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Identifying poles from time–domain data using discrete Laguerre system

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Abstract— **In a recent paper the authors proposed a new frequency–domain approach to identify poles in discrete–time linear systems. The discrete rational transfer function is represented in a rational Laguerre–basis, where the basis elements are expressed by powers of the Blaschke–function. This function can be interpreted as a congruence transform on the Poincaré unit disc model of the hyperbolic geometry. The identification of a pole is given as a hyperbolic transform of the limit of a quotient–sequence formed from the Laguerre–Fourier coefficients. This paper extends this approach for using discrete time–domain data directly.**

I. INTRODUCTION

System identification based on frequency–domain interpretation of discrete–time signals plays significant role in the control theory and design [1]. Estimating the poles associated with input and output signals as well as the transfer functions is an efficient approach of identifying the system dynamics; knowing – exactly or approximately — the location of system poles is sufficient in estimating the whole system dynamics by applying the principles of representations in orthogonal rational bases (GOBs) [2]. However, applying rational orthogonal bases are not sufficient in the derivation of the poles themselves, if no *a priori* knowledge is available on the number and the locations of the poles [3]. Although many efforts were spent in finding adequate initial layout of poles [4] as well as in refining the pole locations [5], the complete identification problem has not been solved. Similar statements are also valid for the classical non–parametric (FFT-based) and parametric (AR and ARMA model–based) spectral identification methods, including the Prony–method [6] of estimating harmonics in signals that sets up essentially the pole–identification problem as its goal. In a recent paper [7] a new method has been proposed that can efficiently be used to identify the poles in a linear system from frequency domain data. The discrete rational transfer function is represented in a rational Laguerre–basis, where the basis elements can be expressed by powers of the Blaschke–function. This function can be interpreted as a congruence transform on the Poincaré unit disc model of the hyperbolic geometry, leading to a nice geometric interpretation of the identification algorithm. The reconstruction of a pole is given as a hyperbolic transform of the limit of a sequence formed of quotients of the Laguerre–Fourier coefficients belonging to the function. The Laguerre-Fourier coefficients can be estimated from frequency domain data by using an efficient FFT-based algorithm.

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The pole–identification algorithm introduced in [7] uses frequency–domain data that can either be direct frequency–domain measurements or spectral estimations based on discrete time–domain samples.

The proposed pole–identification method requires non–uniformly spaces frequency sample points. This allows to use standard FFT algorithm in the estimation of the Laguerre–Fourier coefficients (see [7] for details). The poles are identified from multiple Laguerre–series coefficients corresponding to different Laguerre-parameters, thus multiple measurement sequences are required on non–uniformly spaced frequency scales.

This paper extends the pole–identification method in such a way that discrete time–domain data can directly be used, avoiding the disadvantages arising from the need of using non–uniform Fourier–transform to obtain frequency–domain data. It will be shown that a numerically efficient method is obtained that possesses certain advantages over the algorithm previously reported.

In Section II an outline will be given of the pole–identification method presented in the previous works of the authors. Section III is devoted to the derivation of the method based on the direct use of time–domain data. Finally an example — based on a simulated signal — is presented.

II. OUTLINE OF THE POLE–RECONSTRUCTION METHOD

A key concept in the $\mathrm{H}^2(\mathbb{D})$ system and signal representations is the Blaschke function depending on a parameter $b \in \mathbb{D}$, which can be considered as an *inverse pole* ($b = 1/\overline{p}$) of the function. The Blaschke function is defined as

$$B_{\mathfrak{b}}(z) := \epsilon \frac{z-b}{1-\overline{b}z} \quad (z \in \mathbb{C}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}), \quad (1)$$

where $\mathbb{D}$ and $\mathbb{T}$ denotes the open unit disc and the unit circle, respectively. If $\mathfrak{b} \in \mathbb{B}$, then $B_{\mathfrak{b}}$ is an $1-1$ map on $\mathbb{T}$ and $\mathbb{D}$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or $\mathbb{T}$ with the operation $(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z))$

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form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way $B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$. The group $(\mathbb{B}, \circ)$ will be isomorphic with the group $(\{B_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B}\}, \circ)$. The neutral element of the group $(\mathbb{B}, \circ)$ is $\mathfrak{e} := (0, 1) \in \mathbb{B}$ and the inverse element of $\mathfrak{b} = (b, \epsilon) \in \mathbb{B}$ is $\mathfrak{b}^{-1} = (-b\epsilon, \overline{\epsilon})$.

