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H_∞ Observer Design for Uncertain Discrete-Time Nonlinear Delay Systems: LMI Optimization Approach

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***Abstract*— In this paper, we present a robust H_∞ observer for a class of nonlinear and uncertain time-delayed systems. To design the proposed observer, the time delay does not have to be exactly known. With knowledge about upper and lower bound of the delay term, we can design an H_∞ observer that guarantees asymptotic stability of the estimation error dynamics and is robust against time-varying parametric uncertainties. We show that the described problem can be solved in terms of linear matrix inequalities (LMIs). In addition, the admissible Lipschitz constant of the system is maximized and the disturbance attenuation level is minimized through convex multi-objective optimization. Finally, the proposed observer is illustrated with an example.**

I. INTRODUCTION

Nonlinear observer design for uncertain systems is a research field in control theory that has attracted much attention in recent decades and has been investigated in many aspects [1]–[5]. But, what is the story about nonlinear time-delay systems including uncertainty?

The analysis of nonlinear systems with time-delays, are generally more complicated compared to systems without time delays [6], [7]. Although various aspects of the observability problem for linear time-delay systems have been established in the literature [8]–[10], there are few reports related to nonlinear systems [11]–[13].

Regarding discrete domain, there are some results [14] where the authors propose a Riccati equation approach to design a robust nonlinear H_∞ observer. In [15], they propose to use LMI to design a robust H_∞ observer for a class of Lipschitz nonlinear uncertain systems. In [16], authors converted the robust H_∞ observer problem for uncertain nonlinear systems to a multi-objective problem where finding maximum Lipschitz constant was one of the objectives.

However, to the authors' best knowledge, the problem of multi-objective robust H_∞ nonlinear observer design for Lipschitz time-delay systems including time-varying uncertainties has not been fully investigated yet. So, the main contributions of this paper are summarized as follows. First, we try to design a robust H_∞ observer for a class of uncertain systems with an uncertain time-varying time-delay. Upper and lower bounds for the time-delay is needed to design an observer. Second, we reformulate the problem of robust H_∞ observer design to a multi-objective optimization problem where we, simultaneously, try to find the largest Lipschitz constant and smallest disturbance tuning level. To achieve this goal, we use Pareto multi-objective optimization method [17].

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The rest of this paper is organized as follows. In Section 2, we introduce the class of Lipschitz time-delay systems including time-varying uncertainties and a few preliminaries that are relevant for this study. In Section 3, we propose a new method to design a robust H_∞ observer for the considered systems under study. To do this, based on Lyapunov-Krasovskii functional, a sufficient condition in terms of a set of linear matrix inequalities to make error dynamics asymptotically stable will be proposed. Then, we reformulate the obtained robust H_∞ observer to the multi-objective optimization problem. The proposed observer is illustrated with an example in Section 4, followed by conclusions in Section 5.

Notation. Throughout this paper,

$$\begin{aligned} x_k &\equiv x(k), \\ x_{k-\tau(k)} &\equiv x(k-\tau(k)) \end{aligned} \tag{1}$$

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider the class of discrete-time nonlinear uncertain delay systems:

$$(\Sigma_{d_1}): \quad x_{k+1} = (A+\Delta A(k))x_k + (A_d+\Delta A_d(k))x_{k-\tau(k)} + f(x_k) + B\omega_k, \tag{2a}$$

$$y_k = (C+\Delta C(k))x_k + D\omega_k \tag{2b}$$

where $x \in R^n$, $y \in R^p$ and $\omega_k \in L_2[0,\infty)$ are the state vector, output vector and an unknown exogenous disturbance of the system, respectively. Matrices A, A_d, B, C, and D are real and with appropriate dimensions and $\tau(k)$ is a time-varying delay which satisfies the following assumption,

$$\tau_1 \le \tau(k) \le \tau_2 \tag{3}$$

The function $f : R^n \to R^n$ is nonlinear, differentiable, and verifies the Lipschitz constraint as follows:

$$\|f(x_k) - f(\hat{x}_k)\| \le \gamma \|x_k - \hat{x}_k\| \quad \forall x, \hat{x} \in \Omega \tag{4}$$

where $\gamma > 0$ is the Lipschitz constant. Moreover, ΔA, ΔA_d and ΔC are unknown matrices representing time-varying parameter uncertainties and are assumed to be of the form:

$$\begin{bmatrix} \Delta A(k) \\ \Delta A_d(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} F(k)N \tag{5}$$

where M_1, M_2, M_3 and N are known real constant matrices and $F(k)$ is an unknown real-valued time-varying matrix satisfying,

$$\forall k, \quad F^T(k)F(k) \le I \tag{6}$$

Now, consider a nonlinear observer as follows,

$$\hat{x}_{k+1} = A\hat{x}_k + f(\hat{x}_k) + L(y_k - C\hat{x}_k) \quad (7)$$

where $\hat{x}$ is the observer state and L is the observer gain matrix that should be designed under a few conditions. By defining the observer error as $e_k = x_k - \hat{x}_k$, the error dynamics become,

$$\begin{aligned} e_{k+1} =& (A - LC)e_k + (\Delta A - L\Delta C)x_k \\ &+ (A_d + \Delta A_d)x_{k-\tau(k)} + f(x_k) - f(\hat{x}_k) \\ &(B - LD)\omega_k \end{aligned} \quad (8)$$

