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Analysis of Drilling vibrations: A Time-Delay System Approach

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***Abstract*— The main purpose of this study is the description of the qualitative dynamical response of a rotary drilling system with a drag bit, using a model that takes into consideration the axial and the torsional vibration modes of the bit. The studied model, based on the interface bit-rock, contains a couple of wave equations with boundary conditions consisting of the angular speed and the axial speed at the top additionally to the angular and axial acceleration at the bit whose contain a realistic frictional torque. Our analysis is based on the center manifold Theorem and Normal forms theory whose allow us to simplify the model.**

***Index Terms*— Drilling system, Vibrations analysis, Time-delay systems, Neutral systems, Functional differential equations, Center Manifold, Normal forms.**

I. INTRODUCTION

Interconnected oscillatory systems often display what is called *propagation phenomena*, [13]. In general by Lossless propagation it is understood the phenomenon associated with long transmission lines for physical signals. In engineering, this problem is strongly related to electric and electronic applications, e.g. circuit structures consisting of multipoles connected through LC transmission lines; this can also be seen in steam or gaz flows or pressures and water pipes [21], [9], [22]. The mathematical model is described in all these cases by a mixed initial and boundary value problem for hyperbolic partial differential equations modeling the lossless propagation. The boundary conditions are of special type, being in feedback connection with some system described by ordinary differential equations. This leads to the so-called derivative boundary conditions considered by Cooke & Krumme [6], but also to the even more general boundary conditions of Abolina & Myshkis described by Volterra operators, see [22]. Integration along characteristics of the hyperbolic partial differential equations (here d'Alembert method) allows the association of certain system of functional equations to the mixed problem.

This paper is concerned by an application which can be modeled by such equations; therefore, the above idea is adopted, see [2], [8], [24], [25]. The analysis and modeling of rotary drilling vibrations is a topic whose economical interest has been renewed by recent oilfields discoveries leading to a growing literature, see for instance [23], [10], [20], [17], [19] and [18].

Throughout this paper, we would like first to contribute by improving the modelling of the drilling system, taking into account both axial and torsional vibrations and secondly by extending the qualitative analysis to the investigation of the nonlinear terms in the model. Indeed, the center manifold theorem and normal forms theory are applied to obtain a finite dimensional approximation which conserves the main dynamics of the physical system. Let us consider the following model for the axial vibrations U and torsional vibrations Φ:

$$\begin{cases} \partial_t^2 U(t,s) = c^2\,\partial_s^2 U(t,s) \\ E\,\Gamma\,\partial_s U(t,0) = \alpha \partial_t U(t,0) - H(t) \\ M\,\partial_t^2 U(t,L) = -E\,\Gamma \partial_s U(t,L) + F(\partial_t U(t,L)) \end{cases}, \quad (1)$$

and

$$\begin{cases} \partial_t^2 \Phi(t,s) = \tilde{c}^2\,\partial_s^2 \Phi(t,s) \\ G\,\Sigma\,\partial_s \Phi(t,0) = \beta \partial_t \Phi(t,0) - \Omega(t) \\ J\,\partial_t^2 \Phi(t,L) = -G\,\Sigma\,\partial_s \Phi(t,L) + \tilde{F}(\partial_t U(t,L)) \end{cases}, \quad (2)$$

where, in equation (1), H is the brake motor control and $\alpha \partial_t U(t,0)$ represents a friction force of viscous type. For equation (2), the right hand side of the second equation designates the difference between the motor speed and rotational speed of the first pipe. The physical parameters of the model (1)-(2) are: G is the shear modulus of the drillstring steel and E the elasticity Young's modulus. Then the wave speeds can be expressed by $c = \sqrt{E/\rho}$ and $\tilde{c} = \sqrt{G/\rho}$ and J the inertia $J = M\,r^2$ where r is taken as the averaged radius of drillpipe and Γ is the averaged section of the drillpipe and Σ is the quadratic momentum. Those parameters are taken following the numerical settings presented in the Appendix. The nonlinear aspect of the model is considered by taking functions F and $\tilde{F}$ in the form: $z \mapsto pk\,z/(k^2\,z^2 + \zeta)$ where the parameters p, k, ζ are some positive integer responsible of the sharpness of the top angle of the friction force graph and p is some parameter deciding the amplitude of the friction force such that $0 < \zeta \ll 1$ and $0 < k < 1$. Moreover, the behavior of the chosen friction model is close from the empirical model: the white friction force but is more handleable, which can be very useful in experimental identifications. Note also that the proposed model can be expanded to Taylor sum, which is very important when the aim is to give accurate approximation at any fixed order. The chosen functions have

