the optimal energy consumption per period $U_i, \quad i = 1, \ldots, N$ is known. However, the energy is supplied by a discrete control variable $u(t)$, but the previos stage only provides its integral value, that is, $\int_{\Delta t} u(t)dt$. The discrete control variable is recovered by finding a feasible BIP solution for each period $i = 1, \ldots, N$ with constant electricity price. In this way, period i with constant electricity price is discretized in K parts and a solution is given by any feasible solution of the BIP with constraints

$$\frac{\Delta t_i}{K}\sum_{k=1}^{K} u_m(k) = U_{mi}, \quad m = 1, \ldots, M \tag{18}$$

$$x(k+1) = A_d x(k) + B_{wd} w(k) + B_d u(k) \tag{19}$$

$$x_{min} \leq x(k) \leq x_{max} \tag{20}$$

$$u(k) \in \{0, 1\} \tag{21}$$

where (18) constraints the supplied energy to be equal to the optimal value obtained by the previous LP, equations (19) and (20) are the discrete time version of the continuous system dynamics, and (21) recovers the discrete nature of the control action. Matrices $A_d \in \mathbb{R}^{n\times n}$, $B_{wd} \in \mathbb{R}^{n\times D}$ and $B_d \in \mathbb{R}^{n\times M}$ are obtained by zero order hold discretization from their continuous time counterparts. Discrete time vectors have the same dimensions as their continuous counterparts, hence $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^D$, and $u(k) \in \mathbb{R}^M$.

Remark 1: Dynamic model (19) is the zero order hold discretization of continuous time model (2). Moreover, during sampling times, control action is constant due to its integer nature, as a result both models (19) and (2) yield the same value at sampling times [17], thus the discretization is exact.

The BIP defined by (18)-(21) can be written in standard form as follows. First, the constraint (18) is written in matrix form as

$$\mathbb{1}_{M\times K} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(K-1) \end{bmatrix} = U_i \frac{K}{\Delta t_i}. \tag{22}$$

Next, equation (19) is written, for all $k = 0, \ldots, K$, in matrix form as

$$\begin{aligned} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(K) \end{bmatrix} &= \begin{bmatrix} I_{n\times n} \\ A_d^1 \\ \vdots \\ A_d^K \end{bmatrix} \otimes x(0) + \bar{B}_{wd} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(K-1) \end{bmatrix} \\ &+ \bar{B}_d \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(K-1) \end{bmatrix} \end{aligned} \tag{23}$$

with

$$\bar{B}_{wd} = \begin{bmatrix} 0_{n\times D} & 0_{n\times D} & \ldots & 0_{n\times D} \\ B_{wd} & 0_{n\times D} & \ldots & 0_{n\times D} \\ A_d B_{wd} & B_{wd} & \ldots & 0_{n\times D} \\ \vdots & & & \\ A_d^{K-1} B_{wd} & A_d^{K-2} B_{wd} & \ldots & B_{wd} \end{bmatrix} \tag{24}$$

$$\bar{B}_d = \begin{bmatrix} 0_{n\times M} & 0_{n\times M} & \ldots & 0_{n\times M} \\ B_d & 0_{n\times M} & \ldots & 0_{n\times M} \\ A_d B_d & B_d & \ldots & 0_{n\times M} \\ \vdots & & & \\ A_d^{K-1} B_d & A_d^{K-2} B_d & \ldots & B_d \end{bmatrix} \tag{25}$$

Constraint (20) applied to equation (23) yields

$$\begin{bmatrix} -\bar{B}_d \\ \bar{B}_d \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(K-1) \end{bmatrix} \leq$$

$$\begin{bmatrix} -x_{min} \otimes \mathbb{1}_{K\times 1} + \begin{bmatrix} I_{n\times n} \\ A_d^1 \\ \vdots \\ A_d^K \end{bmatrix} \otimes x(0) + \bar{B}_{wd} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(K-1) \end{bmatrix} \\ x_{max} \otimes \mathbb{1}_{K\times 1} - \begin{bmatrix} I_{n\times n} \\ A_d^1 \\ \vdots \\ A_d^{K-1} \end{bmatrix} \otimes x(0) - \bar{B}_{wd} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(K-1) \end{bmatrix} \end{bmatrix} \quad (26)$$

yielding the standard BIP linear inequality $Ax \leq b$.

