Preprints of the 2012 20th Mediterranean Conference on Control & Automation (MED), Barcelona, Spain, July 3-6, 2012

Nonlinear Feedback Control of a Nutrient Removal Biological Plant

Hicham El bahja[1], Pastora Vega Cruz[1] and Othman Bakka[2]

Abstract— **The aim of this work is to observe and control a biological nitrogen removal process. The paper illustrates the use of a linearized model in the design, an observer (software sensor) and a nonlinear feedback control technique based on a phenomenological model of the process. This model describes the complete dynamics of autotrophic and heterotrophic biomasses, biodegradable organic and nitrogenous matters. The control approach structure is combined with the estimation algorithm, for the on-line reconstruction of unmeasured biological states of the bioprocess. The efficiency of both the control and estimation are demonstrated via computer simulations.**

I. INTRODUCTION

In this paper, the state estimation and control of a nutrient removal plant are considered. The main scope of the of the modelling is to describe the dynamics of the components concentration in the plant. Population dynamics for microorganisms are defined by different equations for each type of biomass (nitrifiers/autotrophics and denitrifiers/heterotrophics). The resulting system is strongly nonlinear. Additionally biological wastewater treatement plants are considered complex nonlinear systems due to large variations in their flow rates and feed concentrations. In addition, the microorganisms that are involved in the process and their adaptive behavior coupled with nonlinear dynamics of the system make the WWTP to be really challenging from the control point of view[1,2,3].

The state space representation is frequently used to form multivariable approach to linear control system synthesis and design. These control schemes are based on the assumption that the system state vector is available for feedback control purposes. In some applications, this assumption is not satisfied because it is either impossible or inappropriate, in practical situations, to measure all elements of system state. To retain many useful properties of the linear state feedback control, it is necessary to overcome the problem of the incomplete state information. The state observation problem is based on the construction of an auxiliary dynamical system, known as the state observer, driven by the inputs and outputs of the original system [4]. The reconstructed state vector is then substituted for the inaccessible one in the usual linear state feedback. Furthermore, and as pointed out above in many practical situations, linear systems are subjected to state and (or) input constraints. Such constraints are generally associated with physical limitations in process variables. The respect of these constraints can be accomplished by designing suitable feedback control laws. In many cases, this can be done by constructing positively invariant domains inside the set of the constraints, [5]. Other important applications were derived from this concept. In particular, one of them consists in using a large set of initial states while the constraints on the control vector are respected, [6].

The remainder of the paper is organized as follows. The modeling of the continuous wastewater treatment plants is detailed in Section 2. In fact, the modeling of the aerated basin, the anoxics basin and the settler are depicted. The control of the process is presented in section 3. The simulation results are then described in Section 4. Finally, section 5 ends the paper with concluding remarks.

II. PROCESS MODELING

In this study we consider six basic components present in the wastewater: autotrophic bacteria X_A, heterotrophic bacteria X_H, readily biodegradable carbonacceous substrates S_S, nitrogen substrates S_{NH}, S_{NO} and dissolved oxygen S_O, where X_A, X_H, S_S, S_{NH}, S_{NO}, and S_O represents the concentrations of these elements. In the formulation of the model, the following assumptions are considered: the physical properties of fluid are constant; there is no concentration gradient across the vessel; substrates and dissolved oxygen are considered as a rate-limiting with a bi-substrate Monod-type Kinetic and finally no bioreaction takes place in the settler which is perfect.

Based on the above description and assumptions, we can formulate the full set of ordinary differential equations (mass balance equations), making up the IWA AS Model NO.1 [6].

A. Modeling of the aerated basin

$$\begin{aligned}\dot{X}_{A,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(X_{A,denit}-X_{A,nit})\\ &+ (\mu_{A,nit}-b_A)\,X_{A,nit} \qquad (1)\end{aligned}$$

$$\begin{aligned}\dot{X}_{H,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(X_{H,denit}-X_{H,nit})\\ &+ (\mu_{H,nit}-b_H)\,X_{H,nit} \qquad (2)\end{aligned}$$

$$\begin{aligned}\dot{S}_{S,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(S_{S,denit}-S_{S,nit})\\ &+ (\mu_{H,nit}+\mu_{Ha,nit})\,X_{H,nit}/Y_H \qquad (3)\end{aligned}$$

$$\begin{aligned}\dot{S}_{NH,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(S_{NH,denit}-S_{NH,nit})\\ &+ (i_{xb}+1/Y_A)\,\mu_{A,nit}X_{A,nit}\\ &- (\mu_{H,nit}+\mu_{Ha,nit})\,i_{xb}X_{H,nit} \qquad (4)\end{aligned}$$

