Nonlinear Control of Magnetic Levitation System with Finite-time P-PI Control

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Abstract— **The finite-time P-PI controller has not been applied to a control problem having large observation noises. In this paper, the controller is applied to a position control of a magnetic levitation system. We show the effectiveness of the finite-time P-PI controller by computer simulation and experiments. Finally, we compare the controller with a traditional P-PI controller.**

I. Introduction

Recent years, finite-time control attracts much attention in nonlinear control theory [1]. Superior performance of the control has been confirmed by experiments [2]. Finite-time P-PI control is one of the nonlinear finite-time control. We confirmed that the control is valid to position control of a robot manipulator [3].

However, the effectiveness of the finite-time P-PI control has not been evaluated under large observation noises. In this paper, we propose a generalized P-PI control and show that the control locally finite-time stabilizes the operating point. We apply the proposed method to a magnetic levitation system having large observation noises. Moreover, we confirm the effectiveness of the controller by computer simulation and experiments.

II. A Traditional P-PI Controller

In this paper, we consider the following system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \theta + lu\\ y &= x_1,\end{aligned} \tag{1}$$

where $(x_1, x_2) \in \mathbb{R}^2$ is a state, $u \in \mathbb{R}$ is an input, $l > 0$ is a known constant, and $\theta \in \mathbb{R}$ is an unknown constant.

The following is a traditional P-PI controller for system (1):

$$\begin{aligned}u &= -k_2 u_0 - k_3 \int_0^t u_0 d\hat{t}\\ u_0 &= x_2 + k_1 x_1\end{aligned} \tag{2}$$

where k_1, k_2 and k_3 are positive constants.

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III. A Finite-time P-PI Controller

Finite-time P-PI control was proposed in [3]. However, a generalized structure is not given. Then, we consider the following generalized control:

$$\begin{aligned}u &= -k_2|u_0|^{\frac{2\alpha-1}{\alpha}}\mathrm{sgn}u_0 - k_3\int_0^t (|u_0|^{\frac{3\alpha-2}{\alpha}}\mathrm{sgn}u_0)d\hat{t}\\ u_0 &= x_2 + k_1|x_1|^{\alpha}\mathrm{sgn}x_1,\end{aligned} \tag{3}$$

where α, k_1, k_2, and k_3 are positive constants. A discontinuous function sgn is defined as follows:

$$\mathrm{sgn}(x) := \begin{cases} 1 & (x > 0)\\ 0 & (x = 0)\\ -1 & (x < 0).\end{cases} \tag{4}$$

Then, the following proposition holds:

Proposition 1: Consider system (1) and controller (3). Then, the origin is finite-time stable if $2/3 < (\alpha - 1) < 1$ and the origin is asymptotically stable.

Proof: According to (1) and (3), we can obtain the following system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -lk_2|u_0|^{\frac{2\alpha-1}{\alpha}}\mathrm{sgn}u_0 + x_3\\ \dot{x}_3 &= -lk_3|u_0|^{\frac{3\alpha-2}{\alpha}}\mathrm{sgn}u_0,\end{aligned} \tag{5}$$

where

$$x_3 = \theta - lk_3\int_0^t (|u_0|^{\frac{3\alpha-2}{\alpha}}\mathrm{sgn}u_0)d\hat{t}. \tag{6}$$

According to *Definition 6* in the appendix, system (5) is homogeneous of degree $(\alpha - 1)$ with dilation exponent $(1, \alpha, 2\alpha - 1)$. Hence if $(\alpha - 1) < 0$, (5) is continuous and the origin of (5) is asymptotically stable, the origin is finite-time stable owing to Lemma 1. The righthand-side of (5) is continuous if $(3\alpha - 2) > 0$. Therefore, if $2/3 < \alpha < 1$ and the origin is asymptotically stable, the origin is finite-time stable. ■

Let $\alpha = 3/4$ and system (5) is homogeneous of degree $-1/4$ with dilation exponent $(1, 3/4, 1/2)$. We can obtain the following controller:

$$\begin{aligned}u &= -k_2 \mid u_0 \mid^{\frac{2}{3}} \mathrm{sgn}u_0 - k_3\int_0^t (\mid u_0 \mid^{\frac{1}{3}} \mathrm{sgn}u_0)d\hat{t}\\ u_0 &= x_2 + k_1 \mid x_1 \mid^{\frac{3}{4}} \mathrm{sgn}x_1.\end{aligned} \tag{7}$$

Let $\alpha = 1$, system (5) is homogeneous of degree 0 with dilation exponent $(1, 1, 1)$. Then, controller (3) is identified with traditional controller (2). In this case, if the origin of

Fig. 1. Magnetic levitation system.