It can be proved that the map

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - \overline{z}_1\, z_2|} = |B_{z_1}(z_2)| \tag{2}$$
$$(B_{z_1} := B_{(z_1,1)}, z_1, z_2 \in \mathbb{D})$$

is a metric on $\mathbb{D}$, called pseudohyperbolic metric (see [8]). Moreover the Blaschke functions B_b $(b \in \mathbb{D})$ are isometries with respect to this metric, i.e.

$$\rho(B_b(z_1), B_b(z_2)) = \rho(z_1, z_2) \quad (b \in \mathbb{D}, z_1, z_2 \in \mathbb{D}). \tag{3}$$

The discrete Laguerre–functions are defined by

$$L_n^b(z) := \frac{\sqrt{1 - |b|^2}}{1 - \overline{b}z} \left(\frac{z - b}{1 - \overline{b}z} \right)^n \tag{4}$$
$$(z \in \overline{\mathbb{D}}, b \in \mathbb{D}, n \in \mathbb{N}).$$

Using the function

$$F_{\mathfrak{b}}(z) := \frac{\sqrt{\epsilon(1 - |b|^2)}}{1 - \overline{b}z} \tag{5}$$
$$(\mathfrak{b} := (b, \epsilon) \in \mathbb{B}, b \in \mathbb{D}, \epsilon \in \mathbb{T}, z \in \overline{\mathbb{D}}),$$

and the Blaschke maps according to (1), the discrete Laguerre–functions can be expressed in the form

$$L_n^b = F_{\mathfrak{b}} B_{\mathfrak{b}}^n \quad (\mathfrak{b} = (b, 1) \in \mathbb{B}, n \in \mathbb{N}). \tag{6}$$

Denote $H = \mathrm{H}^2(\mathbb{T})$ the Hardy space with the usual scalar product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{\mathrm{i}\,t}) \overline{g}(e^{\mathrm{i}\,t})\, dt \quad (f, g \in \mathrm{H}^2(\mathbb{T})). \tag{7}$$

We introduce the collection of operators $U_{\mathfrak{b}} : \mathrm{H}^2(\mathbb{T}) \to \mathrm{H}^2(\mathbb{T})$ $(\mathfrak{b} \in \mathbb{B})$ defined by

$$U_{\mathfrak{b}} f := F_{\mathfrak{b}^{-1}}\, f \circ B_{\mathfrak{b}^{-1}} \quad (f \in \mathrm{H}^2(\mathbb{T}), \mathfrak{b} \in \mathbb{B}). \tag{8}$$

It is known that $U_{\mathfrak{b}}$, $(\mathfrak{b} \in \mathbb{B})$ is a unitary representation of the Blaschke group $\mathbb{B}$ [9], i.e.

(i) $U_{\mathfrak{b}_1}(U_{\mathfrak{b}_2} f)) = U_{\mathfrak{b}_1 \circ \mathfrak{b}_2} f$,
(ii) $\langle U_{\mathfrak{b}} f, U_{\mathfrak{b}} g \rangle = \langle f, g \rangle$ $(f, g \in \mathrm{H}^2(\mathbb{T}))$.

The discrete Laguerre–functions L_n^b can be introduced as image of the power function $h_n(z) := z^n$ by the representation $U_{\mathfrak{b}}$:

$$L_n^b := U_{\mathfrak{b}}^{-1} h_n \quad (n \in \mathbb{N}, \mathfrak{b} = (b, 1) \in \mathbb{T}). \tag{9}$$

Since $U_{\mathfrak{b}}$ is unitary, $U_{\mathfrak{b}}^* = U_{\mathfrak{b}}^{-1} = U_{\mathfrak{b}^{-1}}$ and consequently for any $m, n \in \mathbb{N}$

(i) $\langle L_n^b, L_m^b \rangle = \langle U_{\mathfrak{b}^{-1}} h_n, U_{\mathfrak{b}^{-1}} h_m \rangle = \langle h_n, h_m \rangle = \delta_{mn}$
(ii) $\langle f, L_n^b \rangle = \langle f, U_{\mathfrak{b}^{-1}} h_n \rangle = \langle U_{\mathfrak{b}} f, h_n \rangle$.

Thus the discrete Laguerre–Fourier coefficients of f are equal to the trigonometric Fourier coefficients of the function $U_{\mathfrak{b}} f$. This relation can be used to compute the discrete Laguerre–Fourier coefficients.