Suppose the controlled output for the error state is as follows,

$$z_k = He_k \quad (9)$$

where H is a known matrix. We need to design (7) such that the following objectives can be achieved:
1 - The observer error dynamics are asymptotically stable.
2 - Under the zero initial state condition and for arbitrary $\omega_k \in L_2[0,\infty)$, z_k satisfies $\|z_k\|_2 \leq \eta \|\omega_k\|_2$.
3 - Maximum Lipschitz constant γ and minimum H_∞-norm bound η, simultaneously
If the above conditions can be satisfied, then the observer error dynamics (8) are said to be asymptotically stable with maximum Lipschitz constant γ^* and minimum H_∞-norm η^*. Before ending this section, we present three lemmas which will be used in the proof of our main theorems.

Lemma 1 *[15]. For any $x, y \in R^n$ and any positive-definite matrix $\Gamma \in R^{n\times n}$, we have*

$$2x^T y \leq x^T \Gamma x + y^T \Gamma^{-1} y \quad (10)$$

Lemma 2 *[18]. Let A, D, E, F and Γ be real matrices of appropriate dimensions with $\Gamma > 0$ and F satisfying $F^T F \leq I$. Then for any scalar $\varepsilon > 0$ satisfying $\Gamma^{-1} - \varepsilon^{-1} DD^T > 0$, we have*

$$(A + DFE)^T \Gamma (A + DFE) \leq A^T (\Gamma^{-1} - \varepsilon^{-1} DD^T)^{-1} A + \varepsilon E^T E \quad (11)$$

Lemma 3 (Matrix Inversion Lemma) *Let A, B, C and D be matrices of appropriate dimensions, we have*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \quad (12)$$

if and only if A, C and $C^{-1} + DA^{-1}B$ are nonsingular.

III. Main Results

Assume,

$$\xi_k = \begin{bmatrix} e_k^T & x_k^T \end{bmatrix}^T \quad (13)$$

then, the augmented system dynamics are

$$(\Sigma_{d_2}): \quad \begin{aligned} \xi_{k+1} =& (\tilde{A} + \Delta\tilde{A})\xi_k + (\tilde{A}_d + \Delta\tilde{A}_d)\xi_{k-\tau(k)} \\ &+ \tilde{f}(x_k, \hat{x}_k) + \tilde{B}\omega_k \quad (14a) \\ z_k =& \tilde{H}\xi_k \quad (14b) \end{aligned}$$

where,

$$\tilde{A} = \begin{bmatrix} A - LC & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} 0 & A_d \\ 0 & A_d \end{bmatrix},$$

$$\Delta\tilde{A} = \begin{bmatrix} 0 & \Delta A - L\Delta C \\ 0 & \Delta A \end{bmatrix} = \begin{bmatrix} 0 & M_1FN - LM_3FN \\ 0 & M_1FN \end{bmatrix}$$

$$= \begin{bmatrix} M_1 - LM_3 \\ M_1 \end{bmatrix} F \begin{bmatrix} 0 & N \end{bmatrix} \equiv \tilde{M}_1 F \tilde{N},$$

$$\Delta\tilde{A}_d = \begin{bmatrix} 0 & \Delta A_d \\ 0 & \Delta A_d \end{bmatrix} = \begin{bmatrix} 0 & M_2FN \\ 0 & M_2FN \end{bmatrix}$$

$$= \begin{bmatrix} M_2 \\ M_2 \end{bmatrix} F \begin{bmatrix} 0 & N \end{bmatrix} \equiv \tilde{M}_2 F \tilde{N},$$

$$\tilde{B} = \begin{bmatrix} B - LD \\ B \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} H & 0 \end{bmatrix},$$

$$\tilde{f}(x_k, \hat{x}_k) = \begin{bmatrix} f(x_k) - f(\hat{x}_k) \\ f(x_k) \end{bmatrix}, \quad \|\tilde{f}(x_k, \hat{x}_k)\| \leq \gamma \|\xi_k\| \quad (15)$$

