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Distributed tracking control of an uncertain heat diffusion process

Yury Orlov
CICESE Research Center, Ensenada (Mexico)
Email: yorlov@cicese.mx

Alessandro Pisano, Elio Usai
Dept. of Electrical and Electronic Engineering (DIEE)
University of Cagliari (Italy)
Email:{pisano,eusai}@diee.unica.it

Abstract— **This paper addresses the design of a distributed non smooth tracking controller for a class of uncertain diffusive processes with homogeneous boundary conditions of Neumann type. Uncertainty in the diffusivity parameter, along with the presence of a sufficiently smooth distributed disturbance, characterize the considered class of processes. The paper considers a dynamical controller with a linear PI structure complemented by a certain non-smooth integral term which gives the algorithm its main robustness properties. A constructive Lyapunov-based stability analysis is illustrated which leads to simple tuning conditions for the controller parameters guaranteeing the asympto tic tracking in the space L_2 of sufficiently smooth reference profiles. The main contribution of the present paper over recent related works (cfr. [19], [20]) is the relaxation of a quite restrictive assumption involving the external perturbation and, at the same time, the achievement of the control goal with milder information demand on the system's state. The good performance of the proposed control systems are verified by means of computer simulations.**
Keywords: ***Distributed parameter systems, Diffusion equation, Distributed control systems, Sliding mode control.***

I. Introduction

Sliding-mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and unmodelled dynamics when controlling dynamical systems of finite and infinite dimension (see [25]).

Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented (see [16], [23], [17], [22], [24]) and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterpart. Other robust control paradigms have been fruitfully applied in the infinite dimensional setting such as adaptive and model-reference control (see [11], [7]), geometric and Lyapunov-based design (see [4]), H_∞ and LMI-based design (see [9]). It should be noted that the latter paradigms are capable of **attenuating** vanishing disturbances only, whereas the former discontinuous control is additionally capable of **rejecting** persistent disturbances with an *a priori* known bound on their L_2 norm.

In the present paper we consider an uncertain form of a popular parabolic infinite dimensional dynamics, the heat equation, under the effect of an external smooth disturbance. In some recent authors' publications (see [19], [20])) two finite dimensional robust control algorithms, namely, the "Super-Twisting" and "Twisting" second-order sliding-mode (2-SM) controllers (see [14]) for details on these controllers) have been generalized to the infinite-dimensional setting and applied for controlling heat and wave processes, respectively. The mentioned 2-SM controllers are of special interest because in the finite dimensional setting they significantly improve the performance of sliding-mode control systems, in terms of accuracy and chattering avoidance, as compared to the standard "first-order" sliding mode control techniques (see [3]).

In [19], a quite restrictive assumption was made concerning the external perturbation (this aspect is going to be discussed in dedicated Remark), while in [20], essentially dealing with the same class of processes, such restrictive assumption was not met anymore but the time derivative of the state variable was required for feedback within the framework of a dynamical input-extension methodology.

In this paper we manage to overcome the above mentioned drawbacks by providing a solution to the tracking problem not requiring the restrictive assumption made in [19] and avoiding at the same time the need to measure the time derivative of the state variable. We suggest a control algorithm that appropriately combines three ingredients: a PI feedback control component, a non smooth integral control component, and a feed-forward term. From a different point of view, the proposed controller could be sought as a modified version of the Distributed Super-Twisting controller in [19] with the characteristic square-root static term of the algorithm being replaced by a linear one. This solves certain technical problems associated to the convergence proof, thereby allowing the previously mentioned restrictive assumption to be relaxed without needing to increase the information demand by measuring, as done in [20], the time derivative of the distributed state vector, that would be a quite unpractical requirement.

It must be stressed that the obtained results assume distributed sensing and actuation. Despite this is normally recognized as a critical drawback, it should be put into evidence that, due to advances in technological developments of Micro Electro-Mechanical Systems (MEMS), manufacturing large arrays of micro-sensors and actuators with integrated control circuitry has become feasible (for control applications of

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such devices see [2] and references therein) which could permit an effective utilization of distributed feedback control loops in some practical applications.