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a close behavior to the one used in [3] for modelling the friction. The second originality of the present contribution is an analytical study of the drilling model as functional differential equations of neutral type and based on the qualitative theory; Center Manifold Theorem [5] and Normal Forms Theory [11]. Indeed, most of the references concerned by partial differential equations (PDE) or Delay differential equations (DDE) models for the drilling problem have a numerical analysis character. In this work, we reduce the considered PDE model to a singularly perturbed system of ordinary differential equations (ODE) in Bogdanov-Takens like configuration (double zero eigenvalues). Furthermore, to the best of the authors knowledge, this type of singularity has never been studied for NDDE depending on parameters; thus we extend the methodology for computing the center manifold which allows us to establish the linear stability and the bifurcation elements. Similar results can be found in [12] with the analysis of a physiological control model of DDE with double-zero eigenvalue singularity. This study has the same spirit as the results of Bogdanov and Takens for ODEs. We refer the reader to [14], [11] for elements on the qualitative theory of differential equations and Bifurcation theory.

The remaining part of the paper is organized as follows. The second section is concerned by preliminaries, we describe the standard procedure for reducing the PDE drillstring model to a Neutral Delay differential equations (NDDE). In the third section, entitled Dynamics Analysis, we establish linear stability analysis and apply bifurcation results for Bogdanov-Takens singularity. The methodological scheme described in [1], [4] is extended to the study of the parametrized model of neutral type. For the sake of self-containment, we report in the Appendix a table for the numerical settings for the parameters used in (1)-(2). We refer the reader to [4], [27] for the outlines of a methodology enabling to approximate a system of NDDE by a system of ordinary differential equations (Center Variety) and then to [12] for the study of local bifurcations (Normal Forms) for FDE.

II. Preliminaries

For the sake of self-containment, in the sequel we describe a standard procedure allowing to transform the considered PDE model to a delay system of Neutral type. To the best of the authors knowledge, this was presented for the first time in [6], see also [2] and [16].

A. Axial vibrations

First, let us consider the subsystem (1).
The change of variables $\xi = t + c\,s$ and $\eta = t - c\,s$ gives from the top equation of (1) $\partial^2{}_{\xi\eta}U(t,s) = 0$ which leads to separate the variables, i.e. $U(\xi,\eta) = \varphi(\xi) + \psi(\eta)$.
Then

$$\partial_t U = \frac{\partial}{\partial \xi}(\varphi) + \frac{\partial}{\partial \eta}(\psi), \text{ and } \partial_s U = c\,\frac{\partial}{\partial \xi}(\varphi) - c\,\frac{\partial}{\partial \eta}(\psi)$$

Substituting this into the two last equations (boundary conditions) of (1) and introducing τ such that $\tau = c\,L$ we obtain

$$\frac{E\,\Gamma}{\alpha}\partial_s U(t,0) = \partial_t U(t,0) - \frac{1}{\alpha}H(t). \tag{3}$$

To simplify notations, from now we adopt $(\dot{\cdot})$ for $(\partial_t = \frac{d}{dt})$.

$$H(t) = (\alpha - c\,E\,\Gamma)\dot\varphi(t) + (\alpha + c\,E\,\Gamma)\dot\psi(t) = A\,\dot\varphi(t) + B\,\dot\psi(t) \tag{4}$$

such that $A = \alpha - c\,E\,\Gamma$ and $B = \alpha + c\,E\,\Gamma$.

$$\begin{aligned} M\,\ddot U(t,L) &= M\,\ddot\varphi(t+\tau) + M\,\ddot\psi(t-\tau) \\ &= -E\,\Gamma\,c\,\dot\varphi(t+\tau) + E\,\Gamma\,c\,\dot\psi(t-\tau) \\ &\quad + F(\dot\varphi(t+\tau) + \dot\psi(t-\tau)). \end{aligned} \tag{5}$$