Theorem 1 *Consider the system (2) together with the nonlinear observer (7) and let $\tau_0 = \tau_2 - \tau_1 + 1$. Assume that there exist matrices $P_1 > 0$, $P_2 > 0$ and Υ, and also scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\eta > 0$, and $\alpha > 0$ such that the following LMI optimization problem has a feasible solution:*

$$\begin{aligned} &min. \quad (\alpha + \varepsilon_1) \\ &s.t. \quad \begin{bmatrix} \Psi_1 & \Omega \\ * & \Psi_2 \end{bmatrix} < 0, \quad \Psi_3 < 0 \end{aligned} \quad (16)$$

where Ψ_1, Ψ_2, Ψ_3 and Ω are as in (17). Then, with $L = P_1^{-1}\Upsilon$ the observer error dynamics (8) are asymptotically stable with maximized admissible Lipschitz constant $\gamma^* = \frac{1}{\sqrt{\alpha(1+\varepsilon_1)}}$ and disturbance tuning parameter η.

Proof. Consider a Lyapunov-Krasovskii functional candidate as,

$$V_k = \xi_k^T P \xi_k + \hat{V}_k + \tilde{V}_k \quad (18)$$

where,

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}; \quad P_1 > 0, \ P_2 > 0,$$

$$\hat{V}_k = \sum_{i=k-\tau(k)}^{k-1} \xi^T(i) R \xi(i); \quad R > 0,$$

$$\tilde{V}_k = \sum_{j=-\tau_2+2}^{-\tau_1+1} \sum_{q=k+j-1}^{k-1} \xi^T(q) R \xi(q) \quad (19)$$

Then,

$$\begin{aligned} \hat{V}_{k+1} - \hat{V}_k =& \sum_{i=k-\tau(k+1)}^{k} \xi_i^T R \xi_i - \sum_{i=k-\tau(k)}^{k-1} \xi_i^T R \xi_i \\ =& \sum_{i=k+1-\tau(k+1)}^{k-\tau_1} \xi_i^T R \xi_i + \sum_{k+1-\tau_1}^{k-1} \xi_i^T R \xi_i \\ &+ \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} - \sum_{k+1-\tau(k)}^{k-1} \xi_i^T R \xi_i \end{aligned} \quad (20)$$

$$\Psi_1 = \begin{bmatrix} -P_1 + H^T H & 0 & I \\ * & (\varepsilon_2 + \tau_0 \varepsilon_3) N^T N - P_2 & 0 \\ * & * & -\alpha I \end{bmatrix} < 0,$$

$$\Psi_2 = \begin{bmatrix} \frac{-1}{3}P_1 & 0 & \frac{1}{3}P_1 & 0 & P_1 M_1 - \Upsilon M_3 & 0 & 0 & 0 \\ 0 & -\frac{1}{3}P_2 & 0 & \frac{1}{3}P_2 & P_2 M_1 & 0 & 0 & 0 \\ * & * & \frac{5}{3}P_1 - \varepsilon_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \frac{5}{3}P_2 - \varepsilon_1 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & * & * & \frac{-1}{3}\tau_0^{-1}P_1 & 0 & P_1 M_3 \\ * & * & * & * & * & * & \frac{-1}{3}\tau_0^{-1}P_2 & P_2 M_3 \\ * & * & * & * & * & * & * & -\varepsilon_3 \tau_0 I \end{bmatrix} < 0,$$

$$\Omega = \begin{bmatrix} A^T P_1 - C^T \Upsilon^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & A^T P_2 & 0 & 0 & 0 & A_d^T P_1 & A_d^T P_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} < 0,$$

$$\Psi_3 = \begin{bmatrix} -\eta^2 I & B^T P_1 - D^T \Upsilon^T & B^T P_2 & B^T P_1 - D^T \Upsilon^T & B^T P_2 \\ * & -\frac{1}{3}P_1 & 0 & 0 & 0 \\ * & * & -\frac{1}{3}P_2 & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{17}$$

since $\tau(k) \geq \tau_1$, we have

$$\sum_{k+1-\tau_1}^{k-1} \xi_i^T R \xi_i - \sum_{k+1-\tau(k)}^{k-1} \xi_i^T R \xi_i \leq 0 \tag{21}$$

so,

$$\hat{V}_{k+1} - \hat{V}_k \leq \sum_{i=k+1-\tau(k+1)}^{k-\tau_1} \xi_i^T R \xi_i + \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} \tag{22}$$

Regarding $\tilde{V}_k$, after some calculations,

$$\begin{aligned} \tilde{V}_{k+1} - \tilde{V}_k &\leq \sum_{j=-\tau_2+2}^{-\tau_1+1} [\xi_k^T R \xi_k + \sum_{q=k+j}^{k-1} \xi_q^T R \xi_q - \sum_{q=k+j-1}^{k-1} \xi_q^T R \xi_q] \\ &= (\tau_2 - \tau_1) \xi_k^T R \xi_k - \sum_{i=k+1-\tau_2}^{k-\tau_1} \xi_i^T R \xi_i \end{aligned} \tag{23}$$