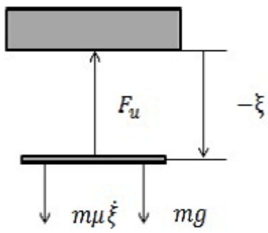

Fig. 2. System setting.

(5) is asymptotically stable, the origin is exponentially stable. (See *Lemma 1*.)

IV. Magnetic Levitation System

We consider a magnetic levitation system illustrated in Fig. 1. The system consists of a magnet with a disc, a glass guide rod, upper and lower magnetic drive coils and two laser-based sensors. A magnetic field is generated by DC currents through the coils. Each sensor measures the position of each magnetic by using the reflection of the disk surface.

In this paper, we consider the stabilization problem of the disc at the desired operating point ξ^*[m] by using attractive force generated by the upper magnetic coil. In this case, the magnetic levitation system can be modeled as Fig. 2 [4], where ξ[m] is the position of the magnet from the upper coil, F_u[N] denotes an attractive force for the disc generated by the upper drive magnetic coil, m[kg] is the mass of disc, μ[-] represents a friction constant and g_0[m/s^2] is the gravitational acceleration.

The dynamical equation for the disc illustrated in Fig. 2 is described by

$$\ddot{\xi} = F_u - m\mu\dot{\xi} - mg_0, \tag{8}$$

where the magnetic force F_u can be modeled by

$$F_u = \frac{u}{a(-\xi + b)^4}, \tag{9}$$

where a and b are constants [4].

Let ξ^*[m] be a desired operation point of the disc and introduce a new variable $x = [x_1, x_2]^T$, where $x_1 = \xi - \xi^*$[m] and $x_2 = \dot{x}_1$[m/s]. According to (8) and (9), we can obtain the following state equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 - g_0 + \tfrac{u}{ma(x_1 - \xi^* + b)^4}. \end{aligned} \tag{10}$$

TABLE I

Identified parameters.

m [kg]	0.12
μ [-]	4.5
g [m/s_2]	9.80665
a [V/(N $\cdot$ m^4)]	21127
b [m]	0.056464

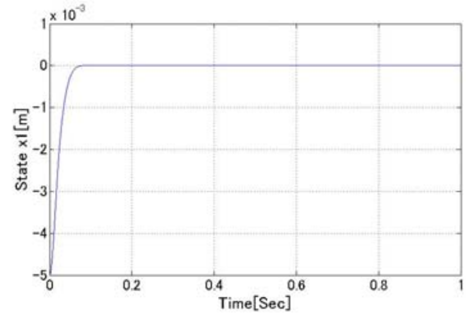

Fig. 3. Simulation result : Finite-time P-PI control state.

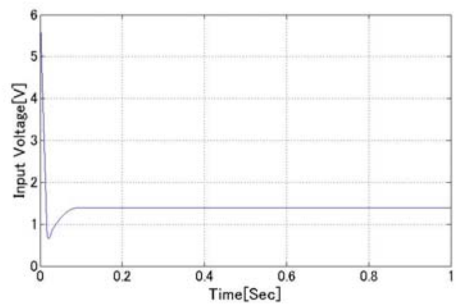

Fig. 4. Simulation result : Finite-time P-PI control input.

The control problem of the paper is to asymptotically stabilizes the origin $x = [0, 0]^T$ of (8). We show parameters in system (10) in Table. I.

Let $x_3 = -g_0$. Compare system (1) with system (10). By using a homogeneous approximation of system (10) with respect to dilation exponent (1,3/4,1/2), we obtain the following system: (See *Definition 5*)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + \tfrac{1}{ma(-\xi^* + b)^4} u. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} \theta &= x_3 \\ l &= \frac{1}{ma(-\xi^* + b)^4}, \end{aligned} \tag{12}$$

and we obtain system (1).

V. Computer Simulation

In this section, we compare finite-time P-PI controller (7) with traditional one (2) in magnetic levitation system (10) by computer simulation. We show simulation results with initial

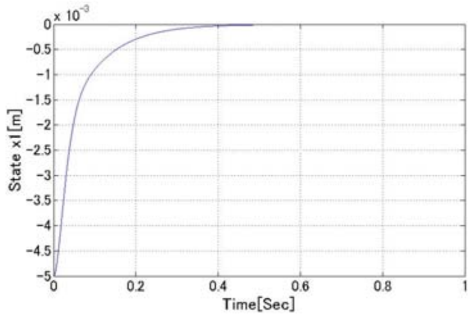

Fig. 5. Simulation result : P-PI control state.