The representation of any function $f \in \mathrm{H}^2(\mathbb{D})$ in the Laguerre–system can be expressed as

$$f(z) = \sum_{n=0}^{\infty} l_n L_n^b(z), \tag{10}$$

where l_n coefficients – i.e. the so called Laguerre–Fourier coefficients belonging to function f – can be computed by the scalar products $\langle f, L_n^b \rangle$ $(n \in \mathbb{N})$.

Let $\mathfrak{R}$ denote the set of rational functions with poles falling outside the closed unit disc. It is obvious that functions

$$r_{n,a}(z) := \frac{z^n}{(1 - \overline{a}z)^{n+1}} \quad (a \in \mathbb{D}, z \in \overline{\mathbb{D}}, n \in \mathbb{N}) \tag{11}$$

belong to $\mathfrak{R}$. $a^* := 1/\overline{a}$ is a pole of multiplicity $(n + 1)$ of the function $r_{n,a}$, which is the inverse map of a with respect to the unit circle. a will be referred as the "inverse pole" of the function $r_{n,a}$ in the subsequent part of the paper. It is well known that the functions of form (11) generate the function class $\mathfrak{R}$, i.e. any $f \in \mathfrak{R}$ can be expressed in the form

$$f := \sum_{k=1}^{P} \sum_{i=0}^{m_k - 1} A_{ki} r_{i,a_k}, \tag{12}$$

where $a_k \in \mathbb{D}$ $(k = 1, \dots, P)$ denote the inverse poles of the function f with their multiplicity m_k.

The following lemma will be used in computing the Laguerre–Fourier coefficients of f.

Lemma 1: For every function $g \in \mathfrak{R}$

$$\langle g, r_{n,a} \rangle = \frac{g^{(n)}(a)}{n!} \quad (n \in \mathbb{N}, a \in \mathbb{D}). \tag{13}$$

The proof of the lemma can be found in [7].

In the case if $m_k = 1$ the associated term will be r_{0,a_k}, and the conjugate of the Laguerre–Fourier coefficients belonging to it are directly given by (13) as

$$\langle L_n^b, r_{0,a_k} \rangle = L_n^b(a_k), \tag{14}$$

that is equal to coefficient l_n. To indicate that l_n coefficients belong to the parameter b used in the Laguerre representation, let us denote them as l_n^b.

Suppose that the system under consideration contains only a single pole of multiplicity 1, in this case the conjugated Laguerre–Fourier coefficients are given as $c_n^b = L_n^b(a)$, and the quotients

$$q_n(b) = \frac{l_{n+1}^b}{l_n^b} = B_b(a) \quad (n \in \mathbb{N}), \tag{15}$$

form a constant sequence and its elements equal to a Blaschke function applied to a. This fact can be used to identify the position of inverse pole a,

$$a = B_{b^{-1}} \left(\frac{l_{n+1}^b}{l_n^b} \right), \tag{16}$$

where $B_{b^{-1}}$ is the inverse of B_b, i.e. a is given by applying a hyperbolic transform corresponding to the inverse group element belonging to b.

This concept can be extended to multiple poles, it will be shown that in the case of multiple poles there exist a region of $\mathbb{D}$ where the sequence of the quotients generated by the conjugated Laguerre–Fourier coefficients converge.

Let the inverse poles $a_1, a_2 \ldots, a_P \in \mathbb{D}$ of function f be fixed. Applying the hyperbolic distance as defined by (2) let us introduce subsets of $\mathbb{D}$ as follows:

$$\begin{aligned} D_i &:= \{b \in \mathbb{D} : \rho(b, a_i) > \max_{1 \le j \le P, i \ne j} \rho(b, a_j)\}, \\ D &:= \bigcup_{j=1}^{P} D_i \quad (i = 1, 2, \ldots, P). \end{aligned} \tag{17}$$

Concerning these sets a rather informative interpretation can be given: the set

$$\mathcal{L}_{ij} := \{b \in \mathbb{D} : \rho(a_i, b) = \rho(a_j, b)\} \tag{18}$$

can be considered as the hyperbolic perpendicular bisector of the points a_i, a_j that divides $\mathbb{D}$ in two hyperbolic half-planes. Let the following notations be introduced:

$$\begin{aligned} D_{ij} &:= \{b \in \mathbb{D} : \rho(a_i, b) > \rho(a_j, b)\} \\ D_{ji} &:= \{b \in \mathbb{D} : \rho(a_i, b) < \rho(a_j, b)\}. \end{aligned} \tag{19}$$

The sets D_i can be generated as an intersection of the hyperbolic half-planes, i.e. according to the definitions in (17) and (19)

$$D_i = \bigcap_{k=1, k \ne i}^{P} D_{ik} \quad (i = 1, 2, \cdots, P). \tag{20}$$

As a consequence the sets D_i are hyperbolically convex regions, i.e. any hyperbolic line segment connecting two points belonging to any D_i is located as a whole in the same region.