C. Sampling Time Selection

The sampling time $T_s = \Delta t_i / K$ to obtain the BIP must be chosen with care because, on the one hand, the smaller the sampling time the larger the number of BIP variables, and, on the other hand, if the sampling time is too large the resulting BIP problem may be infeasible. Moreover, the sampling time selection may be used to meet the integer constraints on equation (18). Recall that the term $\sum_k u_m(k)$ is an integer for $m = 1 \ldots M$. From equation (18), U_{mi}/T_s for $m = 1 \ldots M$ must also be an integer to have a meaningful BIP. We assume that $U_{mi} \in \mathbb{N}$, by rounding U_{mi} up to the desired accuracy degree. By choosing a sampling time equal to $T_s = U_{gcd}/L$, with U_{gcd} the greatest common divisor of $\{U_{1i}, U_{2i}, \ldots, U_{Mi}\}$, and $L \in \mathbb{N}$, we may guarantee the integer value of U_{mi}/T_s for $m = 1 \ldots M$. It is worth to mention that the term L, whenever $K = \Delta t_i L / U_{gcd} \in \mathbb{N}$, does not have influence on the optimality of the energy consumption but it has influence on the feasibility and complexity of the BIP problem. In fact the BIP number of time slots is given by $K = \Delta t_i L / U_{gcd}$. A lower bound on the BIP complexity is provided by $L = 1$ but the BIP may be infeasible. In this case parameter L must be increased to obtain a feasible solution at the cost of increasing the BIP complexity because the BIP variables number is given by MK, where M is the number of loads to be scheduled.

IV. CASE STUDY

Water supply systems present high-energy consumption values, which corresponds to the major expenses of these systems. In this section we consider the optimal operation of the water supply system presented in Figure 2. It consists on three reservoirs and two pumps to be controlled. A constant flow input of $F_e = 1/6$ Kilo-liter per minute, or Kl/min, is entering reservoir 1 (Fig. 2), while reservoirs 2 and 3 must supply a constant flow of $F_{o2} = F_{o2} = 1/12$ Kl/min each. The maximum volume of reservoir 1, 2, and 3 is 0.4 Ml, 0.25 Ml and 0.25 Ml, respectively whereas the minimum volume is 0.02 Ml, 0.02 Ml, and 0.02 Ml. Pump 1 provides a nominal flow of $F_1 = 1/2$ Kl/min and a power consumption of $P_1 = 5$ KW. Pump 2 provides a nominal flow of $F_2 = 3/5$ Kl/min and the power consumption is $P_2 = 6$ KW because not only the flow is higher but also the piezometric height of reservoir 3 is higher than reservoir 2. The objective is to minimize the operation electricity cost while maintaining the reservoirs volume constraints.

The electricity price, in cents of euro per KWh or c €/KWh, has 6 intervals during a 24 hours period, and is given by

$$\begin{aligned} 11.87 \quad c€/KWh, \quad t &= [0,6] \cup [22,24] \\ 14.11 \quad c€/KWh, \quad t &= [6,7] \cup [10,18] \\ 82.05 \quad c€/KWh, \quad t &= [7,10] \cup [18,22] \end{aligned}$$

that shows 6 time intervals of very different time duration, ranging from 1 to 8 hours. Moreover it presents a high cost on the peak hours.

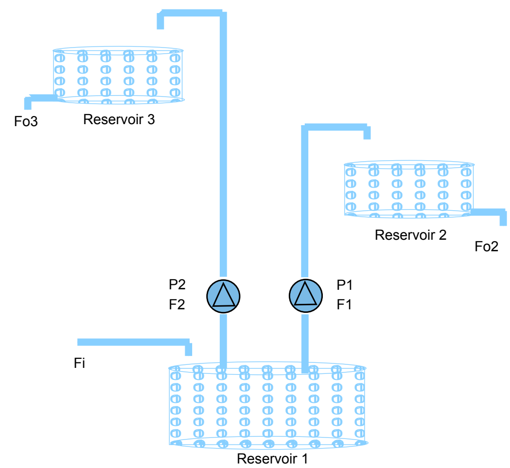

Fig. 2. Water management pumping scheme.