The rest of the paper is structured as follows. Some notations are introduced in the remainder of the Introduction. Section II presents the problem formulation and describes the proposed solution, based on a proper combination of linear and nonlinear control techniques. Section III illustrates some relevant numerical simulations, and, finally, Section IV gives some concluding remarks and draws possible directions of improvement of the proposed results.

A. Notation

The notation used throughout is fairly standard. $L_p(a,b)$, with $p \geq 1$ and $a \leq b$, stands for the Hilbert space of square integrable functions $z(\zeta),\ \zeta \in [a,b]$, with the norm

$$\|z(\cdot)\|_p = \sqrt[p]{\int_a^b z^p(\zeta)d\zeta} \tag{1}$$

$W^{l,2}(a,b)$ stands for the Sobolev space of absolutely continuous scalar functions on $[a,b]$ with square integrable derivatives of the order $l \geq 1$.

II. Tracking control of the Heat equation

A. Problem formulation

Consider the space- and time-varying scalar field $Q(\xi,t)$ evolving in a Hilbert space $H = L_2(0,1)$, where $\xi \in [0,1]$ is the monodimensional (1D) spatial variable and $t \geq 0$ is time. Let it be governed by a perturbed version of the parabolic PDE which is commonly referred to as the "**Heat Equation**":

$$Q_t(\xi,t) = \theta Q_{\xi\xi}(\xi,t) + \psi(\xi,t) + u(\xi,t), \tag{2}$$

where θ ia a positive coefficient called *thermal conductivity* (or, more generally, *diffusivity*), $u(\xi,t) \in L_2(0,1)$ is the distributed control input and $\psi(\xi,t)$ represents a distributed uncertain perturbation term of class $L_\infty(0,\infty;L_2(0,1))$. The initial conditions (IC) are

$$Q(\xi,0) = Q^0(\xi) \in W^{2,2}(0,1). \tag{3}$$

We consider homogeneous boundary conditions (BC's) of Neumann type

$$Q_\xi(0,t) = Q_\xi(1,t) = 0. \tag{4}$$

Consider a space and time-varying reference profile $Q^r(\xi,t) \in W^{2,2}(0,1)$. The actual control task is that of steering to zero the L_2 norm of the deviation variable

$$\tilde{Q}(\xi,t) = Q(\xi,t) - Q^r(\xi,t) \tag{5}$$

in spite of the presence of the external unknown perturbation $\psi(\xi,t)$ and the uncertain value of the diffusivity parameter. Additional underlying regularity assumptions on the external unknown perturbation and on the admitted reference profile will be detailed later on.

B. Main result

The feedback/feedforward dynamical distributed controller

$$u(\xi,t) = Q_t^r(\xi,t) - k_2\tilde{Q}(\xi,t) + v(\xi,t), \tag{6}$$

$$v_t(\xi,t) = -k_1\frac{\tilde{Q}(\xi,t)}{\|\tilde{Q}(\cdot,t)\|_2} - k_3\tilde{Q}(\xi,t), \tag{7}$$

is currently under investigation, where k_1, k_2, k_3 are the tuning constants of the algorithm.

The closed loop deviation variable dynamics is readily written as

$$\tilde{Q}_t(\xi,t) = \theta\tilde{Q}_{\xi\xi}(\xi,t) + \bar{\psi}(\xi,t) - k_2\tilde{Q}(\xi,t) + v(\xi,t), \tag{8}$$

$$v_t(\xi,t) = -k_1\frac{\tilde{Q}(\xi,t)}{\|\tilde{Q}(\cdot,t)\|_2} - k_3\tilde{Q}(\xi,t), \tag{9}$$

with the BCs

$$\tilde{Q}_\xi(0,t) = \tilde{Q}_\xi(1,t) = 0, \tag{10}$$

and where $\bar{\psi}(\xi,t)$ is the "augmented" disturbance

$$\bar{\psi}(\xi,t) = \psi(\xi,t) + \theta Q^r_{\xi\xi}(\xi,t). \tag{11}$$

The precise meaning of the solutions of the non-smooth boundary value-problem (8)-(11) can be defined, in the generalized sense [24], as a limiting result obtained through the regularization procedure, similar to that proposed for finite-dimensional systems [25].