It will be shown that in any point of D the limit

$$(\mathcal{Q}f)(b) := \lim_{n \to \infty} \frac{l_{n+1}^b}{l_n^b} \quad (f \in \mathfrak{R}) \tag{21}$$

does exist, and it can be used to reconstruct the poles of function f. It should be mentioned that operator $\mathcal{Q}$ defined on domain $\mathfrak{R}$ is nonlinear.

Theorem 1: For any rational function f of form (12) in any point b of D the limit (21) exists, and

$$(\mathcal{Q}f)(b) = B_b(a_i), \quad b \in D_i \quad (i = 1, 2, \ldots, P). \tag{22}$$

In the case of poles of multiplicity 1 for the speed of convergence the estimation

$$\left| \frac{l_{n+1}^b}{l_n^b} - B_b(a_i) \right| = O(q_i^n) \ (n \in \mathbb{N}, b \in D_i, q_i < 1)$$

can be given.

A simplified proof for the case of pole multiplicities equal to 1, and the general proof can be found in [7] and [10], respectively.

According to Theorem 1

$$B_b^{-1}((\mathcal{Q}f)(b)) = a_i \quad (b \in D_i, i = 1, 2, \cdots, P) \tag{23}$$

which can be used to reconstruct all the poles with region $D_i \ne \varnothing$ belonging to them.

On the basis of Theorem 1 and its corollary (23) a practically realizable method can be constructed for the reconstruction of system poles by using frequency-domain signal measurements. Two basic steps to be realized are as follow:

1) Estimation of Laguerre–Fourier coefficients.
2) Reconstruction the poles from the Laguerre–Fourier coefficients.

It is assumed that the frequency points where the measurements are to be performed can be assigned arbitrarily. The concept of estimating the Laguerre–Fourier coefficients is based on the unitary representation of the Blaschke group, that implies that the Laguerre–Fourier coefficients of a function f are equal to the Fourier coefficients of the function $U_b f$, see (8). This means that the coefficients can be computed by the evaluation of Fourier-integrals, in discrete case by Fast Fourier Transform (FFT). This results in a non-uniformly spaced sampling scheme in the frequency scale. This non-uniform scale depends on the b parameter of the Laguerre–system and can be constructed from the inverse of the argument-function associated with the Blaschke function B_b. Let the argument function of B_b be denoted by $\beta_b(t)$ and denote the value of β_b by s. The inverse function $t = \beta_b^{-1}(s)$ can be expressed in the form

$$t = \varphi + 2 \arctan\Big(\frac{1-r}{1+r} \tan \frac{s - \gamma}{2}\Big) \tag{24}$$

where $b = re^{\mathrm{i}\,\phi}$ and γ is a parameter chosen such a way that

$$\beta_b : [-\pi, \pi] \to [-\pi, \pi].$$

All the conditions of realizing the algorithm can be realized, see for details in [11]. The estimation of the Laguerre–Fourier coefficients can be performed by executing the following steps:

1) Derive a nonuniform sampling scheme associated to parameter b in the frequency scale and obtain N frequency measurement points.
2) Compute the values of the unitary representation U_b in the nonuniform scale.
3) Compute the Laguerre–Fourier coefficients by applying FFT on the function points U_b.

The algorithmic issues of estimating the pole location on the basis of the Laguerre–Fourier coefficients can be found in [7].