Let v be the axial vibration at the bit

$$v(t) = U(t,L) = \varphi(t+\tau) + \psi(t-\tau), \tag{6}$$

which gives

$$\varphi(t) = v(t-\tau) - \psi(t-2\,\tau). \tag{7}$$

Equality (4) with (7) give

$$H(t-\tau) = A\,(\dot v(t-2\,\tau) - \dot\psi(t-3\,\tau)) + B\dot\psi(t-\tau). \tag{8}$$

Equality (5) with (7) give

$$M\,\ddot v(t) = -E\,\Gamma\,c\,\dot v(t) + 2\,E\,\Gamma\,c\,\dot\psi(t-\tau) + F(\dot v(t)). \tag{9}$$

This last equality gives

$$\dot\psi(t-\tau) = \frac{M\,\ddot v(t) + E\,\Gamma\,c\,\dot v(t) - F(\dot v(t))}{2\,E\,\Gamma\,c}. \tag{10}$$

One obtains the first neutral delay differential equation by substituting (10) into (8):

$$\begin{aligned} \ddot v(t) - \frac{A}{B}\ddot v(t-2\tau) &= -\frac{E\,\Gamma\,c}{M}\,\dot v(t) - \frac{A\,E\,\Gamma\,c}{MB}\,\dot v(t-2\tau) \\ &\quad + \frac{1}{M}F(\dot v(t)) - \frac{A}{B\,M}F(\dot v(t-2\tau)) \\ &\quad + \frac{2\,E\,\Gamma\,c}{B\,M}\,H(t-\tau) \end{aligned} \tag{11}$$

B. Torsional vibrations

Let us deal with the torsional vibrations and consider the subsystem (2). We use the change of variables $\alpha = t + \tilde c\,s$ and $\beta = t - \tilde c\,s$ and we separate the variables $\Phi(\alpha,\beta) = \tilde\varphi(\alpha) + \tilde\psi(\beta)$. Then

$$\partial_t\Phi = \frac{\partial}{\partial\alpha}(\tilde\varphi) + \frac{\partial}{\partial\beta}(\tilde\psi)$$

and

$$\partial_s\Phi = \tilde c\,\frac{\partial}{\partial\alpha}(\tilde\varphi) - \tilde c\,\frac{\partial}{\partial\beta}(\tilde\psi).$$

Now let w be the torsional vibration at the bit, by the same way as for the axial vibrations, we obtain

$$\begin{aligned} \ddot w(t) - \frac{C}{D}\ddot w(t-2\,\tilde\tau) &= -\frac{\tilde c\,G\,\Sigma}{J}\,\dot w(t) - \frac{\tilde c\,C\,G\,\Sigma}{D\,J}\,\dot w(t-2\tilde\tau) \\ &\quad + \frac{1}{J}\tilde F(\dot v(t)) - \frac{C}{D\,J}\tilde F(\dot v(t-2\tilde\tau)) \\ &\quad + \frac{2\tilde c\,G\,\Sigma}{D\,J}\,\Omega(t-\tilde\tau) \end{aligned} \tag{12}$$

such that $C = \beta - \tilde{c}\, G\,\Sigma$ and $D = \beta + \tilde{c}\, G\,\Sigma$. Now let us consider the two obtained neutral equations (11)-(12). To obtain the NDDE dimensionless form of (1)-(2) one may adopt the following units of length, time and torque the quantities L, $T = L/c$ and $E\Gamma/L$, thus system (11)-(12) is written