According to this procedure, the strong solutions of the boundary-value problem are only considered whenever they are beyond the discontinuity manifold $\tilde{Q}(\xi,t) = 0$ whereas in a vicinity of this manifold the original system is replaced by a related system, which takes into account all possible imperfections in the new input function $v^\delta(\tilde{Q}(\xi,t))$ (e.g., delay, hysteresis, saturation, etc.) and for which there exists a strong solution. A generalized solution of the system in question is then obtained by making the characteristics of the new system approach those of the original one. As in the finite-dimensional case, a motion along the discontinuity manifold is referred to as a sliding mode. The paper mainly focuses on the synthesis procedure whereas the detailed investigation of the existence of generalized solutions and their regularity, normally ensured by the parabolic type of the equation, remain beyond the scope of the paper in the light of the space limitations.

The class of admissible (augmented) disturbances is specified by the following restrictions

Assumption 1 There exist *a priori* known constants F_1, F_2 such that

$$\|\bar{\psi}_t(\cdot,t)\|_2 \leq F_1, \tag{12}$$

$$\|\bar{\psi}_{tt}(\cdot,t)\|_2 \leq F_2. \tag{13}$$

It is worth noting that since, by (11), the next relation holds,

$$\|\bar{\psi}_t(\cdot,t)\|_2 \leq \|\psi_t(\cdot,t)\|_2 + \theta\|Q^r_{t\xi\xi}(\xi,t)\|_2, \tag{14}$$

then Assumption 1 actually represents a set of restrictions involving, at the same time, the external perturbation, the uncertain diffusivity and the reference profile.

Remark 1 In [19] a similar control problem was dealt with, and different restrictions on the disturbance as those specified in the Assumption 1 were made. Particularly, relation (12) was replaced by the next one

$$\|\bar{\psi}_t(\cdot,t)\|_2 \le F_1 \frac{|\tilde{Q}(\xi,t)|}{\|\tilde{Q}(\cdot,t)\|_2}, \quad \forall t \ge 0, \ \forall \xi \in [0,1]. \quad (15)$$

Assumption (15) appears to be more restrictive than (12), even if the admissible disturbances fulfilling (15) have a time derivative which is not necessarily vanishing as $|\tilde{Q}(\xi,t)| \to 0$ because the norm in the right-hand side of (15) remains unit since $\left\| \frac{|\tilde{Q}(\xi,t)|}{\|\tilde{Q}(\cdot,t)\|_2} \right\|_2 = 1$. On the other hand, in [20] assumptions (15) and (13) were not made at all, and just restriction (12) was assumed, but availability for measurement of the state error derivative $\tilde{Q}_t(\xi,t)$ was needed for controller implementation.

The main result of the paper is formalized by means of the next Theorem.

Theorem 1 *Consider the perturbed heat process (2) along with the initial and boundary conditions (3) and (4), and with the uncertain perturbation, diffusivity and reference profile satisfying Assumption 1. Then, the distributed control strategy (6)-(7) with the parameters such that*

$$k_1 > F_1 + 2\frac{F_2}{k_2}, \qquad k_3 > k_2^2, \quad (16)$$

guarantees that

$$\|\tilde{Q}(\cdot,t)\|_2 \to 0 \quad \text{as} \quad t \to \infty. \quad (17)$$

Proof of Theorem 1 Consider the Lyapunov functional

$$\begin{aligned} V(t) &= \frac{1}{2}k_3\|\tilde{Q}(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde{Q}_t(\cdot,t)\|_2^2 \\ &+ \frac{1}{2}k_2\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta + k_1\|\tilde{Q}(\cdot,t)\|_2 \\ &- \int_0^1 \bar{\psi}_t(\eta,t)\tilde{Q}(\eta,t)d\eta + \frac{1}{4}k_2\theta\|\tilde{Q}_\xi(\cdot,t)\|_2^2. \end{aligned} \quad (18)$$