From (22) and (23),

$$\begin{aligned} \hat{V}_{k+1} + \tilde{V}_{k+1} - \hat{V}_k - \tilde{V}_k &\leq (\tau_2 - \tau_1 + 1) \xi_k^T R \xi_k - \xi_{k-\tau_k}^T R \xi_{k-\tau_k} \\ &+ \sum_{i=k+1-\tau(k+1)}^{k-\tau_1} \xi_i^T R \xi_i - \sum_{i=k+1-\tau_2}^{k-\tau_1} \xi_i^T R \xi_i \end{aligned} \tag{24}$$

and $\tau(k) \leq \tau_2$,

$$\sum_{i=k+1-\tau(k+1)}^{k-\tau_1} \xi_i^T R \xi_i - \sum_{i=k+1-\tau_2}^{k-\tau_1} \xi_i^T R \xi_i < 0 \tag{25}$$

Therefore, (24) can be rewritten as,

$$\hat{V}_{k+1} + \tilde{V}_{k+1} - \hat{V}_k - \tilde{V}_k \leq \tau_0 \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} \tag{26}$$

where $\tau_0 = \tau_2 - \tau_1 + 1$. By using (26) and for $\omega_k = 0$,

$$\begin{aligned} \Delta V =& V_{k+1} - V_k \\ \leq& \xi_{k+1}^T P \xi_{k+1} - \xi_k^T P \xi_k + \tau_0 \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} \\ =& \xi_k^T (\tilde{A} + \Delta\tilde{A})^T P (\tilde{A} + \Delta\tilde{A}) \xi_k \\ &+ \xi_{k-\tau(k)}^T (\tilde{A}_d + \Delta\tilde{A}_d)^T P (\tilde{A}_d + \Delta\tilde{A}_d) \xi_{k-\tau(k)} \\ &+ 2\xi_{k-\tau(k)}^T (\tilde{A}_d + \Delta\tilde{A}_d)^T P \tilde{f} + 2\xi_k^T (\tilde{A} + \Delta\tilde{A})^T P \tilde{f} \\ &+ 2\xi_k^T (\tilde{A} + \Delta\tilde{A})^T P (\tilde{A}_d + \Delta\tilde{A}_d) \xi_{k-\tau(k)} + \tilde{f}^T P \tilde{f} \\ &- \xi_k^T P \xi_k + \tau_0 \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} \end{aligned} \tag{27}$$

Based on Lemma 1,

$$\begin{aligned} 2\xi_{k-\tau(k)}^T (\tilde{A}_d + \Delta\tilde{A}_d)^T P \tilde{f} \leq& \xi_{k-\tau(k)}^T (\tilde{A}_d + \Delta\tilde{A}_d)^T P \\ &\times (\tilde{A}_d + \Delta\tilde{A}_d) \xi_{k-\tau(k)} + \tilde{f}^T P \tilde{f} \end{aligned} \tag{28}$$

from (28) and Assumption 1,

$$\begin{aligned} 2\xi_k^T (\tilde{A} + \Delta\tilde{A})^T P \tilde{f} + 2\tilde{f}^T P \tilde{f} =& 2\xi_k^T (\tilde{A} + \Delta\tilde{A})^T P \tilde{f} - \tilde{f}^T Q \tilde{f} \\ &+ \varepsilon_1 \tilde{f}^T \tilde{f} \\ \leq& \xi_k^T (\tilde{A} + \Delta\tilde{A})^T P Q^{-1} P (\tilde{A} + \Delta\tilde{A}) \xi_k + \varepsilon_1 \gamma^2 \xi_k^T \xi_k \end{aligned} \tag{29}$$

where $Q = \varepsilon_1 - 2P > 0$, thus

$$\begin{aligned} &\xi_k^T (\tilde{A} + \Delta\tilde{A})^T P (\tilde{A} + \Delta\tilde{A}) \xi_k + 2\xi^T (\tilde{A} + \Delta\tilde{A})^T P \tilde{f} \\ &\quad + 2\xi_{k-\tau(k)}^T (\tilde{A}_d + \Delta\tilde{A}_d)^T P f + \tilde{f}^T P \tilde{f} \\ &\quad + \tau_0 \xi_k^T R \xi_k - \xi_{k-\tau(k)}^T R \xi_{k-\tau(k)} \\ &\quad \leq \xi_k^T [(P\tilde{A} + P\Delta\tilde{A})^T (P^{-1} + Q^{-1})(P\tilde{A} + P\Delta\tilde{A}) \\ &\quad - P + \tau_0 R + \varepsilon_1 \gamma^2] \xi_k \\ &\quad + \xi_{k-\tau(k)}^T [(\tilde{A}_d + \Delta\tilde{A}_d)^T P (\tilde{A}_d + \Delta\tilde{A}_d) - R] \xi_{k-\tau(k)} \end{aligned} \tag{30}$$