$$\begin{cases} \ddot{v}(t) - \dfrac{\alpha-1}{\alpha+1}\ddot{v}(t-2) = \\ -\dfrac{1}{M}\,\dot{v}(t) - \dfrac{\alpha-1}{M\,(\alpha+1)}\,\dot{v}(t-2) + \dfrac{2}{M\,(\alpha+1)}\,H(t-1) \\ +\dfrac{1}{M}F(\dot{v}(t)) - \dfrac{\alpha-1}{M\,(\alpha+1)}F(\dot{v}(t-2)) \\ \ddot{w}(t) - \dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma}\ddot{w}(t-2\,\tilde{\tau}) = \\ -\dfrac{\tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,J}\,\dot{w}(t) - \dfrac{\tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,J}\dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma}\,\dot{w}(t-2\tilde{\tau}) \\ +\dfrac{1}{J}\tilde{F}(\dot{v}(t)) - \dfrac{c\,E\,\Gamma\,\beta - \tilde{c}\,G\,\Sigma}{J\,(c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma)}\tilde{F}(\dot{v}(t-2\tilde{\tau})) \\ +\dfrac{2\tilde{c}\,G\,\Sigma}{J\,(c\,E\,\Gamma\,\beta + \tilde{c}\,G\,\Sigma)}\,\Omega(t-\tilde{\tau}) \end{cases} \tag{13}$$

where $\tilde{\tau}$ is the ratio of the speeds $\tilde{\tau} = \frac{\tilde{c}}{c}$.

III. Dynamics Analysis of the Uncontrolled Drilling System

Let us consider the normalized and uncontrolled system, i.e. $\Omega = H = 0$.

A. Linearized Stability and Bifurcation Analysis

Let denote by x_1 the axial vibrations speed $x_1 = \dot{v}$ and by x_2 the angular vibrations speed $x_2 = \dot{w}$ and adopt the matrix representation of the linear part of the above system where $x = (x_1, x_2)^T$

$$\begin{cases} \dot{x}(t) = D_1\,\dot{x}(t-2) + D_2\,\dot{x}(t-\dfrac{2\tilde{c}}{c}) + A_0\,x(t) + A_1\,x(t-2) \\ \qquad + A_2\,x(t-\dfrac{2\tilde{c}}{c}) + \mathcal{F}(x(t), x(t-2), x(t-\dfrac{2\tilde{c}}{c})) \end{cases} \tag{14}$$

where $\mathcal{F}$ is the nonlinear part of the system (13)

$$D_1 = \begin{bmatrix} d_{1,1,1} & 0 \\ 0 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 0 \\ 0 & d_{2,2,2} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} a_{0,1,1} & 0 \\ a_{0,2,1} & a_{0,2,2} \end{bmatrix},\ A_1 = \begin{bmatrix} a_{1,1,1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ a_{2,2,1} & a_{2,2,2} \end{bmatrix}$$

where the matrices coefficients are $d_{1,1,1} = \frac{\alpha-1}{\alpha+1}$, $d_{2,2,2} = \frac{c_1 E(\Gamma)\,\beta - c_2 G\Sigma}{c_1 E(\Gamma)\,\beta + c_2 G\Sigma}$, $a_{0,1,1} = \frac{pk-\zeta}{M\zeta}$, $a_{0,2,1} = \frac{a_2 k}{J\zeta}$, $a_{0,2,2} = -\frac{c_2 G\Sigma}{c_1 E(\Gamma)\,J}$, $a_{1,1,1} = -\frac{(\alpha-1)(\zeta+pk)}{(\alpha+1)M\zeta}$, $a_{2,2,1} = -\frac{pk(c_1 E(\Gamma)\,\beta - c_2 G\Sigma)}{\zeta\,(c_1 E(\Gamma)\,\beta + c_2 G\Sigma) J}$, $a_{2,2,2} = -\frac{c_2 G\Sigma\,(c_1 E(\Gamma)\,\beta - c_2 G\Sigma)}{J c_1 E(\Gamma)\,(c_1 E(\Gamma)\,\beta + c_2 G\Sigma)}$.

Recall that in the above quoted references (concerned by PDE models), the studies were concerned only by the torsional vibrations. Thus the associated NDDE (governing the speed of such vibrations) is scalar, which is easier to study compared with (14). And for the physiological model considered in [12], $A_2 = D_i = 0$ for $i \in \{1, 2\}$ since the model is DDE with one delay.

Setting the numerical values of the physical parameters given in the Appendix and the parameter p (the parameter deciding for the amplitude of the friction forces F and $\tilde{F}$) is left free allows us to establish the following result.