Let us show that the above functional is positive definite. The sign-indefinite integral terms in (18) can be estimated by Young and Cauchy-Schwartz inequalities, respectively, as

$$\frac{1}{2}k_2\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta \ge -\frac{1}{2}k_2^2\|\tilde{Q}(\cdot,t)\|_2^2 - \frac{1}{8}\|\tilde{Q}_t(\cdot,t)\|_2^2 \quad (19)$$

$$\left|\int_0^1 \bar{\psi}_t(\eta,t)\tilde{Q}(\eta,t)d\eta\right| \le \|\bar{\psi}_t(\cdot,t)\|_2\|\tilde{Q}(\cdot,t)\|_2 \le F_1\|\tilde{Q}(\cdot,t)\|_2 \quad (20)$$

Considering (19) and (20) into (18) yields the further estimation

$$\begin{aligned} V(t) &\ge \frac{1}{2}(k_3 - k_2^2)\|\tilde{Q}(\cdot,t)\|_2^2 + \frac{3}{8}\|\tilde{Q}_t(\cdot,t)\|_2^2 \\ &+ (k_1 - F_1)\|\tilde{Q}(\cdot,t)\|_2 + \frac{1}{4}k_2\theta\|\tilde{Q}_\xi(\cdot,t)\|_2^2. \end{aligned} \quad (21)$$

Considering the tuning conditions (16), it can be concluded that the Lyapunov functional is positive definite.

The time derivative of $V(t)$ is given by

$$\begin{aligned} \dot{V}(t) &= k_3\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta + \int_0^1 \tilde{Q}_t(\eta,t)\tilde{Q}_{tt}(\eta,t)d\eta \\ &+ \frac{1}{2}k_2\|\tilde{Q}_t(\cdot,t)\|_2^2 + \frac{1}{2}k_2\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_{tt}(\eta,t)d\eta \\ &+ \frac{k_1}{\|\tilde{Q}(\cdot,t)\|_2}\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta \\ &- \int_0^1 \bar{\psi}_{tt}(\eta,t)\tilde{Q}(\eta,t)d\eta - \int_0^1 \bar{\psi}_t(\eta,t)\tilde{Q}_t(\eta,t)d\eta \\ &+ \frac{k_2\theta}{2}\int_0^1 \tilde{Q}_\xi(\eta,t)\tilde{Q}_{\xi t}(\eta,t)d\eta. \end{aligned} \quad (22)$$

To evaluate the time derivative $\dot{V}(t)$ along the trajectories of the closed loop system, let us differentiate in time (8)-(9) and write down the corresponding second-order closed loop dynamics

$$\begin{aligned} \tilde{Q}_{tt}(\xi,t) &= \theta\tilde{Q}_{t\xi\xi}(\xi,t) + \bar{\psi}_t(\xi,t) - k_2\tilde{Q}_t(\xi,t) \\ &- k_1\frac{\tilde{Q}(\xi,t)}{\|\tilde{Q}(\cdot,t)\|_2} - k_3\tilde{Q}(\xi,t) \end{aligned} \quad (23)$$

with the boundary conditions

$$\tilde{Q}_{t\xi}(0,t) = \tilde{Q}_{t\xi}(1,t) = 0. \quad (24)$$

The underlying assumptions on the initial conditions, external disturbances and desired profile ensure the existence of the generalized solutions for the boundary value problem (8)-(11) (in the sense previously defined) and its regularity. By the way, in this regard relation (3) should be specified with the Sobolev space $W^{4,2}(0,1)$ to ensure the aforementioned regularity of solutions of the boundary value problem (23)-(24). Substituting (23) into (22) we get