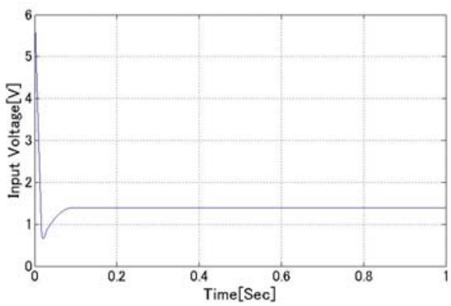

Fig. 6. Simulation result : P-PI control input.

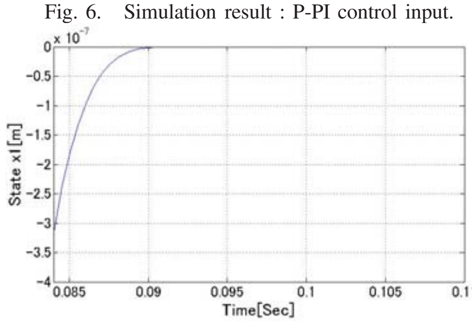

Fig. 7. Simulation result : Finite-time P-PI control state between 0.084[s] and 0.11[s].

state $x = [-5.0 \times 10^{-3}$[m],0[m/s]$]^T$ ($\xi = 3.5 \times 10^{-2}$[m], $\xi^* = 3.0 \times 10^{-2}$[m]) of finite-time P-PI controller (7) in Figs. 3 and 4 and those of traditional one (2) in Figs. 5 and 6. We designed the following parameter by the trial-and-error method.

1) We obtain gains of the finite-time P-PI controller as follows:

$$k_1 = 12.6, k_2 = 16, k_3 = 180. \tag{13}$$

2) We obtain gains of the traditional P-PI controller as follows:

$$k_1 = 40, k_2 = 18, k_3 = 180. \tag{14}$$

We show a history of x_1 from -4.0×10^{-7}[m] to 0[m] of a simulation result of the finite-time P-PI control state in Fig. 7 and those of traditional P-PI control state in Fig. 8. Finite-time P-PI control shows that state x_1 converges to $\pm 4.435 \times 10^{-21}$[m] after 0.092[s]. This means the limit of numerical calculation.

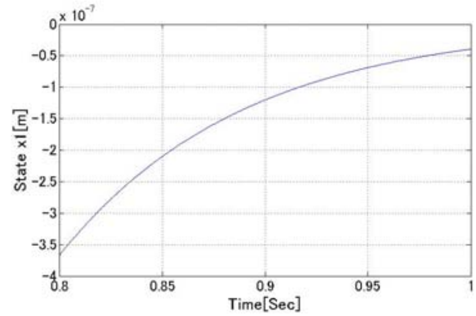

Fig. 8. Simulation result : P-PI control state between 0.8[s] and 1[s].

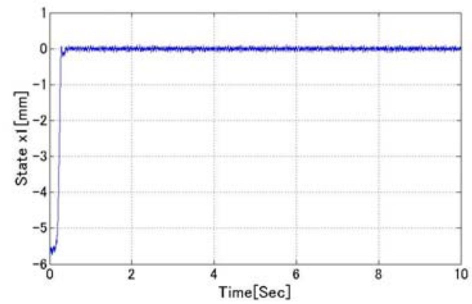

Fig. 9. Experimental result : finite-time P-PI control state between 0[s] and 10[s].

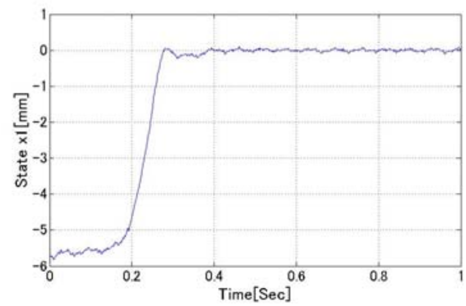

Fig. 10. Experimental result : finite-time P-PI control state between 0[s] and 1[s].

The result of computer simulation confirms that finite-time P-PI controller (7) has higher accuracy and faster convergence than traditional controller (2).