*This work was not supported by any organization

[1]H. El bahja and Pastora Vega Cruz are with Faculty of science, Automatic and Computer Science, University of Salamanca, Plaza de los Cados s/n, 37008, Salamanca Spain elbahja.hicham@gmail.com and pvega@usal.es

[2]O. Bakka is with the Faculty of science semlalia Marrakech, B.P. 2390, 40000 Marrakesh, Morocco bakka.othman@yahoo.fr

978-1-4673-2529-5/12/$31.00 ©2012 IEEE

$$\begin{aligned}\dot{S}_{NO,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(s_{NO,denit}-s_{NO,nit}) \\ &+ \mu_{A,nit}\frac{X_{A,nit}}{Y_A} \\ &- \frac{1-Y_H}{2.86Y_H}\mu_{Ha,nit}X_{H,nit} \qquad (5)\end{aligned}$$

$$\begin{aligned}\dot{S}_{O,nit}(t) &= (1+r_1+r_2)\,D_{nit}\,(S_{O,denit}-S_{O,nit}) \\ &+ a_0Q_{air}(C_S-S_{O,nit}) \\ &- \frac{4.57-Y_A}{Y_A}\mu_{A,nit}X_{A,nit} \\ &- \frac{1-Y_H}{Y_H}\mu_{Ha,nit}X_{H,nit} \qquad (6)\end{aligned}$$

Where:

$$\mu_{A,nit} = \mu_{max,A}\frac{S_{NH,nit}}{(K_{NH,A}+S_{NH,nit})}\frac{S_{O,nit}}{(K_{O,A}+S_{O,nit})}$$
$$\mu_{H,nit} = \mu_{max,H}\frac{S_{S,nit}}{(K_S+S_{S,nit})}\frac{S_{NH,nit}}{(K_{NH,H}+S_{NH,nit})}\times\frac{S_{O,nit}}{(K_{O,H}+S_{O,nit})}$$
$$\mu_{Ha,nit} = \mu_{max,H}\frac{S_{S,nit}}{(K_S+S_{S,nit})}\frac{S_{NH,nit}}{(K_{NH,H}+S_{NH,nit})}\times\frac{K_{O,H}}{(K_{O,H}+S_{O,nit})}\frac{S_{NO,nit}}{(K_{NO}+S_{NO,nit})}.\eta_{NO}$$

$\mu_{A,nit}$ and $\mu_{H,nit}$ are the growth rates of autotrophs and heterotrophs in aerobic conditions and $\mu_{Ha,nit}$ is the growth rate of heterotrophs in anoxic conditions.

B. Modeling of the anoxic basin

$$\begin{aligned}\dot{X}_{A,denit}(t) &= Ddenit(X_{A,in}+r_1X_{A,nit})+\alpha.r_2D_{denit} \\ &\times X_{rec}-(1+r_1+r_2)\,D_{denit}X_{A,denit} \\ &+ (\mu_{A,denit}-b_A)X_{A,denit} \qquad (7)\end{aligned}$$

$$\begin{aligned}\dot{X}_{H,denit}(t) &= D_{denit}(X_{H,in}+r_1X_{H,nit})-(1+r_1+r_2) \\ &\times D_{denit}X_{H,denit}+(1-\alpha)r_2D_{denit}X_{rec} \\ &+ (\mu_{H,denit}-b_H)X_{H,denit} \qquad (8)\end{aligned}$$

$$\begin{aligned}\dot{S}_{S,denit}(t) &= -(\mu_{H,denit+\mu_{Ha,denit}})\frac{X_{H,denit}}{Y_H}-(1+r_1+r_2) \\ &\times D_{denit}S_{S,denit}+D_{denit}(S_{S,in}-r_1S_{S,nit}) \qquad (9)\end{aligned}$$

$$\begin{aligned}\dot{S}_{NH,denit}(t) &= D_{denit}(S_{NH,in}+r_1S_{NH,nit})-(1+r_1+r_2) \\ &\times D_{denit}S_{NH,denit}-(i_{xb}+1/Y_A)\,\mu_{A,denit} \\ &\times X_{A,denit}-(\mu_{H,denit}+mu_{Ha,denit}) \\ &\times i_{xb}X_{H,denit} \qquad (10)\end{aligned}$$

$$\begin{aligned}\dot{S}_{NO,denit}(t) &= D_{denit}(S_{NO,in}+r_1S_{NO,nit})-(1+r_1+r_2) \\ &\times D_{denit}S_{NO,denit}+\frac{\mu_{A,denit}X_{A,denit}}{Y_A} \\ &- \frac{1-Y_H}{2.86Y_H}\mu_{Ha,denit}X_{H,denit} \qquad (11)\end{aligned}$$