A. Solution of the LP

The LP cost function is

$$J = P_1 \sum_{i=1}^{6} C_i U_{1i} + P_2 \sum_{i=1}^{6} C_i U_{2i} \quad (27)$$

The constraints (14)-(16) are defined by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -F_1 & -F_2 \\ F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \quad (28)$$

$$W_i = \begin{bmatrix} F_e \Delta t_i \\ F_{o2} \Delta t_i \\ F_{o3} \Delta t_i \end{bmatrix} \quad (29)$$

and the decision variables are

$$U_i = \begin{bmatrix} U_{1i} \\ U_{2i} \end{bmatrix}, \quad i = 1, \ldots, 6 \quad (30)$$

Two distinct optimization problems are considered

1) Case 1: a 24 hours scheduling with initial state given by $x(0) = [2 \;\; 1 \;\; 1]^T 10^{-1}$ Ml.
2) Case 2: a 48 hours scheduling with initial state given by $x(0) = [2 \;\; 1 \;\; 1]^T 10^{-1}$ Ml.

Both, Case 1 and Case 2 have the same initial conditions. Case 2 is included in order to show how optimization period can be enlarged and still yield a solvable problem. Tables I and II presents the minimization results for Case 1 and Case 2, respectively. Case 1 is a LP with 12 decision variables, whereas Case 2 requires the solution of a LP with 24 decision variables. Both problems are solved in less than 0.5 seconds in a desktop PC (2.66 GHz, 2.49 GB RAM) using the standard Matlab function "linprog".

TABLE I
CASE 1 OPTIMAL ENERGY PER PERIOD GIVEN BY LP SOLUTION.

Interval i	1	2	3	4	5	6	Energy (min)
U_{1i} (min)	227	14	0	103	0	40	384
U_{2i} (min)	191	11	0	85	0	33	320

B. Solution of the BIP

Tables I and II provide the optimal energy deployment per period with constant electricity price. In order to obtain the discrete control action it is necessary to solve 4 feasibility BIP's in Case 1 and 8 feasibility BIP's in Case 2, because in all solutions, during the time intervals with peak electricity price both pumps are kept off. The sampling time used for discretization is show in the second row of tables III and IV, for the two cases considered, whereas the first row is the selected L value and the last row shows the number of intervals K used in each feasibility BIP. Consider for instance Case 1. The largest K is equal to 7, which corresponds to the last time interval. The feasibility BIP has 14 variables, because there are 2 pumps. All cases are solved in less than 1.5 seconds in a standard desktop PC using Matlab functions "linprog" and "bintprog". Note that the commonly used approach of direct discretization of cost function (1) subject to (2)-(4) with one hour sampling time yields an optimization BIP with 48 variables. The procedure was implemented by means of the Matlab "bintprog" and stopped without finding any solution because of "maximum nodes reached without converging". Finally Figure 3 shows the pumps control action together with the electricity price for Case 1. As can be seen during the peak intervals both pumps are kept off. Figure 4 shows the reservoir volume variations which are always in accordance with reservoir capacity constraints. The results for Case 2 can be seen in Figures 5 and 6.

V. CONCLUSION

In this article we have presented a new approach to the solution of optimization problems with discrete control variables which yield binary integer programs (BIP). The approach is to transform a single BIP optimization by a LP and N feasibility BIP with less number of variables. In this way it is possible to solve optimization problems with longer

TABLE III
CASE 1 PARAMETER L, SAMPLING TIME T_s, AND NUMBER OF INTERVALS K FOR EACH FEASIBILITY BIP. ZEROS INDICATE THAT NO FEASIBILITY BIP IS CALCULATED BECAUSE THERE IS NO ENERGY TO DEPLOY IN THE INTERVAL.