$$\begin{aligned} \dot{V}(t) &= k_3\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta + \frac{1}{2}k_2\|\tilde{Q}_t(\cdot,t)\|_2^2 \\ &+\int_0^1 \tilde{Q}_t(\eta,t)\Big(\theta\tilde{Q}_{t\xi\xi}(\eta,t) + \bar{\psi}_t(\eta,t) - k_2\tilde{Q}_t(\eta,t) \\ &\qquad -k_1\frac{\tilde{Q}(\eta,t)}{\|\tilde{Q}(\eta,t)\|_2} - k_3\tilde{Q}(\eta,t)d\eta\Big) \\ &+\frac{k_2}{2}\int_0^1 \tilde{Q}(\eta,t)\Big(\theta\tilde{Q}_{t\xi\xi}(\eta,t) + \bar{\psi}_t(\eta,t) - k_2\tilde{Q}_t(\eta,t) \\ &\qquad -k_1\frac{\tilde{Q}(\eta,t)}{\|\tilde{Q}(\eta,t)\|_2} - k_3\tilde{Q}(\eta,t)d\eta\Big)d\eta \\ &+\frac{k_1}{\|\tilde{Q}(\cdot,t)\|_2}\int_0^1 \tilde{Q}(\eta,t)\tilde{Q}_t(\eta,t)d\eta \quad (25) \\ &-\int_0^1 \bar{\psi}_{tt}(\eta,t)\tilde{Q}(\eta,t)d\eta - \int_0^1 \bar{\psi}_t(\eta,t)\tilde{Q}_t(\eta,t)d\eta + \\ &\frac{k_2\theta}{2}\int_0^1 \tilde{Q}_\xi(\eta,t)\tilde{Q}_{\xi t}(\eta,t)d\eta. \end{aligned}$$

Manipulating (25) we obtain

$$
\begin{aligned}
\dot V(t) &= k_3\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta + \frac{1}{2}k_2\|\tilde Q_t(\cdot,t)\|_2^2 \\
&+ \theta\int_0^1 \tilde Q_t(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta \\
&+ \int_0^1 \tilde Q_t(\eta,t)\bar\psi_t(\eta,t)d\eta - k_2\|\tilde Q_t(\cdot,t)\|_2^2 \\
&- \frac{k_1}{\|\tilde Q(\cdot,t)\|_2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta \\
&- k_3\int_0^1 \tilde Q_t(\eta,t)\tilde Q(\eta,t)d\eta \\
&+ \frac{k_2\theta}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta \\
&+ \frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\bar\psi_t(\eta,t)d\eta \\
&- \frac{k_2^2}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta - \frac{k_1k_2}{2}\|\tilde Q(\cdot,t)\|_2 \\
&- \frac{k_2k_3}{2}\|\tilde Q(\cdot,t)\|_2^2 \\
&+ \frac{k_1}{\|\tilde Q(\cdot,t)\|_2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta \qquad (26)\\
&- \int_0^1 \bar\psi_{tt}(\eta,t)\tilde Q(\eta,t)d\eta - \int_0^1 \bar\psi_t(\eta,t)\tilde Q_t(\eta,t)d\eta \\
&+ \frac{k_2\theta}{2}\int_0^1 \tilde Q_\xi(\eta,t)\tilde Q_{\xi t}(\eta,t)d\eta.
\end{aligned}
$$

Simplifying and reordering (26) it yields

$$
\begin{aligned}
\dot V(t) &= -\frac{1}{2}k_2\|\tilde Q_t(\cdot,t)\|_2^2 + \theta\int_0^1 \tilde Q_t(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta \\
&+ \frac{k_2\theta}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta \\
&+ \frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\bar\psi_t(\eta,t)d\eta - \frac{k_2k_3}{2}\|\tilde Q(\cdot,t)\|_2^2 \\
&- \frac{k_2^2}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta - \frac{k_1k_2}{2}\|\tilde Q(\cdot,t)\|_2 \\
&- \int_0^1 \bar\psi_{tt}(\eta,t)\tilde Q(\eta,t)d\eta \\
&+ \frac{k_2\theta}{2}\int_0^1 \tilde Q_\xi(\eta,t)\tilde Q_{\xi t}(\eta,t)d\eta. \qquad (27)
\end{aligned}
$$