In addition,

$$2\xi_k^T(\tilde{A}+\Delta\tilde{A})^T P(\tilde{A}_d+\Delta\tilde{A}_d)\xi_{k-\tau(k)} \le \xi_k^T[(P\tilde{A}+P\Delta\tilde{A})^T \times P^{-1}(P\tilde{A}_d+P\Delta\tilde{A}_d)]\xi_k + \xi_{k-\tau(k)}^T(P\tilde{A}_d+P\Delta\tilde{A}_d)^T \times P^{-1}(P\tilde{A}_d+P\Delta\tilde{A}_d)\xi_{k-\tau(k)} \quad (31)$$

So, from (30) and (31),

$$\Delta V \le \xi_k^T\Omega_1\xi_k + \xi_{k-\tau(k)}^T\Omega_2\xi_{k-\tau(k)} \quad (32)$$

where,

$$\Omega_1 = (P\tilde{A}+P\tilde{M}_1F\tilde{N})^T(2P^{-1}+Q^{-1})(P\tilde{A}+P\tilde{M}_1F\tilde{N}) - P + \tau_0 R + \varepsilon_1\gamma^2$$
$$\Omega_2 = (P\tilde{A}_d+P\tilde{M}_2F\tilde{N})^T(\frac{1}{2}P)^{-1}(P\tilde{A}_d+P\tilde{M}_2F\tilde{N}) - R \quad (33)$$

It is obvious that if Ω_1 and Ω_2 are negative definite then the observer error dynamics (8) are asymptotically stable. Now, we want to show that the observer error dynamics (8) are asymptotically stable with a minimum H_∞ -norm bound η^* and maximum Lipschitz constant γ^*. To this end, the index function J is defined as,

$$J = \sum_{k=1}^{\infty}[z_k^T z_k - \eta^2\omega_k^T\omega_k] \quad (34)$$

For the zero initial condition and for any $\omega \neq 0$, the following equation can be obtained,

$$J \le \sum_{k=1}^{\infty}[z_k^T z_k - \eta^2\omega_k^T\omega_k + \Delta V] \le \sum_{k=1}^{\infty}[\xi_k^T\tilde{H}^T\tilde{H}\xi_k - \eta^2\omega_k^T\omega_k + \xi_k^T\Omega_1\xi_k + \xi_{k-\tau(k)}^T\Omega_2\xi_{k-\tau(k)} + 2\xi_k^T(\tilde{A}+\Delta\tilde{A})^T P\tilde{B}\omega + 2\xi_{k-\tau(k)}^T(\tilde{A}_d+\Delta\tilde{A}_d)^T P\tilde{B}\omega + 2\tilde{f}^T P\tilde{B}\omega + \omega^T\tilde{B}^T P\tilde{B}\omega] \quad (35)$$

Similarly, using Lemma 1

$$2\xi_k^T(\tilde{A}+\Delta\tilde{A})^T P\tilde{B}\omega + 2\xi_{k-\tau(k)}^T(\tilde{A}_d+\Delta\tilde{A}_d)^T P\tilde{B}\omega + 2\tilde{f}^T P\tilde{B}\omega + \omega^T\tilde{B}^T P\tilde{B}\omega \le \xi_k^T(\tilde{A}+\tilde{M}_1F\tilde{N})^T P(\tilde{A}+\tilde{M}_1F\tilde{N})\xi_k + \xi_{k-\tau(k)}^T(\tilde{A}_d+\tilde{M}_2F\tilde{N})^T P(\tilde{A}_d+\tilde{M}_2F\tilde{N}) \times \xi_{k-\tau(k)}^T + \gamma^2\xi^T\xi + \omega^T\tilde{B}^T PP\tilde{B}\omega + 3\omega^T\tilde{B}^T P\tilde{B}\omega \quad (36)$$

from (33), (36), and also using Lemma 2, a sufficient condition for $J<0$ is as follows,

$$\xi_k^T[\tilde{A}^T P[(3P^{-1}+Q^{-1})^{-1} - \varepsilon_2^{-1}\tilde{M}_1\tilde{M}_1^T]^{-1}P\tilde{A} + \varepsilon_2\tilde{N}^T\tilde{N} - P + \tau_0 R + (1+\varepsilon_1)\gamma^2 + \tilde{H}^T\tilde{H}]\xi_k + \xi_{k-\tau(k)}^T[\tilde{A}_d^T P(\frac{1}{3}P - \varepsilon_3^{-1}\tilde{M}_2\tilde{M}_2^T)^{-1}P\tilde{A}_d + \varepsilon_3\tilde{N}^T\tilde{N} - R]\xi_{k-\tau(k)} + \omega_k^T[\tilde{B}^T PP\tilde{B} + 3\tilde{B}^T P\tilde{B} - \eta^2 I]\omega_k \le 0 \quad (37)$$