III. USING TIME–DOMAIN DATA

For the purpose to use time–domain data consider the expression of deriving Laguerre-Fourier coefficients associated with parameter $b \in \mathbb{D}$ and of the function $f \in \mathrm{H}^2(\mathbb{D})$ expressed in the form of a complex contour integral:

$$l_n^b = \frac{\sqrt{1 - |b|^2}}{2\pi\,\mathrm{i}} \oint_{\mathbb{T}} f(z) \frac{(1 - \overline{b}z)^n}{(z - b)^{n+1}} \, dz \tag{25}$$

for indices $n \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the nonnegative integers. Apply the substitution corresponding to a transform with the inverse Blaschke group element

$$z = \frac{w+b}{1+\bar{b}w} \tag{26}$$

To perform the substitution computing the derivative of (26) with respect to variable w is needed:

$$\frac{dz}{dw} = \frac{(1+\bar{b}w) - \bar{b}(w+b)}{(1+\bar{b}w)^2} = \frac{1-|b|^2}{(1+\bar{b}w)^2}.$$

With the substitution the expression of the Laguerre–Fourier coefficient of index n is given as

$$l_n^b = \frac{\sqrt{1-|b|^2}}{2\pi\,\mathrm{i}} \oint_{\mathbb{T}} \frac{f\left(\frac{w+b}{1+\bar{b}w}\right)}{1+\bar{b}w} w^{-(n+1)}\,dw. \tag{27}$$

Let f be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} x_k z^k. \tag{28}$$

where $\{x_k\}$ are uniformly spaces time–domain samples of the signal belonging to the function f. If $f \in \mathrm{H}^2(\mathbb{D})$, $\{x_k\}$ belongs to the sequence–space ℓ^2. The expression (28) is the well–known z–transform of the discrete–time function x. In practical cases $\{x_k\}$ contains finite elements, i.e. finite sample record of the physical signals is used, let the number of the data points $K \in \mathbb{N}$ in the forthcoming discussion.

By substituting the expression of the inverse Blaschke–transform (26)

$$f\left(\frac{w+b}{1+\bar{b}w}\right) = \sum_{k=0}^{K-1} x_k \left(\frac{w+b}{1+\bar{b}w}\right)^k$$

is given. Applying this form in the expression of the Laguerre-Fourier coefficient (27), as well as changing the order of summing and integration and bringing the constants x_k outside of the integral, the form

$$l_n^b = \sum_{k=0}^{K-1} x_k \frac{\sqrt{1-|b|^2}}{2\pi i} \oint_{\mathbb{T}} \frac{(w+\,b)^k}{(1+\bar{b}w)^{k+1}} w^{-(n+1)}\,dw \tag{29}$$

is obtained.

Equation (29) has fundamental importance since the coefficients belonging to to the time–domain samples are independent of the samples themselves; this fact implies that the coefficients can be computed in advance, separately of the measurements.

Introduce the notation

$$\lambda_{n,k}^b = \frac{\sqrt{1-|b|^2}}{2\pi i} \oint_{\mathbb{T}} \frac{(w+\,b)^k}{(1+\bar{b}w)^{k+1}} w^{-(n+1)}\,dw \tag{30}$$

for the coefficients, the Laguerre-Fourier coefficients belonging to the parameter b can be computed as

$$l_n^b = \sum_{k=0}^{K-1} x_k \lambda_{k,n}^b.$$

This form can be considered as a discrete time-domain Laguerre–Fourier transform.

Since in practical cases finite number of the Laguerre-Fourier coefficients can be handled, let $N \in \mathbb{N}$ be this number, the coefficients $\lambda_{n,k}^b$ can be arranged in a matrix $\mathbf{\Lambda}_{N\times K}^b$. By fixing K and N, i.e. the number of the Laguerre-Fourier coefficients, and the number of time–domain data samples, $\mathbf{\Lambda}_{N\times K}^b$ matrices should be generated for every parameter value $b \in \mathbb{D}$, and in this case the finite set of the Laguerre–Fourier coefficients can be computed as

$$\mathbf{l}^b = \mathbf{\Lambda}^b \mathbf{x}$$

where $\mathbf{l}^b$ and $\mathbf{x}$ denotes the column–vectors of the Laguerre–Fourier coefficients and the time–domain data samples, respectively.

The number of the instructions needed to compute a finite set of Laguerre–Fourier coefficients can be simply be estimated. By considering N number of the coefficients and K number of time–domain data N number of convolution–type procedures of size K should be performed, hence $N \cdot K$ instructions – consisting of one real with complex multiplication and one complex addition – are needed. The original algorithm consists of two steps, computing a Fourier-transform of K real data in N nonuniformly spaced points, furthermore computing the Laguerre–Fourier coefficients by a N-point FFT. The first step can be realized by using $N \cdot K$ instructions – also consisting of one real with complex multiplication and one complex addition. The second step requires $K \cdot \log_2 K$ compound instructions, i.e. the butterflies belonging to FFT. Hence the method introduced above uses less instructions under the same conditions. Further advantages can arise from the structure of the matrix $\mathbf{\Lambda}^b$ that will be considered as follow.