Now let us integrate by parts the next integral terms by taking into account the boundary conditions (24)

$$
\begin{aligned}
\theta\int_0^1 \tilde Q_t(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta &= \theta\tilde Q_t(1,t)\tilde Q_{t\xi}(1,t) \\
&-\theta\tilde Q_t(0,t)\tilde Q_{t\xi}(0,t) - \theta\|\tilde Q_{t\xi}(\cdot,t)\|_2^2 \\
&= -\theta\|\tilde Q_{t\xi}(\cdot,t)\|_2^2, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\frac{k_2\theta}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_{t\xi\xi}(\eta,t)d\eta &= \frac{k_2\theta}{2}\tilde Q(1,t)\tilde Q_{t\xi}(1,t) \\
&-\frac{k_2\theta}{2}\tilde Q(0,t)\tilde Q_{t\xi}(0,t)] \\
&-\frac{k_2\theta}{2}\int_0^1 \tilde Q_\xi(\eta,t)\tilde Q_{\xi t}(\eta,t)d\eta \\
&= -\frac{k_2\theta}{2}\int_0^1 \tilde Q_\xi(\eta,t)\tilde Q_{\xi t}(\eta,t)d\eta. \qquad (29)
\end{aligned}
$$

By taking into account (28) and (29) one can further simplify (27) as

$$
\begin{aligned}
\dot V(t) &= -k_2\left[\frac{k_3}{2}\|\tilde Q(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde Q_t(\cdot,t)\|_2^2\right. \\
&+ \left.\frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta\right] \\
&- \theta\|\tilde Q_{t\xi}(\cdot,t)\|_2^2 - \frac{k_1k_2}{2}\|\tilde Q(\cdot,t)\|_2 \qquad (30)\\
&+ \frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\bar\psi_t(\eta,t)d\eta - \int_0^1 \bar\psi_{tt}(\eta,t)\tilde Q(\eta,t)d\eta
\end{aligned}
$$

Now by the Cauchy Schwartz inequality and the Assumption 1 let us estimate the integral terms in the last line of (30) as

$$
\begin{aligned}
\left|\frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\bar\psi_t(\eta,t)d\eta\right| &\le \frac{k_2}{2}\|\tilde Q(\cdot,t)\|_2\|\bar\psi_t(\cdot,t)\|_2 \\
&\le \frac{k_2}{2}F_1\|\tilde Q(\cdot,t)\|_2, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_0^1 \bar\psi_{tt}(\eta,t)\tilde Q(\eta,t)d\eta\right| &\le \|\tilde Q(\cdot,t)\|_2\|\bar\psi_{tt}(\cdot,t)\|_2 \\
&\le F_2\|\tilde Q(\cdot,t)\|_2. \qquad (32)
\end{aligned}
$$

The Lyapunov functional derivative becomes

$$
\begin{aligned}
\dot V(t) &\le -k_2\left[\frac{k_3}{2}\|\tilde Q(\cdot,t)\|_2^2 + \frac{1}{2}\|\tilde Q_t(\cdot,t)\|_2^2\right. \\
&+ \left.\frac{k_2}{2}\int_0^1 \tilde Q(\eta,t)\tilde Q_t(\eta,t)d\eta\right] \\
&- \theta\|\tilde Q_{t\xi}(\cdot,t)\|_2^2 - \frac{k_2}{2}\left[k_1 - F_1 - \frac{2F_2}{k_2}\right]\|\tilde Q(\cdot,t)\|_2, \qquad (33)
\end{aligned}
$$

and, by exploiting (19), one finally receives that

$$
\begin{aligned}
\dot V(t) &\le -\frac{1}{2}k_2(k_3-k_2^2)\|\tilde Q(\cdot,t)\|_2^2 - \frac{3}{8}k_2\|\tilde Q_t(\cdot,t)\|_2^2 \\
&- \theta\|vQ_{t\xi}(\cdot,t)\|_2^2 - \frac{k_2}{2}\left[k_1 - F_1 - \frac{2F_2}{k_2}\right]\|\tilde Q(\cdot,t)\|_2. \qquad (34)
\end{aligned}
$$