Where:

$$\mu_{A,denit} = \mu_{max,A}\frac{S_{NH,denit}}{(K_{NH,A}+S_{NH,denit})}$$

$$\mu_{H,denit} = \mu_{max,H}\frac{S_{S,denit}}{(K_S+S_{S,denit})}\frac{S_{NH,denit}}{(K_{NH,H}+S_{NH,denit})}$$

$$\mu_{Ha,denit} = \mu_{max,H}\frac{S_{S,denit}}{(K_S+S_{S,denit})}\frac{S_{NH,denit}}{(K_{NH,H}+S_{NH,denit})}\times\frac{S_{NO,denit}}{(K_{NO}+S_{NO,denit})}\eta_{NO}$$

C. Modeling of the setller

D. Modeling of the setller

A mass balance on the setler leads to the following equations:

$$\dot{X}_{rec} = (1+r_2)D_{dec}(X_{A,nit}+X_{H,nit})-(r_2+w)D_{dec}X_{rec} \qquad (12)$$

where r_1,r_2 and w represent respectively, the ratio of the internal recycled flow Q_{r1} to influent flow Q_{in}, the ratio of the recycled flow Q_{r2} to the influent flow and the ratio of waste flow Q_w to influent flow, C_S is the maximum dissolved oxygen concentration. D_{nit}, D_{denit} and D_{dec} are the dilution rates in respectively, nitrification, denitrification, denitrification basins and settler tank; X_{rec} is the concentration of the recycled biomass. The other variables and parameters of the system equations (1)-(12) are also defined.

III. THE CONTROL PROBLEM

A. Linearization

Bioligical Wastewater treatment models can be generally represented as follows:

$$\begin{cases}\frac{dx}{dt}=f(u,x)\\ y(t)=g(u,x)\end{cases} \qquad (13)$$

Where x is a vector of variables reflecting the systems state, called state variables, u is a vector of input variables, and y a vector of outputs or measured variables. The linear model is formed by numerically evaluating the change in function values (f and g) resulting from small changes in model variables (x,u) at any particular operating point:

$$A=\frac{\partial f}{\partial x}\,|_{OP}\quad ; \qquad B=\frac{\partial f}{\partial u}\,|_{OP} \qquad (14)$$

$$C=\frac{\partial g}{\partial x}\,|_{OP}\quad ; \qquad D=\frac{\partial g}{\partial u}\,|_{OP} \qquad (15)$$

where all of the above partial derivatives are evaluated at the chosen operating point and will, therefore, change depending on the operating point. This results in the well known linear "state-space" model format:

$$\begin{cases}\frac{dx}{dt}=Ax+Bu\\ y(t)=Cx+Du\end{cases} \qquad (16)$$

For the Activated Sludge Model No.1 (ASM1) simplified trough linearization, the state, input and output vectors are depend by equations (17)-(19) :

$$\begin{aligned}x(t) = [&X_{A,nit}(t)\;\; X_{H,nit}(t)\;\; S_{S,nit}(t) \\ &S_{NH,nit}(t)\;\; S_{NO,nit}(t)\;\; S_{O,nit}(t) \\ &X_{A,denit}(t)\;\; X_{H,denit}(t)\;\; S_{S,denit}(t) \\ &S_{NH,denit}(t)\;\; S_{NO,denit}(t)\;\; X_{rec}(t) \\ ]^T & \qquad (17)\end{aligned}$$

$$y(t) \;=\; [S_{NH,nit}(t)\;\; S_{NO,nit}(t)\;\; S_{O,nit}(t)]^T \quad (18)$$

$$u(t) = [Q_{r1} \;\; Q_{r2} \;\; Q_{air}]^T \quad (19)$$

For the steady-state operating point:

$$\begin{aligned} x(t) \;=\; & [69.6 \;\; 623 \;\; 13.5 \;\; 3.2 \;\; 10.4 \;\; 2.4 \\ & 68.9 \; 624.6 \;\; 20.9 \;\; 8.9 \;\; 5.3 \;\; 1356.8]^T \end{aligned} \quad (20)$$

$$\bar{u} = [\bar{Q}_{r1} \;\; \bar{Q}_{r2} \;\; \bar{Q}_{air}]^T \quad (21)$$

With the following limitations

$$\begin{cases} -\bar{Q}_{r1} \leq Q_{r1} \leq 4\bar{Q}_{r1} \\ -\bar{Q}_{r2} \leq Q_{r2} \leq \bar{Q}_{r2} \\ -\bar{Q}_{air} \leq Q_{air} \leq 2\bar{Q}_{air} \end{cases} \quad (22)$$