Since we try to maximize γ, we define a new variable α as,

$$\alpha \equiv \frac{1}{(1+\varepsilon_1)\gamma^2} \quad \Rightarrow \quad \gamma = \frac{1}{\sqrt{\alpha(1+\varepsilon_1)}} \quad (38)$$

Maximization of γ can be done by minimization of α and ε_1 at the same time. In fact, by combining the two objective functions, we will minimize the scalar linear objective function $\alpha+\varepsilon_1$. On the other hand, by using Lemma 3,

$$(3P^{-1}+Q^{-1})^{-1} = [(\frac{1}{3}P)^{-1} + IQ^{-1}I]^{-1} = \frac{1}{3}P - \frac{1}{3}P(\varepsilon_1 I - \frac{5}{3}P)^{-1}\frac{1}{3}P \quad (39)$$

now, let

$$R = \tilde{A}_d^T P(\frac{1}{3}P - \varepsilon_3^{-1}\tilde{M}_2\tilde{M}_2^T)^{-1}P\tilde{A}_d + \varepsilon_3\tilde{N}^T\tilde{N} \quad (40)$$

from (39), and also substituting (40) into (37) yields,

$$J \le \xi_k^T\Pi_1\xi_k + \omega_k^T\Pi_2\omega_k < 0 \quad (41)$$

where,

$$\Pi_1 = \tilde{A}^T P[\frac{1}{3}P - \frac{1}{3}P(\varepsilon_1 I - \frac{5}{3}P)^{-1}\frac{1}{3}P - \varepsilon_2^{-1}\tilde{M}_1\tilde{M}_1^T]^{-1}P\tilde{A} + (\varepsilon_2+\tau_0\varepsilon_3)\tilde{N}^T\tilde{N} - P + \alpha^{-1}I + \tilde{H}^T\tilde{H} + \tau_0\tilde{A}_d^T P(\frac{1}{3}P - \varepsilon_3^{-1}\tilde{M}_2\tilde{M}_2^T)^{-1}P\tilde{A}_d$$
$$\Pi_2 = \tilde{B}^T PP\tilde{B} + 3\tilde{B}^T P\tilde{B} - \eta^2 I \quad (42)$$

we also have,

$$(\varepsilon_2+\tau_0\varepsilon_3)\tilde{N}^T\tilde{N} = \begin{bmatrix} 0 & 0 \\ 0 & (\varepsilon_2+\tau_0\varepsilon_3)N^T N \end{bmatrix},$$
$$P\tilde{M}_1 = \begin{bmatrix} P_1M_1 - \Upsilon M_3 \\ P_2M_1 \end{bmatrix}, \quad P\tilde{M}_2 = \begin{bmatrix} P_1M_2 \\ P_2M_2 \end{bmatrix},$$
$$\tilde{A}^T P = \begin{bmatrix} A^T P_1 - C^T\Upsilon^T & 0 \\ 0 & A^T P_2 \end{bmatrix}, \quad \tilde{A}_d^T P = \begin{bmatrix} 0 & A_d^T P_1 \\ 0 & A_d^T P_2 \end{bmatrix},$$
$$P\tilde{B} = \begin{bmatrix} P_1B - \Upsilon D \\ P_2B \end{bmatrix}, \quad \tilde{H}^T\tilde{H} = \begin{bmatrix} H^T H & 0 \\ 0 & 0 \end{bmatrix} \quad (43)$$

substituting (43) into (42), and also using Schur complements, the LMIs in Theorem 1 are achievable. □

Now, we want to reformulate the problem of robust H_∞ observer design in Theorem 1 to the multi-objective optimization problem where finding the maximum Lipschitz constant and also minimum disturbance tuning level are objectives. As we mentioned before, in order to achieve this goal, we use Pareto multi-objective optimization, i.e. optimization of a linear combination of several optimality criteria [17] resulting in a trade-off between these.