The elements of matrix $\mathbf{\Lambda}^b$ can be computed as an option by evaluating the Fourier–transform corresponding to (30). Namely, by applying the substitution $w = e^{\mathrm{i}\,\omega}$

$$\lambda_{k,n}^b = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{(\mathrm{i}\,\omega}+b)^k}{(1+e^{\mathrm{i}\,\omega}\bar{b})^{k+1}} e^{-\,\mathrm{i}\,n\omega}\,d\omega,$$

with $dw = \mathrm{i}\,e^{\mathrm{i}\,\omega} d\omega$, furthermore by realizing the discretization

$$\omega = \ell\frac{2\pi}{N} \quad \ell = 0, 1, 2 \ldots, N-1$$

the form

$$\lambda_{n,k}^b \approx \frac{1}{N} \sum_{\ell=0}^{N-1} \frac{e^{(\mathrm{i}\,2\pi\frac{\ell}{N}}+b)^k}{(1+e^{\mathrm{i}\,2\pi\frac{\ell}{N}}\bar{b})^{k+1}} e^{-\,\mathrm{i}\,2\pi\frac{\ell n}{N}}$$

is given, that defines a discrete Fourier–transform, which can be computed by using FFT.

Another option is to compute the coefficients $\lambda_{n,k}^b$ analytically starting from the form (30). Apply the substitution $z = 1/w$, with $dw = -dz/z^2$

$$\lambda_{n,k}^b = -\frac{\sqrt{1-|b|^2}}{2\pi i} \oint_{\mathbb{T}} \frac{(1+bz)^k}{(z+\bar{b})^{k+1}} z^n\,dz.$$

The contour integral can be computed by applying the Cauchy integral formula in the form

$$\frac{k!}{2\pi i}\oint_{\mathbb{T}} \frac{(1+bz)^k z^n}{(z+\overline{b})^{k+1}}\,dz = \left[\frac{d^k}{dz^k}(1+bz)^k z^n\right]_{z=-\overline{b}}$$

resulting in

$$\lambda^b_{n,k} = -\frac{\sqrt{1-|b|^2}}{k!}\left[\frac{d^k}{dz^k}(1+bz)^k z^n\right]_{z=-\overline{b}}. \qquad (31)$$

The form (31) implies the Rodrigues formula belonging in the Jacobi polynomials of the form

$$(1-x)^\alpha(1+x)^\beta P_k^{(\alpha,\beta)} =$$
$$\frac{(-1)^k}{2^k k!}\frac{d^k}{dx^k}\left[(1-x)^{k+\alpha}(1+x)^{k+\beta}\right] \quad (\alpha,\beta \geq -1),$$

(see p. 62 in [12]). By applying the substitution $x = 1+2bz$, with $\alpha = n-k$ and $\beta = 0$

$$z^{n-k}P_k^{(n-k,0)}(1+2bz) = \frac{1}{k!}\frac{d^k}{dz^k}\left[z^n(1+bz)^k\right]$$

is obtained, which implies that the coefficients $\lambda^b_{n,k}$ – by applying also $z = -\overline{b}$ – can be computed on the basis of Jacobi polynomials $P_k^{(n-k,0)}(x)$ in the form

$$\lambda^b_{n,k} = -\sqrt{1-|b|^2}\,(-1)^{n-k}\,\overline{b}^{n-k}P_k^{(n-k,0)}(1-2|b|^2).$$

A numerically robust solution to compute the coefficients arises from using the recurrence formula belonging to the Jacobi polynomials; according to p. 71 of [12]

$$\begin{aligned} 2kn(k+n-2)\,P_k^{(n-k,0)}(x) = {} & (k+n-1)\cdot \\ \cdot\left[(k+n)(k+n-2)x+(n-k)^2\right] & P_{k-1}^{(n-k,0)}(x) - \\ -2(n-1)(k-1)(k+n) & \,P_{k-2}^{(n-k,0)}(x) \end{aligned} \qquad (32)$$

for indices $k = 2,3,4\ldots$ and starting with

$$P_0^{(n-k,0)}(x) = 1 \quad P_1^{(n-k,0)}(x) = \tfrac{1}{2}(n-k+2)x + \tfrac{1}{2}(n-k).$$

The analytic computation of the Laguerre–Fourier coefficients indicates significant advantages over the FFT-based method, since the latter one suffers from serious drawbacks such as follow:

- The FFT-based method uses an approximation of the Fourier–integral, hence the coefficients computed with some approximation error.
- The FFT algorithms — depending on the implementations — can be inaccurate for extremely small data. This effect is very frequent in Laguerre representations with fast convergence — that is just the desirable case.