B. Decomposition

In order to apply the concept of positive-invariance, the studied system must be controllable and observable. However, our system do not completely satisfy the two later conditions. For this reason, we used the decomposition process that allow us to extract only the controllable and observable part.

Any representation in the state space can be transformed into the equivalent form by using the transformation $Z = T_o x$ [8]:

$$\begin{cases} \dot{Z} = \bar{A}Z + \bar{B}u \\ y(t) = \bar{C}Z \end{cases} \quad (23)$$

with:

$$\bar{A} = \begin{pmatrix} A_{no} & A_{12} \\ 0 & A_o \end{pmatrix} ; \; \bar{B} = \begin{pmatrix} B_{no} \\ B_o \end{pmatrix}$$
$$\bar{C} = \begin{pmatrix} 0 & C_o \end{pmatrix} ; \quad Z = \begin{pmatrix} Z_{no} \\ Z_o \end{pmatrix}$$

where (A_o, C_o) is observable and the pair (A_o, B_o) is controllable.

So we obtain the following system of equations:

$$\begin{cases} \dot{Z}_{no} = A_{no}Z_{no} + A_{12}Z_o + B_{no}u \\ \dot{Z}_o = A_o Z_o + B_o u \\ y = C_o Z_o \end{cases} \quad (24)$$

C. Luenberger observer

Since the objective of the use of an observer is to reconstruct the unavailable states, the presence of p linear combinations of the state in the output suggests that the remaining $n-p$ linear combinations may be reconstructed by an observer of order no greater than $n-p$. Such an observer is called a minimal-order observer [7]. Thus, we wish to generate the remaining state combinations as follow:

$$z(.) = TZ_o(.) \quad (25)$$

where matrix T is chosen in such a way that the matrix $\begin{pmatrix} C_o \\ T \end{pmatrix}$ is invertible. Using this linear combination, with the matrix T of dimension $(n-p, n)$, the estimated state is obtained from:

$$\hat{Z}_o = \begin{pmatrix} C_o \\ T \end{pmatrix}^{-1} \begin{pmatrix} y(.) \\ z(.) \end{pmatrix} = \begin{pmatrix} V & P \end{pmatrix} \begin{pmatrix} y(.) \\ z(.) \end{pmatrix} \quad (26)$$

Furthermore the amount $TZ_o(.)$ can be measured which leads us to generate z(.) from a auxiliary dynamical system as follows:

$$\dot{z}(.) = Dz(.) + Ey(.) + Gu(.) \quad (27)$$

where z (.) is the state of the observer dynamics. Note here that the matrices V, C_o, T, P, satisfy

$$VC_o + PT = \mathbb{I}. \quad (28)$$

The control problem with constraints via an observer of minimal order may be stated in the following way

$$u(.) = sat(F\hat{Z}_o(.)) \quad (29)$$

We choose the state feedback F and matrices D, E and G so that the asymptotic stability and the constraints on the inputs are respected.

The observation error in this case is given by

$$\epsilon(.) = z(.) - TZ_o(.) \quad (30)$$

recall that the matrices of the observer of minimal order are given by [7]

$$D = TA_oP, \quad E = TA_oV, \quad G = TB_o \quad (31)$$

which is equivalent to write that matrices satisfy the following relation

$$TA_o - EC_o = DT \quad (32)$$

where the matrices T and P are chosen to ensure asymptotic stability of the matrix D, and cancel the observation error asymptotically, indeed: [7]

$$\begin{aligned} \dot{\epsilon}(.) &= \dot{z}(.) - T\dot{Z}_o \\ &= Dz(.) + Ey(.) + Gu(.) - T(A_oZ_o(.) + B_ou(.)) \\ &= Dz(.) - DTZ_o(.) \\ &= D\epsilon(.) \end{aligned}$$