VI. Experimental Results

In the preceding section, we confirmed that finite-time P-PI controller (7) is valid to magnetic levitation system (10) by computer simulation. In this section we apply the controller to a magnetic levitation experimental facility shown in Fig. 1 with initial state $x = [-5.0 \times 10^{-3}$[m],0[m/s]$]^T$ ($\xi = 3.5 \times 10^{-2}$[m], $\xi^* = 3.0 \times 10^{-2}$[m]). The controller is implemented by using MATLAB/SIMULINK. Their sampling time is 1[msec]. Note that the variances of measurement of $x_1 = 0$ is 1.87×10^{-5}.

A. Finite-time P-PI control

We designed finite-time P-PI controller (2) with gains (13) in the preceding section. However, the gains does not work in

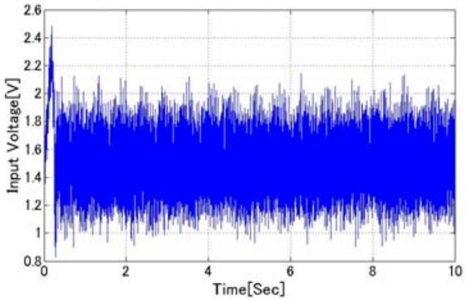

Fig. 11. Experimental result : finite-time P-PI control input.

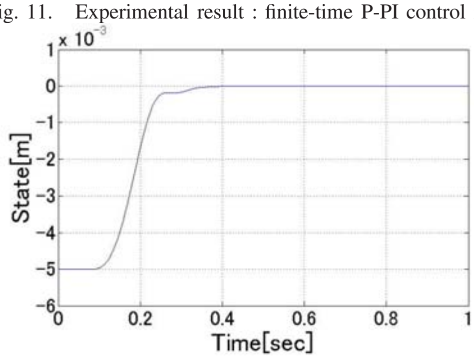

Fig. 12. Simulation result : finite-time P-PI control state with new gains.

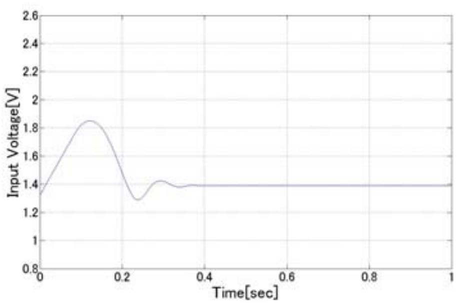

Fig. 13. Simulation result : finite-time P-PI control input with new gains.

an experiment. Hence, we designed the following new gains:

$$k_1 = 12.65, k_2 = 3.45, k_3 = 8.35. \qquad (15)$$

We show experimental results of finite-time P-PI controller (7) in Figs. 9, 10, and 11.

B. Traditional P-PI control

We design gains (14) for P-PI controller (2) in the preceding section. We show experimental results of P-PI controller (2) in Figs. 14, 15, 16. We show the averages and variances of errors between 8[s] and 10[s] and the supplied mechanical energy by settling time in Table. II. The settling time is $\pm 5\%$ of initial state $x_1 = -5.0 \times 10^{-3}$[m]. The supplied mechanical energy is written as follows.

$$E = \int_{r(0)}^{r(t)} F_u(t) dx_1 = \sum_{i=0}^{t} F_u(i) \cdot (x_1(i+1) - x_1(i)), \qquad (16)$$

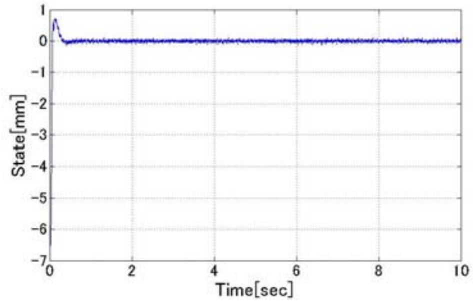

Fig. 14. Experimental result : P-PI control state between 0[s] and 10[s].

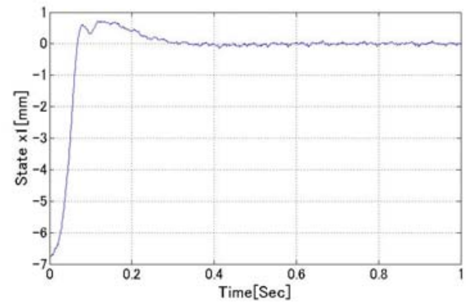

Fig. 15. Experimental result : P-PI control state between 0[s] and 1[s].