Proposition 1:

- When α is left free and $\alpha = 30\,p$ then zero is an eigenvalue of algebraic and geometric multiplicity 1. Moreover, zero is the only eigenvalue with zero real part and the remaining eigenvalues are with negative real parts. We then have a Pitchfork-like bifurcation occurring in ODE, which comes from the Z2 symmetry structure of the system.
- When α takes its physical values α_c given in the appendix and $\alpha_c = 30\,p_c$ ($p_c = 6.6667$), then zero is an eigenvalue of algebraic multiplicity 2 and of geometric multiplicity 1. Zero is the only eigenvalue with zero real part and the remaining eigenvalues are with negative real parts. The zero eigenvalue is non-semisimple and the singularity is of Bogdanov-Takens like, see [11].
- The system (14) is formally stable but not asymptotically stable (although there are no characteristic roots with positive real parts).

Analogously to [12] which considers a singular delay system linearly dependent on a parameter, and in the same spirit to the decomposition established in [7] in the goal of computing the normal form for delay systems depending on a parameter, we extend the scheme of computing the center manifold to the case of NDDE depending on parameters and thus look for the system (14) as a perturbation of

$$\frac{d}{dt}\mathcal{D}\,x_t = \mathcal{L}_0\,x_t, \quad \text{where} \qquad \mathcal{L}_0 = \mathcal{L}|_{\{p=p_c,\,\mu=0\}} \tag{15}$$

Indeed, system (14) can be written as

$$\begin{aligned} \frac{d}{dt}\mathcal{D}\,x_t &:= \mathcal{L}_0\,x_t + \tilde{\mathcal{F}}_{\mu,p_\epsilon}(x_t) \\ &= \mathcal{L}_0\,x_t + (\mathcal{L} - \mathcal{L}_0)\,x_t + \mathcal{F}_{\mu,p_\epsilon}(x_t) \end{aligned} \tag{16}$$

such that

$$\mathcal{F}_{\mu,p} = \begin{bmatrix} -0.006750\,px_1^3(t) + 0.006682px_1^3(t-2) \\ -1.875\,px_1^3(t) + 1.874998\,px_1^3(t-1.264911064) \end{bmatrix}$$

Here we follow the theoretical schemes briefly presented in [4], [12] and give computations steps for the equation of the evolution of the problem's solutions on the center variety for system (15).

First, we compute the basis of the generalized eigenspace corresponding to the double eigenvalue $\lambda_0 = 0$.

$$\Phi(\theta) = \begin{bmatrix} 1-\theta & 1 \\ 104351600 - 104351600\,\theta & 104351600 \end{bmatrix},$$

where $\theta \in [-2, 0]$. Recall that the adjoint linear equation associated to (14) is

$$\begin{cases} \dot{u}(t) = D_1\dot{u}(t+2) + D_2\dot{u}(t+\dfrac{2\tilde{c}}{c}) \\ \qquad - A_0u(t) - A_1u(t+2) - A_2u(t+\dfrac{2\tilde{c}}{c}) \end{cases} \tag{17}$$

with a basis for the generalized eigenspace associated to the double eigenvalue zero is given by

$$\Psi(\theta) = \begin{bmatrix} -0.5025082 + 0.005025011\,\xi & 0 \\ 1.004179 + 0.4924583\,\xi & 0 \end{bmatrix}, \xi \in [0, 2].$$

The associated bilinear form is

$$\begin{aligned}(\psi, \varphi) =& \psi(0)(\varphi(0) - D_1\varphi(-2) - D_2\varphi(-1.264911)) \\ &+ \int_{-2}^{0} \psi(\xi+2)A_1\varphi(\xi)d\xi \\ &+ \int_{-1.264911}^{0} \psi(\xi+1.264911)A_2\varphi(\xi)d\xi \\ &- \int_{-2}^{0} \psi'(\xi+2)D_1\varphi(\xi)d\xi \\ &- \int_{-1.264911}^{0} \psi'(\xi+1.264911)D_2\varphi(\xi)d\xi.\end{aligned}$$