The analytic method introduced above can produce accurate coefficients until high indices of the Laguerre–Fourier coefficients, hence it can be used in wider range of pole–reconstruction problems.

The pole–reconstruction process can be realized on the basis of the Laguerre–Fourier coefficients by using the formuli (21) and (23), see a detailed description in [7].

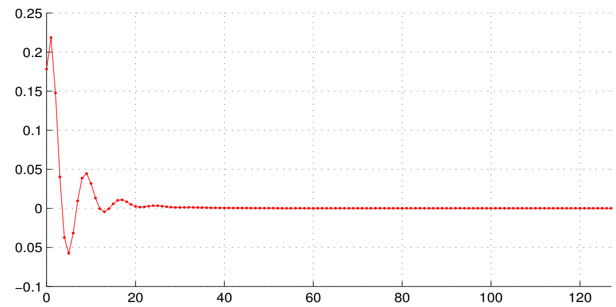

Fig. 1. Example signal.

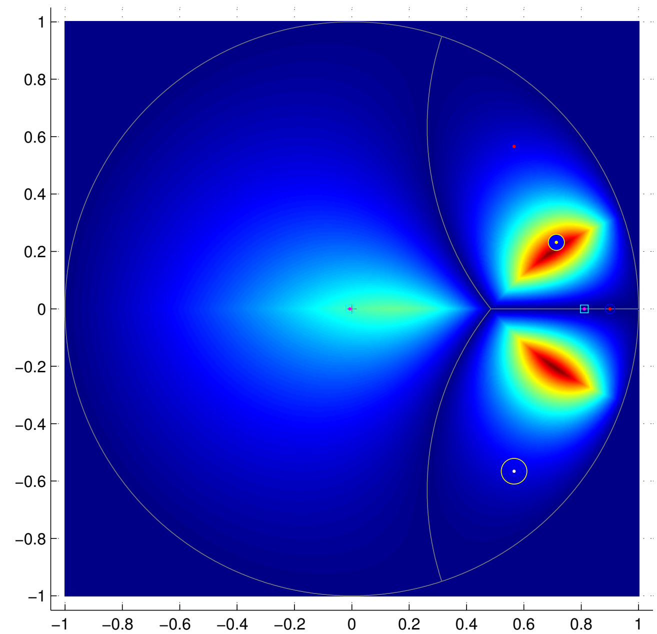

Fig. 2. Reconstructing pole p_1 starting from $b = 0.75e^{\mathrm{i}\,\frac{\pi}{10}}$.

IV. EXAMPLE

An illustrative example based on a simulated discrete–time signal is presented. The signal has been generated as the impulse–response of a 3^{rd}order digital filter with poles $p_0 = 0.9$, and $p_{1,2} = 0.8e^{\pm\,\mathrm{i}\,\frac{\pi}{4}}$, see Fig. 1).

Fig. 2 contains a visualization of the pole–reconstruction. The Laguerre–parameter $b = 0.75e^{\mathrm{i}\,\frac{\pi}{10}}$ is used to compute the Laguerre–Fourier coefficients. The Jacobi polynomials–based method has been used. The modulus and the phase of the transformed quotient sequence (i.e. the sequence of the pole–estimation) computed from the coefficients can be seen in Figure 3. The reconstruction error falls in the range of 10^{-12}.

Figure 4 presents the same example with additive noise applied to the signal ba keeping SNR in level approx. 1000. It can be observed that Laguerre–Fourier coefficients of large indices have been distorted. The highlighted region on the diagram indicates the index-range where the pole estimation could be realized. The estimation error in this case is in the range of 10^{-3}.