To complete the proof it remains to demonstrate that

$$
\|\tilde Q(\cdot,t)\|_2 \to 0 \text{ as } t\to\infty. \qquad (35)
$$

Since, by (34), $\dot V(t) \le 0$, the domain $V(t) \le R_1$ is invariant for $R_1 \ge V(0)$ sufficiently large. From the inequality $V(t) \le R_1$ and taking into account the expression

(18) of the chosen Lyapunov functional, it can be readily concluded that

$$\|\tilde{Q}_t(\cdot,t)\|_2^2 \leq 2R_1. \tag{36}$$

Let us integrate the relation

$$\dot{V}(t) \quad \leq \quad -\frac{k_2}{2}\left[k_1 - F_1 - \frac{2F_2}{k_2}\right]\|\tilde{Q}(\cdot,t)\|_2, \tag{37}$$

straightforwardly resulting from (34), to conclude that

$$\int_0^{\infty} \|\tilde{Q}(\cdot,t)\|_2 dt < \infty. \tag{38}$$

which follows from the boundedness of $V(t)$. Since due to (36) the integrand $\eta(t) = \|\tilde{Q}(\cdot,t)\|_2$ of (38) possesses a uniformly bounded time derivative

$$\dot{\eta}(t) = \frac{\int_0^1 \tilde{Q}(\xi,t)\tilde{Q}_t(\xi,t)d\xi}{\|\tilde{Q}\|_2} \leq \|\tilde{Q}_t(\cdot,t)\|_2 \leq \sqrt{2R_1} \tag{39}$$

on the semi-infinite time interval $[0,\infty)$, convergence (17) is then verified by applying the Barbalat lemma [10]. Theorem 1 is proved. □

III. Simulation results

Consider the perturbed heat equation (2) with diffusivity $\theta = 1$ and homogeneous Neumann-type BCs as in (4). The initial conditions are set to

$$Q(\xi,0) = cos(2\pi\xi), \tag{40}$$

which meets the actual BCs. The system is supposed to be corrupted by the space- and time-varying perturbation

$$\psi(\xi,t) = 2sin(2\pi\xi)sin(2\pi t), \tag{41}$$

and the space and time-varying set-point

$$Q^r(\xi,t) = 10 + cos(\pi\xi)sin(\pi t) \tag{42}$$

is selected. Let us evaluate the L_2 norm bounds F_1 and F_2 of the derivatives $\bar{\psi}_t(\xi,t)$ and $\bar{\psi}_{tt}(\xi,t)$, defined in the Assumption 1, which are involved in the controller tuning inequalities (16). The next, slightly conservative, estimates are made

$$\begin{array}{l}\psi_t(\xi,t) = 4\pi sin(2\pi\xi)cos(2\pi t)\\ \quad\Rightarrow \quad \|\psi_t(\cdot,t)\|_2 = \frac{4\pi}{\sqrt{2}}\,|cos(2\pi t)| \leq 9\end{array} \tag{43}$$

$$\begin{array}{l}\theta Q^r_{\xi\xi t}(\xi,t) = -\pi^3 cos(\pi\xi)cos(\pi t)\\ \quad\Rightarrow \quad \theta\|Q^r_{\xi\xi t}(\cdot,t)\|_2 = \frac{\pi^3}{\sqrt{2}}\,|cos(\pi t)| \leq 22\end{array} \tag{44}$$

Therefore, according to (14), one has that

$$\|\bar{\psi}_t(\cdot,t)\|_2 \leq \|\psi_t(\cdot,t)\|_2 + \|Q^r_{\xi\xi t}(\cdot,t)\|_2 \leq F_1 \equiv 31 \tag{45}$$

By proceeding analogously with the second-order time derivative $\bar{\psi}_{tt}(\xi,t)$, one obtains that

$$\|\bar{\psi}_{tt}(\cdot,t)\|_2 \leq (8\pi^2 + \pi^4)/\sqrt{2} \leq F_2 \equiv 125 \tag{46}$$