By the introduced bilinear form we can easily check that $(\Psi, \Phi) = I_d$, thus the space C can be decomposed as $C = P \bigoplus Q$, where $P = \{\varphi = \Phi z;\ z \in \mathbb{R}^2\}$ and $Q = \{\varphi \in C;\ (\Psi, \varphi) = 0\}$. Recall that each of those subspaces is invariant under the semigroup $T(t)$ and that the matrix B (introduced in the previous section concerned by the theoretical settings) satisfying $\mathcal{A}\Phi = \Phi B$ is given by

$$B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{18}$$

Let us first set the following decomposition $x_t = \Phi y(t) + z(t)$ where $z(t) \in Q$ and $y(t) \in \mathbb{R}^2$, $z(t) = h(y(t))$ and h is some analytic function $h : P \to Q$. Thus the explicit solution on the center manifold can be obtained by the use of the proven formula in [4], [12] that is

$$\dot{y}(t) = By(t) + \Psi(0)\mathcal{F}[\Phi(\theta)y(t) + h(\theta, y(t))] \tag{19}$$

$$\begin{aligned}&\frac{\partial h}{dy}\{By + \Psi(0)\mathcal{F}[\Phi(\theta)y + h]\} + \Phi(\theta)\Psi(0)\mathcal{F}[\Phi(\theta)y + h] \\ &= \begin{cases} \dfrac{\partial h}{d\theta}, & -2 \le \theta \le 0 \\ \mathcal{L}(h(\theta, y)) + \mathcal{F}[\Phi(\theta)y + h(\theta, y)], & \theta = 0 \end{cases}\end{aligned} \tag{20}$$

where $h = h(\theta, y)$ and $\tilde{\mathcal{F}}$ is defined in (16).

Simple computations show that for (15), the evolution of solutions on the center manifold is determined by solving (20) (restricted to $p = p_c$, $\mu = \mu_c = 0$) for $h(\theta, y)$ and then (19) for $y(t)$ (this is done order by order of truncation). It is of important note that $\mathbb{F}$ is an odd function. This fact implies that there is no need to compute h. Thus, the third order ODE reduction of the system (13) at $p = p_c$, $\mu = \mu_c = 0 \Rightarrow \alpha = \alpha_c$ is given by

$$\begin{aligned}\dot{y}(t) =& \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &+ \begin{bmatrix} \begin{pmatrix} -0.5818668\, {y_1}^3 - 0.5366406\, {y_1}^2 y_2 \\ -0.1336538\, y_1 {y_2}^2 + 0.0002250037\, {y_2}^3 \end{pmatrix} \\ \begin{pmatrix} 1.162764\, {y_1}^3 + 1.072387\, {y_1}^2 y_2 \\ +0.2670850\, y_1 {y_2}^2 - 0.0004496326\, {y_2}^3 \end{pmatrix} \end{bmatrix}\end{aligned} \tag{21}$$

Since our aim is to study the parameter bifurcations, the computation of the evolution equation of of the problem's solutions on the center variety for system (16) is required. The same principle being applied (i.e. formulas (19) and (20) are used) with a change in the expression of $\mathcal{F}$; indeed, the above approximation is made for the system (15) for which $p = p_c$, but for the following approximation for (16), p is taken $p = p_c + \mu$ where μ is a small parameter. In the next step, we introduce a small parameter r as a scaling parameter for making a zoom into the neighborhood of the singularity. We introduce the following changes of coordinates $\{\mu := 1326.69991\,\gamma\, r^2, p = p_c + \mu, y_1 = r^2 z_1, y_2 = r z_2\}$ and we scale the time by $t_{old} = r\, t_{new}$ which allows us to the following cubic normal form reduction of (14),

$$\begin{cases} \dot{z}_1 = \gamma\, z_1 - z_1 {z_2}^2 \\ \dot{z}_2 = -z_1 \end{cases} \tag{22}$$

IV. Concluding Remarks

The main purpose of this paper is a qualitative analysis of a PDE drilling model based on a time-delay system approach. In this study, we establish the linear stability analysis of the steady-state at the origin for system (14) as well as a qualitative approximation of solutions of the infinite dimensional system (PDE) by solutions of a finite dimensional one (ODE). For system (22), we establish a first integral $I(z_1, z_2) = (z_1 + \gamma\, z_2 - \frac{1}{3}\, z_2^3)^2$ leading to a Lyapunov function $V(z_1, z_2) = z_1^2 + I(z_1, z_2)$ asserting that the system is locally asymptotically stable in the attraction region $D = \{(z_1, z_2), \text{such that} \gamma < z_2^2\}$. Our future aim will concern the design of appropriate control laws guaranteeing a desired drilling dynamical behavior.