For the observation error, we define the field $D(\mathbb{I}, \epsilon_{\gtrdot\partial\curvearrowleft}, \epsilon_{\gtrdot\sqsupset\ltimes})$ that give us the limits within which we allow change of the error $\epsilon(.)$. The reconstruction error is always given by

$$e(.) = \hat{Z}_o(.) - Z_o(.) \quad (33)$$

and is related to the observation error in the following way:

$$\begin{aligned} e(.) &= Vy(.) + Pz(.) - Z_o(.) \\ &= VC_oZ_o(.) + Pz(.) - (VC_o + PT)Z_o(.) \\ &= P\epsilon(.) \end{aligned}$$

lemma[2]. The field $D(\mathbb{I}, \approxeq_{\gtrdot\partial\curvearrowleft}, \approxeq_{\gtrdot\sqsupset\ltimes}) \times \mathbb{D}(\mathbb{I}, \epsilon_{\gtrdot\partial\curvearrowleft}, \epsilon_{\gtrdot\sqsupset\ltimes})$ is positively invariant with respect to the system trajectory $\begin{pmatrix} u(.) \\ \epsilon(.) \end{pmatrix}$ if and only if, there exists a matrix

$H \in R^{m\times m}$ such that:

$$\begin{cases} HF = FA_o + FB_oF \\ \widetilde{M}q_\epsilon \leq 0 \end{cases} \tag{34}$$

where

$$M = \begin{pmatrix} H & L_r \\ 0 & D \end{pmatrix};\ q_\epsilon = \begin{pmatrix} u_{max} \\ \epsilon_{max} \\ u_{min} \\ \epsilon_{min} \end{pmatrix};\ L_r = -FVC_oA_oP$$

for every pair:
$(u(0), \epsilon(0)) \in D(\mathbb{I}, \approxeq_{\gtrdot\partial\frown}, \approxeq_{\gtrdot\sqsupset\ltimes}) \times \mathbb{D}(\mathbb{I}, \epsilon_{\gtrdot\partial\frown}, \epsilon_{\gtrdot\sqsupset\ltimes})$
with

$$m_{ij}^+ = sup(m_{ij}, 0); \quad m_{ij}^- = sup(-m_{ij}, 0)$$

We start by writing the equation for the evolution of the control $u(t)$ always in the case of a linear behavior using previous relationship (28), (31), (32):

$$\begin{aligned} \dot{u} &= F\dot{\hat{Z}}_o \\ &= FP\dot{z}(.) + FVC_o\dot{Z}_o(.) \\ &= F(PT + VC_o)A_o\hat{Z}_o(.) + FB_oF\hat{Z}_o(.) - FVC_oA_oe(.) \\ &= HF\hat{Z}_o(.) - FVC_oA_oP\epsilon(.) \\ &= Hu(.) + L_r\epsilon(.) \end{aligned}$$

Therefore the system formed by the control $u(.)$ and the error $\epsilon(.)$,can be expressed as:

$$\begin{pmatrix} \dot{u}(.) \\ \dot{\epsilon}(.) \end{pmatrix} = \begin{pmatrix} H & L_r \\ 0 & D \end{pmatrix} \begin{pmatrix} u(.) \\ \epsilon(.) \end{pmatrix}$$

D. The nonlinear feedback control

The objective here is to designa nonlinear feedback control law for the system (16) with the constraints (22) that will cause the output to track a step input rapidly without experc-ing large overshoot respecting the constraints bellow. The following assumptions on the system matrices are requierde:
1)(A_o, B_o) is stabilizable.
2)(A_o, B_o, C_o) is invertible and has no zero at s=0.
In this section, we follow the idea of the work of [7] to develop a nonlinear feedback control technique for the case whe we have $(n-p)$ states of the plant (16) are measurable as montioned befor. We have the following stap-by-step design procedure.
Step 1: Design a linear feedback law

$$u_L = Fx + Gr \tag{35}$$

where r is a step command input and F is chosen such that $1)A_o + B_oF$ is an asymptotically stable matrix (see the section befor). Furthermore, G is a scalar and is given by

$$G = -[C_o(A_o + B_oF)^{-1}B_o]^{-1} \tag{36}$$

Here, we note that G is well defined because $A_o + B_oF$ is stable, and the triple (A_o, B_o, C_o) is invertible and has no invariant zeros at $s = 0$.
Step 2: Next, we compute