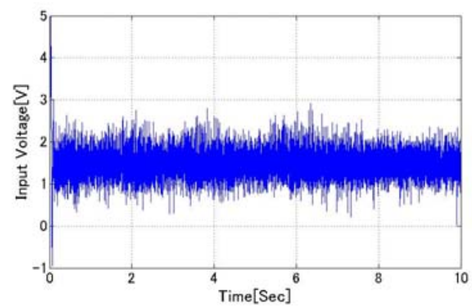

Fig. 16. Experimental result : P-PI control input.

TABLE II
COMPARISON OF CONTROLLERS.

Controller	Finite-time P-PI	Traditional
Settling Time [s]	0.27	0.22
Mean Residual [m]	2.32×10^{-5}	4.87×10^{-5}
Standard deviation [m^2]	1.00×10^{-5}	9.13×10^{-6}
Mechanical Energy [J]	3.1559	3.4207

where $F_u(t)$ and $x_1(t)$ is the state at time t. $x_1(t)$ be comes a constant after the settling time. We suppose that the supplied mechanical energy is the sum of the energy until the settling time.

C. Comparison with Computer Simulation

We compare the experimental results with the computer simulation. Then we show computer simulation of finite-time P-PI controller (7) with new gains (15) in Figs. 12 and 13.

In finite-time P-PI controller (7), comparing Fig. 10 with Fig. 12 and Fig. 13 with Fig. 11, experimental results is similar to computer simulation. In traditional P-PI controller (2), comparing Fig. 5 with Fig. 15, we can view the wave form of them are different and settling time of them are

TABLE III

COMPARISON OF CONTROLLERS (TEN TIMES).

Controller	Finite-time P-PI	Traditional
Settling time [s]	0.226	0. 211
Mean Residual [m]	7.62 $\times 10^{-5}$	3.13 $\times 10^{-5}$
Standard deviation [m^2]	1.14 $\times 10^{-5}$	1.10 $\times 10^{-5}$

almost the same. Comparing Fig. 6 with Fig. 16, we can view the wave form of them are almost the same.

VII. DISCUSSION

We confirmed that finite-time P-PI controller (7) performs smaller averages of errors than traditional one (2). Table. II shows that finite-time P-PI controller (7) supplies 7.7% smaller mechanical energy, and has about a half average of errors and input voltage around 0[s] comparing with traditional one (2).

We conducted experiments ten times under the same conditions in the preceding section. We summarize averages and variances of errors between 8[s] and 10[s] in Table. III. However, finite-time controller (7) has a bigger variance of average of errors and is slower settling time than traditional one (2).

To shorten the settling time of finite-time P-PI controller (7), we enlarge the gain k_1 that affects state x_1 directly; however, this results in longer settling time and a bigger overshoot. Overshooting is necessary for short settling time for traditional P-PI controller. However, the finite-time P-PI controller has little overshoot under the minimum settling time. However the finite-time P-PI controller is slower than the traditional one, because the finite-time P-PI controller has long rise time.

To summarize the above, comparing the finite-time P-PI controller with traditional one, the finite-time P-PI control achieves smaller overshoot and energy consumption and higher convergence accuracy as the same as computer simulation. On the other hand, traditional P-PI controller is faster settling time and has a small variability of average of errors.

VIII. CONCLUSION

In this paper, we proposed a generalized structure of finite-time P-PI controllers system for system (1). The controller is appled to a magnetic levitation system having big observation noises. The finite-time P-PI controller has confirmed higher convergence accuracy than a traditional P-PI controller.

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APPENDIX

A. Homogeneous Stabilization and Finite-time Stable

We consider the following input affine nonlinear system[5]:

$$\dot{x} = f(x) + g(x)u, \tag{17}$$

where $x \in \mathbb{R}^n$ is state vector, $u \in \mathbb{R}^m$ is an input vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$are continuous mappings, and $f(0) = 0$. Let $g_i(x)$ and $g^j(x)$ respectively denote the ith row vector and jth column vector of $g(x)$, $\mathbb{R}_{>0} := (0, \infty)$, and $\mathbb{R}_{\geq 0} := [0, \infty)$.

Definition 1: (dilation): The mapping $\triangle_\epsilon^r x = (\epsilon^{r_1} x_1,\cdots,\epsilon^{r_n} x_n)^T (\forall \epsilon x \in \mathbb{R}^n \setminus \{0\})$ is said to be a dilation on $\mathbb{R}^n$, where $r = (r_1,\cdots,r_n)^T$ is a constant vector satisfying $0 < r_i < \infty (i = 1,\cdots,n)$.