References

[1] M. Ait Babram, O. Arino, and M. L. Hbid. Computational scheme of a center manifold for neutral functional differential equations. *J. Math. Anal. Appl.*, 258(2):396–414, 2001.

[2] A. G. Balanov, N.B. Janson, P.V.E. McClintock, and C. H. T. Wang. Bifurcation analysis of a neutral delay differential equation modelling the torsional motion of a driven drill-string. *Chaos, Solitons and Fractals*, 15:381–394, 2002.

[3] D. A. W. Barton, B. Krauskopf, and R. E. Wilson. Nonlinear dynamics of torsional waves in a drill string model with spacial extent. *Journal of Vibration and Control*, 16:1049–1065, 2007.

[4] S.A. Campbell. Calculating centre manifolds for delay differential equations using maple. In *In Delay Differential Equations: Recent Advances and New Directions*. Springer-Verlag, 2009.

[5] J. Carr. *Application of Center Manifold Theory*. Springer, 1981.

[6] K. Cooke et D. Krumme. Differential-difference equations and nonlinear initial boundary value problems for linear hyperbolic partial differential equations. *J. Math. Anal. Appl.*, 24:372–387, 1968.

[7] T. Faria and L.T. Magalhães. Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation. *J. Differential Equations*, 122(2):181–200, 1995.

[8] E. Fridman, S. Mondié, and B. Saldivar. Bounds on the response of a drilling pipe model. *IMA Journal of Mathematical Control and Information*, pages 1–14, 2010.

[9] P. Fu, S-I. Niculescu, and J. Chen. On the Stability of Linear Delay-Differential Algebraic Systems: Exact Conditions via Matrix Pencil Solutions. *IEEE Transaction on Automatic Control*, 51:1063–1069, 2006.

[10] C. Germay, N. Van De Wouw, H. Nijmeijer, and R. Sepulchre. Nonlinear drilling dynamics analysis. *SIAM J. Dynamical Systems*, 8(2):527–553, 2005.

[11] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcation of vector fields*. Springer, 2002.

[12] J. K. Hale and W. Huang. Period doubling in singularly perturbed delay equations. *Journal of Differential Equations*, 114:1–23, 1994.

[13] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*, volume 99 of *Applied Mathematics Sciences*. Springer Verlag, New York, 1993.

[14] Y. Kuznetsov. *Elements of applied bifurcation theory; Second edition*, volume 112 of *Applied Mathematics Sciences*. Springer, New York, 1998.

[15] W. Michiels and S-I. Niculescu. *Stability and stabilization of time-delay systems*, volume 12 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. An eigenvalue-based approach.

[16] H. Mounier. Propriétés structurelles des systèmes linéaires a retard : Aspects théorique et pratique. *Thèse de l'Université Paris-Sud*, (3888):148, 1995.

[17] E. M. Navarro-López and D. Cortés. Sliding-mode control of a multi-dof oilwell drillstring with stick-slip oscillations. *Proceeding of the 2007 American Control Conference*, pages 3837 – 3842, 2007.

[18] E. M. Navarro-López and R. Suárez. Modelling and analysis of stick-slip behavior in a drillstring under dry friction. *Congreso anual de la AMCA 2004*, pages 330 – 335, 2004.

[19] E. M. Navarro-López and R. Suárez. Practical approach to modelling and controlling stick-slip oscillations in oilwell drillstring. *Proceeding of the 2004 IEEE, International Conference on Control Applications*, 10(5):3162 – 3174, 2004.

[20] Eva M. Navarro-López. An alternative characterization of bit-sticking phenomena in a multi-degree-of-freedom controlled drillstring. *Nonlinear Analysis: Real World Applications*, 10(5):3162 – 3174, 2009.

[21] S-I. Niculescu. Delay Effects on Stability, A Robust Control Approach. *Springer, Lecture Notes in Control And Information Sciences*, 2001.