Accordingly, the controller gains are set as follows

$$k_1 = 73, \quad k_2 = 6, \quad k_3 = 40 \tag{47}$$

Figure 1 depicts the spatiotemporal solution profile $Q(\xi,t)$, and Figure 2 shows the L_2 norm of the corresponding tracking error $\tilde{Q}(\xi,t) = Q(\xi,t) - Q^r(\xi,t)$, whose vanishing transient confirms the good performance of the proposed controller. Figure 3 finally shows the applied distributed control $u(\xi,t)$, which, as expected, is a continuous funciton of time and space.

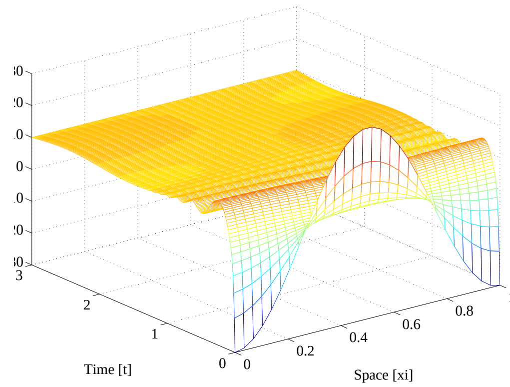

Fig. 1. The solution $Q(\xi,t)$.

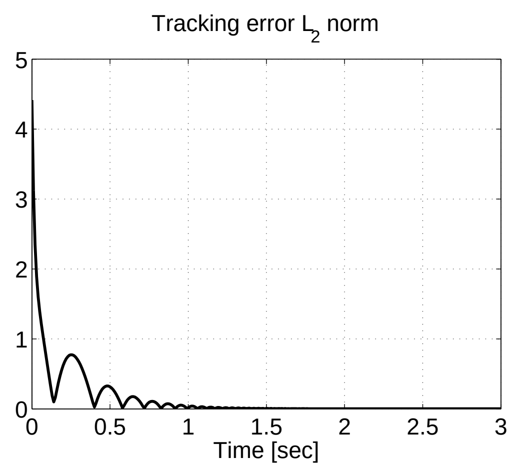

Fig. 2. L_2 norm of the tracking error $\tilde{Q}(\xi,t)$.

IV. Conclusions

A novel non smooth control algorithm has been used in conjunction with linear PI control in a distributed parameters setting involving a class of uncertain infinite dimensional processes. Particularly, the tracking control problem for a class of diffusive processes with uncertain parameters and homogeneous Neumann boundary conditions, subject to a persistent smooth disturbance of arbitrary shape, is tackled. By means of an ad-hoc Lyapunov functional analysis, which, rather unconventionally, explicitly considers the external perturbations in the Lyapunov functional definition, the stability in the L_2 space of the resulting error dynamics is demonstrated. The presented achievement improves some recent results by reducing the information demand and generalizing

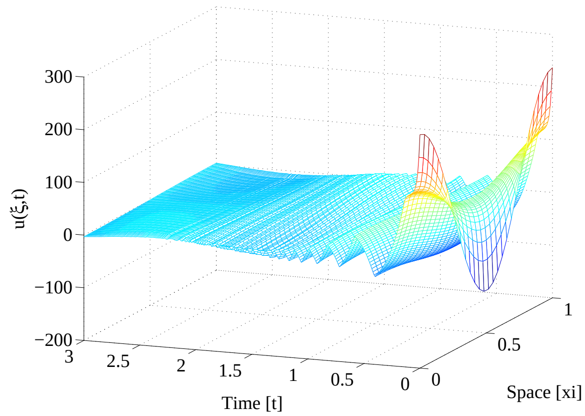

Fig. 3. The distributed input $u(\xi, t)$.

the class of admitted perturbations. Related boundary control problems for similar classes of plants will be addressed in the framework of future research, seeking for a profitable application of the considered type of Lyapunov functionals in the - usually more challenging - boundary control scenario.

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