Theorem 2 *Consider the system (2) together with the nonlinear observer (7) and let $\tau_0 = \tau_2 - \tau_1 + 1$. Assume that there exist matrices $P_1 > 0$, $P_2 > 0$ and Υ, and also scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\alpha > 0$, $\rho > 0$, and $0 \le \lambda \le 1$ such that the following*

LMI optimization problem has a feasible solution:

$$\begin{aligned} min. \quad & [\lambda(\alpha+\varepsilon_1)+(1-\lambda)\ \rho] \\ s.t. \quad & \begin{bmatrix} \Psi_1 & \Omega \\ * & \Psi_2 \end{bmatrix} < 0, \end{aligned}$$

$$\begin{bmatrix} -\rho I & B^TP_1-D^T\Upsilon^T & B^TP_2 & B^TP_1-D^T\Upsilon^T & B^TP_2 \\ * & -\frac{1}{3}P_1 & 0 & 0 & 0 \\ * & * & -\frac{1}{3}P_2 & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{44}$$

where Ψ_1, Ψ_2, *and* Ω *are as in (17). Then, with* $L = P_1^{-1}\Upsilon$ *the observer error dynamics (8) are asymptotically stable with maximized admissible Lipschitz constant* $\gamma^* = \frac{1}{\sqrt{\alpha(1+\varepsilon_1)}}$ *and minimized disturbance tuning parameter* $\eta^* = \sqrt{\rho}$*, simultaneously.*

Proof. First assume $\rho = \eta^2$. In order to optimize two parameters γ and η simultaneously, a scalarization method including two optimality criteria is used. According to [17], since each of these optimization problems is convex, the scalarized problem is also convex. The rest of the proof is the same as the proof of Theorem 1. □

IV. Example

Assume that the parameters of a system (2) are given by,

$$A = \begin{bmatrix} -0.1 & 0.2 \\ -0.3 & -0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0.1 \\ -0.2 & -0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad D = 0.7,$$

$$M_1 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}^T, \quad M_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T, \quad M_3 = 0.1,$$

$$N = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad f(x_1,x_2) = \begin{bmatrix} 0.3\sin(x_2(k)) \\ 0.2\cos(x_1(k)) \end{bmatrix} \tag{45}$$

and also assume,

$$H = 0.15I_2, \quad 0 \le \tau(k) \le 2,$$

$$F(k) = sin(k), \quad \omega(k) = \frac{1}{k+1} \tag{46}$$

Now, using the Matlab LMI toolbox, we can solve the LMIs defined in Theorem 2 for $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.1$, and $\lambda = 0.9$. One feasible solution is found as follows,

$$\gamma^* = 0.335, \quad \eta^* = 2.379,$$

$$P_1 = \begin{bmatrix} 0.38613 & -0.025 \\ -0.025 & 0.309 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.468 & 0.019 \\ 0.019 & 0.277 \end{bmatrix},$$

$$\Upsilon = \begin{bmatrix} 0.408 \\ -0.334 \end{bmatrix}, \quad L = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{47}$$

To validate the designed observer, simulation results are presented in Fig. 1 and Fig. 2. Fig. 1 shows the real and estimated states simultaneously for initial conditions given by, $x(0) = (2,2)^T$ and $\hat{x}(0) = (0,0)^T$. Fig. 2 shows the error dynamics for the same initial conditions. Based on these simulations, one can see that the observer performs as expected and the state estimation errors for the initial values do tend to zero asymptotically with an H_∞-norm bound $\eta^* = 2.379$. Fig. 3 shows the values of γ^* and η^*, and the optimal trade-off curve between them over the range of λ.

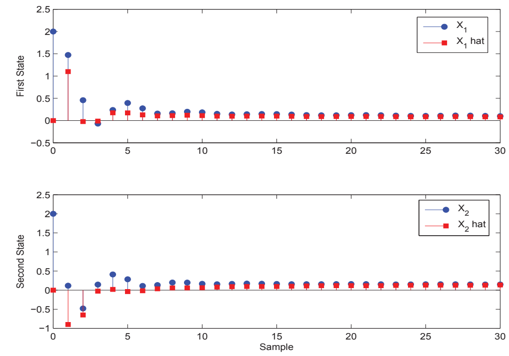

Fig. 1. Real and Estimated States

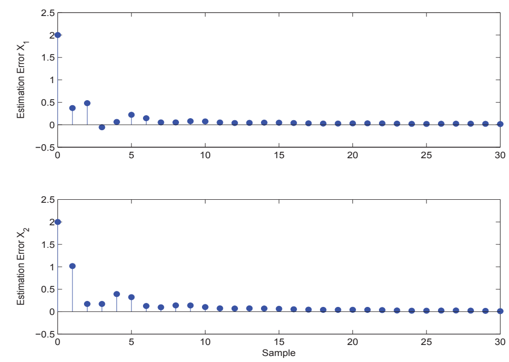

Fig. 2. Estimation Error Dynamics

V. Conclusions

In this note, we present a new approach of robust H_∞ observers for a class of nonlinear uncertain time-delayed systems. There are three main contributions that could be highlighted as follows. First, in the proposed structure for the observer, the designer doesn't need to have exact knowledge about the delay but only of some upper and lower bound. Second, we derive a sufficient condition in terms of LMIs for existence of the proposed nonlinear observer so that the error dynamics are asymptotically stable with an H_∞-norm bound η and maximum Lipschitz constant γ^*. Third, we convert the observer structure under study to a multi-objective optimization problem where we maximize γ and minimize η, simultaneously. We finally illustrate with a numerical example how we can calculate the gain matrices of the proposed observer.