Finally the reconstruction of pole p_0 is presented in Figure 5 in the presence of noise with SNR in the magnitude

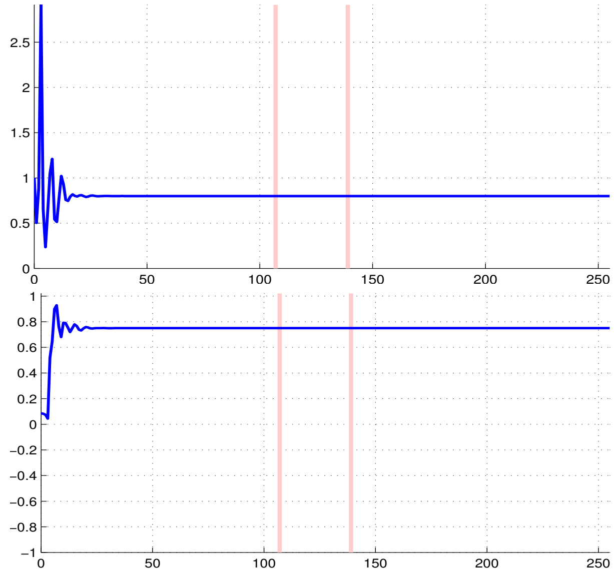

Fig. 3. Reconstructing pole 1: modulus and phase of the estimated pole-sequence

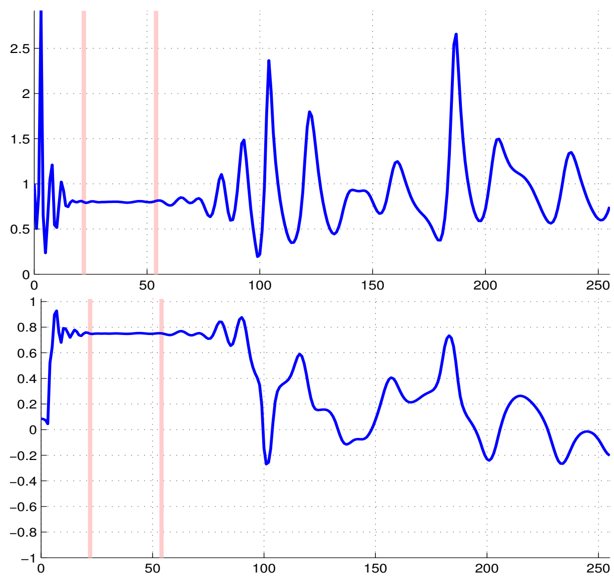

Fig. 4. Reconstructing pole 1 with noise: modulus and phase of the estimated pole-sequence

of 100 starting from parameter $b = 0.25e^{\mathrm{i}\,\frac{\pi}{2}}$. Increased variance in the estimated sequence is experienced, however the estimation error remains in the magnitude of 10^{-2}.

V. CONCLUSIONS AND FUTURE WORKS

The pole–reconstruction method introduced by the authors has been extended to be applied directly on discrete time–domain signals. The new method possesses computational advantages, and produces accuracy at least as good as that of the originally published one, hence it can form an adequate basis for estimating poles of systems as a significant part of an identification process.

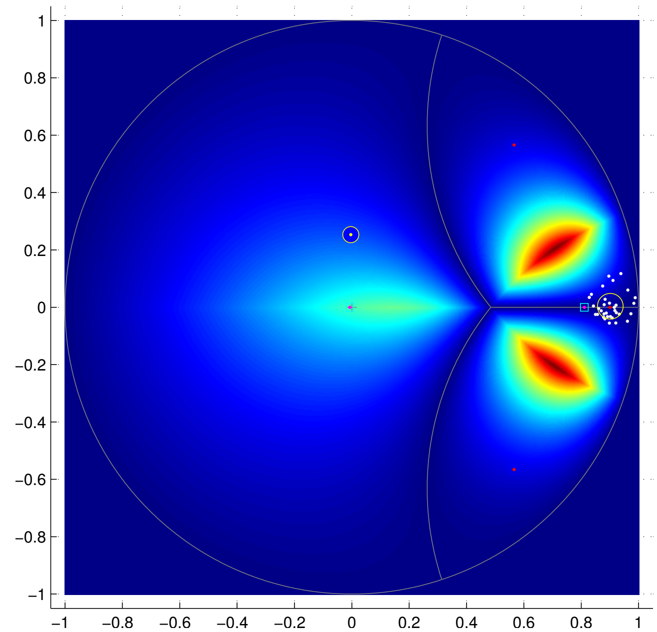

Fig. 5. Reconstructing pole p_0 starting from $b = 0.25e^{\mathrm{i}\,\frac{\pi}{2}}$.

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