$$H := [1 - F(A_o + B_oF)^{-1}B_o]G \tag{37}$$

and

$$x_e := G_er := -(A_o + B_oF)^{-1}B_oGr \tag{38}$$

Given a positive-definite matrix $W \in \mathbb{R}^{\ltimes\times\ltimes}$, solve the following Lyapunov equation:

$$(A_o + B_oF)^TP + P(A_o + B_oF) = -W \tag{39}$$

for $P > 0$. Note that such a P exists since $A_o + B_oF$ is asymptotically stable. Then, the nonlinear feedback control law $u_N(t)$ is given by

$$u_N = \rho(r, y)B^TP(x - x_e) \tag{40}$$

where ρ is any nonpositive function locally Lipschitz in y,which is used to change the system cloosed-loop damping ratio as the output approaches the step command input.
Step 3. The linear and nonlinear feedback laws derived in the previous step are now combined to form a composite nonlinear feedback controller

$$\begin{aligned} u &= u_L + u_N \\ &= Fx + Gr + \rho(r, y)B^TP(x - x_e). \end{aligned} \tag{41}$$

The following theorem shows that the closed-loop system comprising the given plant in (16) and the nonlinear feedack control law in (41) is asymptotically stable. It also deter-mines the magnitude of r that can be tracked by such a control law without exceeding the control limit.
Theorm: Cosider the given sytem in (16), the linear control law of (35) and the nonlinear feedback control law of (41). For any $\alpha \in (0, 1)$, let $c_\alpha > 0$ be the largest positive scalar satisfying the following condition:

$$|Fx| \leq u_{max}(1 - \alpha)\forall x \in X_{alpha} := x : x^TPx \leq c_\alpha \tag{42}$$

Then the linear control law of (35) is capaple of driving the system controller output $y(t)$ to track asymptotically a step command input r, provided that the initial state x_0 and r satisfy

$$\tilde{x}_0 := (x_0 - x_e) \in X_{alpha}, |Hr| \leq u_{max}. \tag{43}$$

Furthermore, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y, the composite nonlinear feedback law in (41) is capable of diriving the system controller output $y(t)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy (43).
(Proof:) Let $\tilde{x} = x - x_e$. It is simple to verify that the linear control law of (35) can be rewritten as

$$\begin{aligned} u_L &= F\tilde{x}(t) + [1 - F(A_o + B_oF)^{-1}B_o]Gr \\ &= F\tilde{x}(t) + Hr. \end{aligned}$$

Hence, for all $\tilde{x} \in X_{alpha}$ and, provided that $|Hr| \leq u_{max}, |F\tilde{x} + Hr| \leq u_{max}$ and the closed-loop system is linear and it is given by

$$\dot{\tilde{x}} = (A_o + B_oF)\tilde{x} + A_ox_e + B_oHr. \tag{44}$$

Noting that

$$\begin{aligned} A_o x_e + B_o H r &= B_o[1 - F(A_o + B_o F)^{-1} B_o] G r \\ &\quad - A_o (A_o + B_o F)^{-1} B_o G r \\ &= [I - B_o F (A_o + B_o F)^{-1} B_o] G r \\ &\quad - A_o (A_o + B_o F)^{-1} B_o G r \\ &= 0. \end{aligned}$$

the closed-loop system in (44) can then be simplified as

$$\dot{\tilde{x}} = (A_o + B_o F)\tilde{x} \tag{45}$$

Similary, the clossed-loop system comprising the given plant in (16) and the nonlinear feedback control (41) can be expressed as

$$\dot{\tilde{x}} = (A_o + B_o F)\tilde{x} + B_o w \tag{46}$$

where

$$w = sat(F\tilde{x} + Hr + u_N) - F\tilde{x} - Hr. \tag{47}$$

Clearly, for the given $x - 0$ satisfaying (43), we have $\tilde{x}_0 = (x_0 - x_e) \in X_\alpha$. We not that (46) is reduced to (45) if $\rho = 0$. Thus, we can prove the results, respectively, under the linear control and the non linear feedback control in one shot.
Next, we define a Lyapunov function $V = \tilde{x}^T P \tilde{x}$, and evaluate the derivation of V along the trajectories of the closed-loop system in (46), i.e.,

$$\begin{aligned} \dot{V} &= \dot{\tilde{x}}^T P \tilde{x}^T P \dot{\tilde{x}} + 2\tilde{x}^T P B_o w \\ &= \tilde{x}^T (A_o + B_o F)^T P \tilde{x} + \tilde{x}^T P (A_o + B_o F)\tilde{x} \\ &\quad + 2\tilde{x}^T P B_o w \\ &= -\tilde{x}^T W \tilde{x} + 2\tilde{x}^T P B_o w \end{aligned} \tag{48}$$