References

[1] Y. Xiong, and M. Saif, "Sliding mode observer for nonlinear uncertain systems," IEEE *Transactions on Automatic Control*, vol. 46, pp. 20122017, 2001.

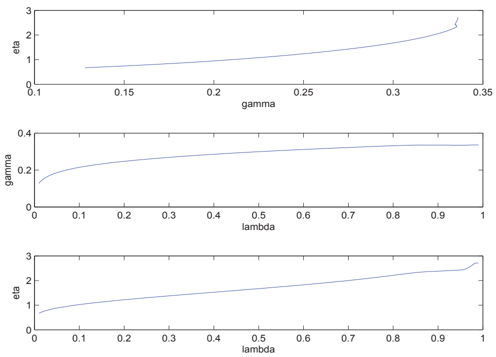

Fig. 3. The trade-off curve $\gamma^* - \eta^*$, $\lambda - \gamma^*$, and $\lambda - \eta^*$

[2] Q. P. Ha, and H. Trinh, "State and input simultaneous estimation for a class of nonlinear systems," *Automatica*, vol. 40, pp. 17791785, 2004.

[3] J-X. Xu , and J. Xu, "Observer based learning control for a class of nonlinear systems with time-varying parametric uncertainties," IEEE *Transactions on Automatic Control*, vol. 49, pp. 275281, 2004.

[4] R. Aguilar-Lopez, and R. Maya-Yescas, "State estimation for nonlinear systems under model uncertainties: a class of sliding-mode observers," *Journal of Process Control*, vol. 15, pp. 363370, 2005.

[5] M. Abbaszadeh, and H. J. Marquez, "A robust observer design method for continuous-time Lipschitz nonlinear systems," *Proceedings of the* 45^{th} *IEEE Conference on Decision and Control*, San Diego, U.S.A., pp. 38953900, 2006.

[6] O. Sename, "New trends in design of observers for time-delay systems," *Kybernetica*, vol. 37, no. 4, pp. 427458, 2001.

[7] J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 16671694, 2003.

[8] M. Boutayeb, "Observers design for linear time-delay systems," *Systems and Control Letters*, vol. 44, pp. 103109, 2001.

[9] A. Seuret, T. Floquet, J. P. Richard, and S. K. Spurgeon, "A sliding mode observer for linear systems with unknown time-varying delay," *in Proc. of IEEE American Control Conference*, 2007.

[10] S. Mondie, and V. Kharitonov, "Exponential estimates for retarded timedelay systems: An LMI approach," IEEE *Trans. Automatic Control*, vol. 50, pp. 268273, 2005.

[11] S. Ibrir, "Adaptive observers for time-delay nonlinear systems in triangular form," *Automatica*, vol. 45, pp. 23922399, 2009.

[12] A. Germani, C. Manes, and P. Pepe, "A new approach to state observation of nonlinear systems with delayed output," IEEE *Trans. Automatic Control*, vol. 47, pp. 96101, 2002.

[13] S. S. Delshad, and T. Gustafsson, "Nonlinear observer design for a class of Lipschitz time-delay systems with unknown inputs: LMI approach," *XXIII IEEE Symposium on Information, Communication and Automation Technologies (ICAT)*, Bosnia and Herzegovina, Sarajevo, pp. 1-5, 2011.

[14] L. Xie, C. D. Souza, and Y. Wang, "Robust filtering for a class of discrete-time uncertain nonlinear systems: an H_∞ approach," *Int. J. Robust Nonlinear Control*, vol. 6, pp. 297-312, 1996.

[15] S. Xu, "Robust H_∞ filtering for a class of discrete-time uncertain nonlinear systems with state delay," IEEE *Transactions on Circuits and Systems*, vol. 49, pp. 1853-1859, 2002.

[16] M. Abbaszadeh, and H. J. Marquez, "LMI optimization approach to robust H_∞ observer design and static output feedback stabilization for discrete-time nonlinear uncertain systems," *Int. J. Robust Nonlinear Control*, vol. 19, pp. 313-340, 2009.

[17] S. Boyd, and L. Vandenberghe, "Convex Optimization," cambridge university press, 2004.

[18] Y. Wang, L. Xie, and C. de Souza CE, "Robust control of a class of uncertain nonlinear systems," *Systems and Control Letters*, vol. 19, pp. 139-149, 1992.