Definition 2: (Homogeneous Function): A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation $\triangle_\epsilon^r x$ if $V(\triangle_\epsilon^r x) = \epsilon^k V(x)$.

Definition 3: (Homogeneous System): System system (17) is said to be homogeneous of $\tau \in \mathbb{R}$ with respect to the dilations $\triangle_\epsilon^r x$ and $\triangle_\epsilon^s u$ if $f(\triangle_\epsilon^r x) + g(\triangle_\epsilon^r x)\triangle_\epsilon^s u = \epsilon^\tau \triangle_\epsilon^r f(x) + g(x)u$.

Definition 4: (Homogeneous Norm): The function $\| x \|_{\{r,p\}} := (\sum_{j=1}^n |x|^{p/r_j})^{1/p}$ is said to be a homogeneous p-norm.

Definition 5: (Stability): We consider that following system is homogeneous of degree τ and the origin is asymptotically stable :

$$\dot{x} = f(x), \tag{18}$$

where $x \in \mathbb{R}^n$ is a state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$is a continuous mapping, and $f(0) = 0$. Theorigin of system (18) is said to be

1) stable if for any $\epsilon > 0$, there exists $\delta > 0$ satisfying

$$\| x(0) \| < \delta \Longrightarrow \| x(t) \| < \epsilon, \forall t \geq 0 \tag{19}$$

2) globally asymptotically stable if it is stable ans all the solutions x(t) satisfy

$$\lim t \to \infty \| x(t) \| = 0 \tag{20}$$

Definition 6: (Exponential Stability): The origin of system (18) is said to be exponentially stable if there exist positive constants $M, D > 0$ such that for any $x_0 \in X \setminus \{0\}$ the solution $x(t)$ with $x(0) = x_0$ is defined on [0,∞) and satisfies

$$\| x \| \geq Me^{-Dt} \| x_0 \|_{\{r,p\}}, \forall t \geq 0 \tag{21}$$

Definition 7: (Finite-time Stability): The origin of system (18) is said to be finite-time stable if it is stable and there exists an open neighborhood X of the origin and a function

$T : X \setminus \{0\} \to \mathbb{R}_{>0}$ such that for any $x_0 \in X \setminus \{0\}$, the solution $x(t)$ with $x(0) = x_0$ is defined on $[0, T(x))$, $x(t) \in X \setminus \{0\}$, $t \in [0, T(x))$, and $\lim_{t \to T(x)} x(t) = 0$.

Lemma 1: We suppose that system (18) is homogeneous of degree τ and the origin is asymptotically stable. Then, the following statements are true:

1) If $\tau > 0$, any $x \in \mathbb{R}^n$ for $t \to \infty$ is $x \to 0$.
2) $\tau = 0$, the origin is exponentially stable.
3) If $\tau < 0$, the origin is finite-time stable.

Definition 8: (Homogeneous Approximation):

$$\dot{x} = f_h(x) + g_h(x)u \tag{22}$$

of degree τ with respect to $\triangle_\epsilon^r x$ and $\triangle_\epsilon^s u$ is said to be homogeneous approximation of (17) if there exists $f_0(x)$ and $g_0(x)$ satisfying

$$f(x) + g(x)u = f_h(x) + g_h(x)u + f_o(x) + g_o(x)u \tag{23}$$

and

$$\lim_{\epsilon \to 0} \frac{f_{o,i}(\triangle_\epsilon^r x) + g_{o,i}(\triangle_\epsilon^r x)\triangle_\epsilon^s u}{\epsilon^{\tau + r_i}} = 0, \forall i = 1, \cdots, n \tag{24}$$

uniformly on $S^{n+m-1} := \{(x, u) \in \mathbb{R}^{n+m} | \parallel (x, u) \parallel = 1\}$.

Lemma 2: Consider system (18) such that

$$f(x) = f_h(x) + f_o(x). \tag{25}$$

where $f_h(x)$ is homogeneous of degree τ with respect to $\triangle_\epsilon^r(x)$ and

$$\lim_{\epsilon \to 0} \frac{f_{o,i}(\triangle_\epsilon^r x)}{\epsilon^{\tau + r_i}} = 0, \forall i = 1, \cdots, n \tag{26}$$

Then, if the origin is asymptotic stable, states in *Lemma 1* are still valid for trajectories of (25).