Note that for all

$$\tilde{x} \in X_\alpha = \tilde{x} : \tilde{x}^T P \tilde{x} \leq c_\alpha \Rightarrow |F\tilde{x}| \leq u_{max}(1 - \alpha) \tag{49}$$

We next study the $\dot{V}$ for the different case of the constraints on the input.
Case 1) If $|F\tilde{x} + Hr + u_N| \leq u_{max}$, then $w = u_N =^T P\tilde{x}$ thus

$$\dot{V} = -\tilde{x}^T W \tilde{x} + 2\rho \tilde{x}^T P B_o B_o^T P \tilde{x} \leq -\tilde{x}^T W \tilde{x} \tag{50}$$

Case 2) If $F\tilde{x} + Hr + u_N \succ u_{max}$, and by construction $|F\tilde{x} + Hr| \leq u_{max}$, we have

$$0 \prec w = u_{max} - F\tilde{x} - Hr \prec u_N = \rho B_o^T P \tilde{x} \tag{51}$$

wich implies that $\tilde{x}^T P B_o \prec 0$ and hence $\dot{V} = -\tilde{x}^T W \tilde{x} + 2\tilde{x}^T P B_o w \leq -\tilde{x}^T W \tilde{x}$.
Case 3) Finally, if $F\tilde{x} + Hr + u_N \leq -u_{min}$, we have

$${}_o^T P \tilde{x} = u_n \prec -u_{min} - F\tilde{x} - Hr \prec 0 \tag{52}$$

implying $\tilde{x}^T P B_o \succ 0$ and hence $\dot{V} \leq -\tilde{x}^T W \tilde{x}$
In conclusion, we have shown that

$$\dot{V} \leq -\tilde{x}^T W \tilde{x} \quad \tilde{x} \in X_{alpha} \tag{53}$$

which implies that X_{alpha} is an invariant set of the closed-loop system in (46). This, in turn, indicates that, for all initial states x_0 and the step command input of amplitude r that satisfy (43)

$$lim_{t\to\infty} x(t) = x_e \Rightarrow lim_{t\to\infty} y(t) = r \tag{54}$$

This completes the proof.

IV. SIMULATION RESULTS

Therafter decomposition, the obtained system represented by (24), A_o, B_o, C_o are injected in closed loop and coupled with a minimal order Luenberger observer which has the role of estimating the non-measurable states from the mesurable ones $(S_{NH,nit}, S_{NO,nit}, S_{O,nit})$. We assume that in our case the control constraints are such as

$$u_{max} = \begin{pmatrix} 9200 \\ 18446 \\ 200 \end{pmatrix}, \quad u_{min} = \begin{pmatrix} 2300 \\ 18446 \\ 100 \end{pmatrix}$$

and reconstruction errors limits are such as:

$$\epsilon_{max} = \begin{pmatrix} 1 & 1 & 0.5 & 1 & 1 & 1 \end{pmatrix}; \ \epsilon_{min} = \begin{pmatrix} 0.5 & 0.5 & 0.25 & 0.8250 & 0.5 & 0.5 \end{pmatrix}$$

we choose the matrix H assigning spectrum $\{-170; -55; -51\}$ as follows:

$$H = \begin{pmatrix} -170 & 0 & 0 \\ 0 & -55 & 0 \\ 0 & 0 & -51 \end{pmatrix}$$

Simulation results are given in figure 2. The evolution of the perturbationS_{in} is presented in the figure 1. The output variables evolution, that are $S_{NH,nit}$, $S_{NO,nit}$ and the dissolved oxygen concentrations, and their corresponding reference trajectones are 3.2, 10.4 and 2.4 , respectively.

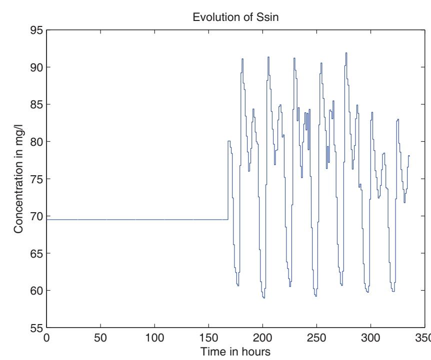

Fig. 1. Evolution of the perturbation Sin.