BIP	1	2	3	4	5	6
L	3	1	0	1	0	2
T_s (min)	60	12	0	80	0	17.14
K	6	5	0	6	0	7

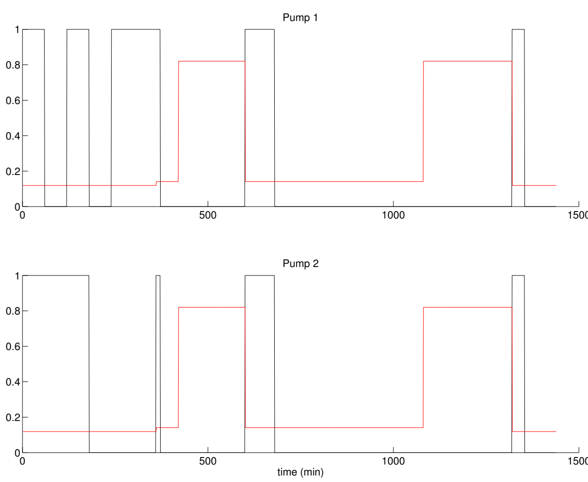

Fig. 3. Case 1 pumps control actions.

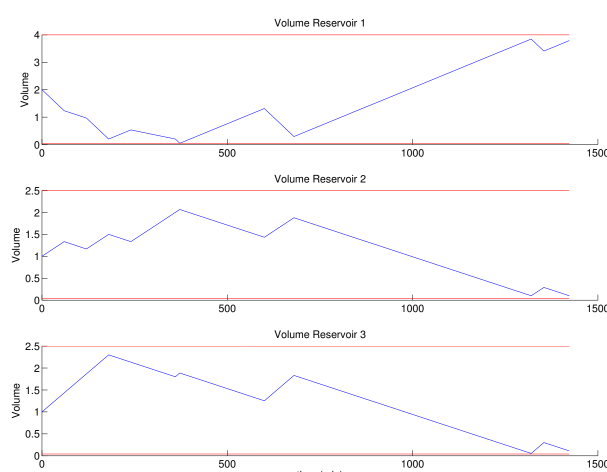

Fig. 4. Case 1 volume variation and capacity constraints.

TABLE II

CASE 2 OPTIMAL ENERGY PER PERIOD GIVEN BY LP SOLUTION.

Interval i	1	2	3	4	5	6	7	8	9	10	11	12	Energy (min)
U_{1i} (min)	224	13	0	112	0	91	265	11	0	113	0	40	869
U_{2i} (min)	188	11	0	93	0	82	213	9	0	95	0	33	724

TABLE IV

CASE 2 PARAMETER L, SAMPLING TIME T_s, AND NUMBER OF INTERVALS K FOR EACH FEASIBILITY BIP. ZEROS INDICATE THAT NO FEASIBILITY BIP IS CALCULATED BECAUSE THERE IS NO ENERGY TO DEPLOY IN THE INTERVAL.

BIP	1	2	3	4	5	6	7	8	9	10	11	12
L	3	1	0	1	0	2	3	1	0	1	0	2
T_s (min)	60	12	0	96	0	40	72	8.57	0	96	0	17.14
K	6	5	0	5	0	3	5	7	0	5	0	7

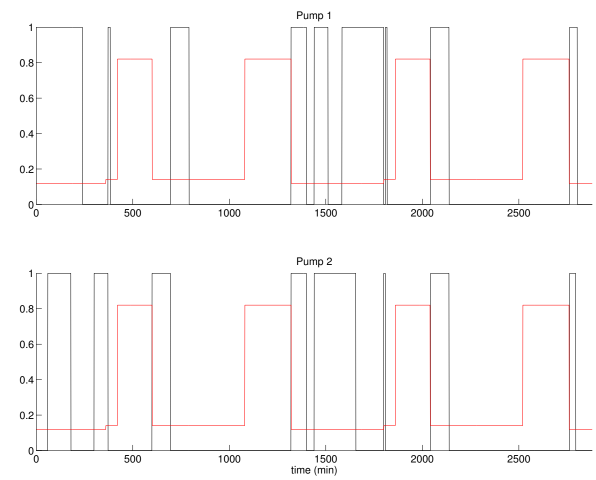

Fig. 5. Case 2 pumps control actions.