[22] V. Rasvan and S-I. Niculescu. Oscillations in lossless propagation models: a liapunov-krasovskii approach. *IMA J. of Math. Contr. and Info.*, 19:157–172, 2002.

[23] Th. Richard, Ch. Germay, and E. Detournay. A simplified model to explore the root cause of stick-slip vibrations in drilling system with drag bits. *Appl. Math. Comput.*, 305:432–456, 2007.

[24] P. Rouchon. Flatness and stick-slip stabilization. *Technical Report*, 492:1–9, 1998.

[25] B. Saldivar, S. Mondié, J-J. Loiseau, and V. Rasvan. Stick-slip oscillations in oillwell drilstrings: Distributed parameter and neutral type retarded model approaches. *IFAC 18th World Congress Milano (Italy) August 28 - September 2, 2011*, pages 283–289, 2011.

[26] M. Weedermann. Normal forms for neutral functional differential equations. In *Topics in functional differential and difference equations (Lisbon, 1999)*, volume 29 of *Fields Inst. Commun.*, pages 361–368. Amer. Math. Soc., Providence, RI, 2001.

[27] M. Weedermann. Hopf Bifurcation calculations for scalar delay differential equations. *Nonlinearity*, 19:2091–2102, 2006.

APPENDIX

A. Graphical Illustration

The projection of the dynamics on the center manifold for the critical value $p = p_c$ is given in Figure 1 and $p = p_c + \mu$ is given in Figure 2 and 3. Figure 4 gives the state z_1 response for various values of γ.

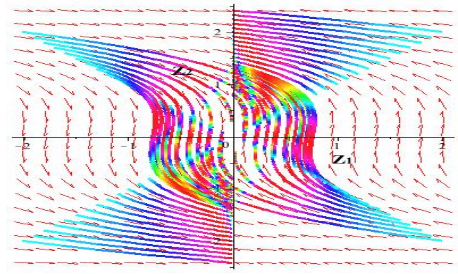

Fig. 1. Phase portrait of the system $(22)|_{\gamma=0}$ that is $p = p_c$

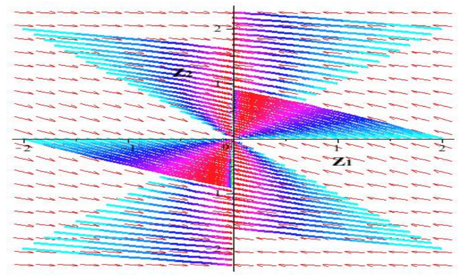

Fig. 2. Phase portrait of the system $(22)|_{\gamma=-2}$ that is $p = p_c + \mu$

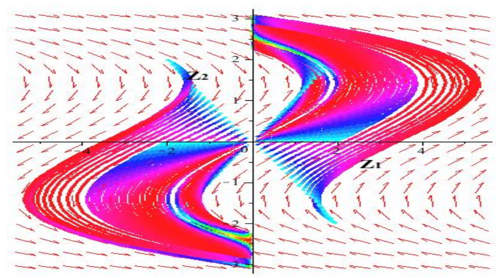

Fig. 3. Phase portrait of the system $(22)|_{\gamma=2}$ that is $p = p_c + \mu$

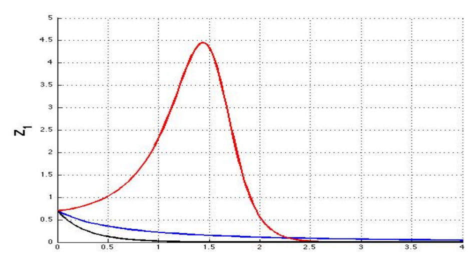

Fig. 4. The State response z_1 for (22) (red) $\gamma = 2$, (black) $\gamma = 0$ and (blue) $\gamma = -2$

B. Numerical Settings

Parameter	Value	Parameter	Value
G	80 GPa	E	200 GPa
ρ	8000 Kg/m^3	r	6 cm
Γ	35 cm^2	Σ	19 cm^4
L	3000 m	M	40000 Kg
α	200,025 kg/s	β	2000 Nms
k	0,3	ζ	0,01