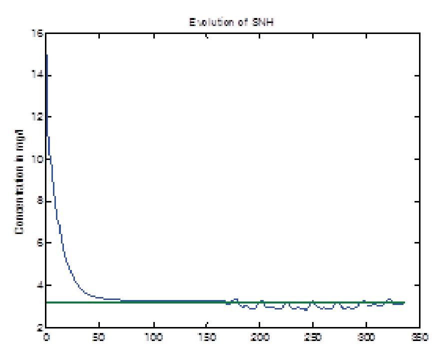

Fig. 2. Evolution of SNH.

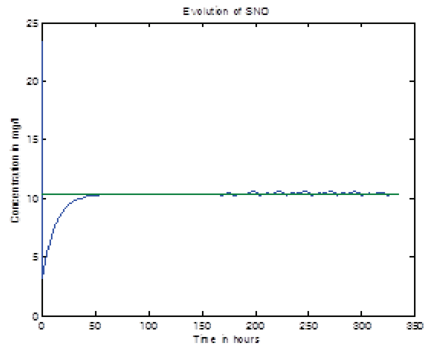

Fig. 3. Evolution of SNO.

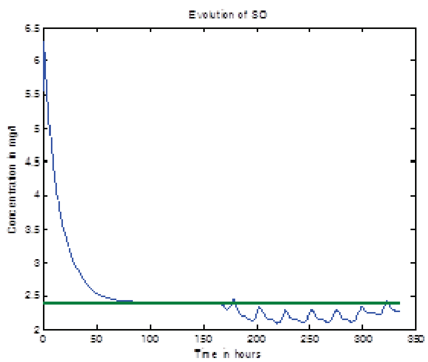

Fig. 4. Evolution of SO.

The figures 2, 3 and 4 shows the performance and the effectiveness of the regulator. In particular, one can appreciate the ability of the controller to track the desired values of the controlled variables.

V. CONCLUSION

In this paper, we introduced the nonlinear feedback control of a non linear system with input constraints. In fact, positive invariance techniques together with minimal order observer (software sensor) are used to control the linearized model of a WWTP. For this process, modeled as a linear process, some state variables are unavailable to measure and more than that no adequate sensor exists. Hence, the introduction of the observer is of great interest. Further, in general case, linearizing a non linear process leads the variables (control in our case) to be limited within neighborhood of the steady point functioning. The positive invariance techniques that had emerged as very efficient to handle similar problems of constrained control is successfully used to control the nitrogen removal process. The observer based constrained control, as presented above may compete with approaches in easiness, applicability and computing effort.

ACKNOWLEDGMENT

The authors gratefully acknowledge the support of the Spanish Government through the MINCINN project $DPI2009-14410-C02-01$.

REFERENCES

[1] H.El bahja, Othman Bakka, Pastora Isabel Vega Cruz,Fouad Mesquine, Non Linear GPC Of a Nuttrient Romoval Biological Plant. Proceeding of ETFA09, Mallorca, Spain.

[2] Fouad Mesquine, Othman Bakka, Hicham El bahja, Pastora Isabel Vega Cruz, Observer based regulator problem for WWTP with constraints on control. Proceeding of ETFA10, Bilbao, Spain.

[3] D. Dochain, Design of adaptive controllers for nonlinear stirred tank bioreactors: extension to the MIMO situation, J. Proc. Cont. 1 (1991) 41-48.

[4] F.Nejjari, Modlisation, Estimation et commande d'un bioprocd de traitement des eaux uses, Thesis Report, Faculty of Sciences, Marrakesh, Morocco, June 1997.

[5] Henze, M., Leslie Grady JR., C.P., Gujerm, W., Maraism, G.V.R. and Matsnom, T.,"Activated sludge Model No.1", I.A.W.Q., scientific and technical Report No.1, 1987.

[6] F. Nejjari, A. Benhammou, B. Dahhou, G. Roux, Nonlinear multivariable control of a biological wastewater treatment process, in: Proceedings of ECC 97, Brussels, Belgium, 1-4 July 1997.

[7] J.Oreilly. Observer for linear systems. Academic-press edition, 1983. Contributed Works

[8] Fatiha Nejjari and Joseba Quevedo, predictive control of a nutrient removal biological plant. Proceeding of the 2004 Americain Conferance Boston.

[9] Z.Lin, M. Pachter, and S. Banda, Toward improvement of tracking performance-nonlinear feedback for linear systems, Int. J. Control, vol. 70, pp.1-11, 1998.