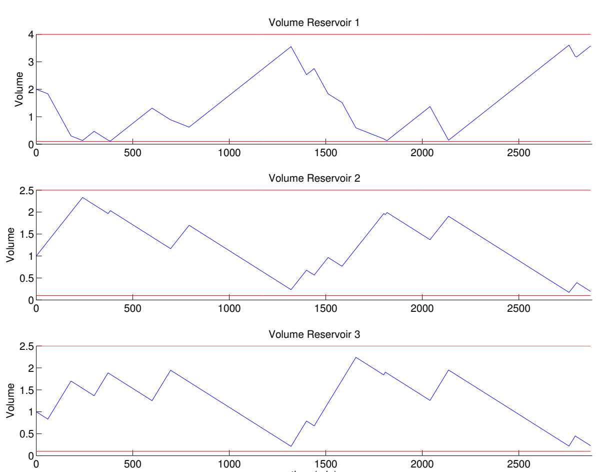

Fig. 6. Case 2 volume variation and capacity constraints.

time intervals and with higher number of actuators, because the BIP number of variables is reduced, improving the algorithmic behavior in providing a solution. Moreover the optimization problem is translated to a feasibility problem, in this way only a feasible solution is needed. Finally the complexity of the feasibility BIPs does not depend on the problem time horizon.

REFERENCES

[1] D. Bertsimas and J. Tsitsiklis, *Introduction to Linear Optimization*. Athena Scientific, 1997.

[2] C. Papadimitriou and K. Steiglitz, *Combinatorial Optimization. Algorithms and Complexity*, 1982.

[3] S. Ashok and R. Banerjee, “An optimization mode for industrial load management,” *IEEE Trans. on Power Syst.*, vol. 16, pp. 879–884, 2001.

[4] A. Middelberg, J. Zhang, and X. Xia, “An optimal control model for load shifting - with application in the energy management of a colliery,” *Appl. Energ.*, vol. 86, pp. 1266–1273, 2009.

[5] E. Castronuovo and J. P. Lopes, “On the optimization of the daily operation of a wind-hydro power plant,” *IEEE Trans. on Power Syst*, vol. 19, pp. 1599–1606, 2004.

[6] F. Vieira and H. Ramos, “Optimization of operational planning for wind/hydro hybrid water supply systems,” *Renew. Energ.*, vol. 34, pp. 928–936, 2009.

[7] C. C. M.B. Liu and W. Huang, “Reactive power and voltage control in distribution systems with limited switching operations,” *IEEE Trans. on Power Syst*, vol. 24, pp. 889–899, 2009.

[8] E. Gómez-Villalba and A. Ramos, “Optimal energy management of an industrial consumer in liberalized markets,” *IEEE Trans. on Power Syst*, vol. 18, pp. 716–723, 2003.

[9] H. Williams, *Logical and Integer Programming*. Springer, 2009.

[10] S. Sager, “Reformulations and algorithms for the optimization of switching decisions in nonlinear optimal control,” *J. Process Contr.*, vol. 19, pp. 1238–1247, 2009.

[11] J. Zhang and X. Xia, “A model predictive control approach to the periodic implementation of the solutions of the optimal dynamic resource allocation problem,” *Automatica*, vol. 47, pp. 358–362, 2011.

[12] K. Ogata, *Modern Control Engineering*. Prentice Hall, 1996.

[13] S. Ashok, “Peak-load management in steel plants,” *Appl. Energ.*, vol. 83, pp. 413–424, 2006.

[14] W. Badenhort, J. Zhang, and X. Xia, “Optimal hoist scheduling of a deep level mine twin rock winder system for demand side management,” *Electr. Pow. Syst. Res.*, vol. 81, pp. 1088–1095, 2011.

[15] M. Moradi-Jalal and B. Karney, “Optimal design and operation of irrigation pumping stations using mathematical programming and genetic algorithms,” *J. Hydraul. Res.*, vol. 46, pp. 237–246, 2008.

[16] D. Bernstein, *Matrix Mathematics*. Princeton University Press, 2009.

[17] K. Ogata, *Discrete-Time Control Systems*. Prentice Hall, 1995.