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Constrained Power Control in Mobile Wireless Networks using Nonlinear Control Techniques

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Abstract— **The paper approaches the problem of power control in mobile wireless networks. We aim at solving the trade-off between conflicting constraints on the bounds of the two relevant variables of the system - the power and the signal-to-interference ratio (SIR) - that is to keep the SIR at a high value with a low power. One of the challenges refers to the partial controlability of the system. We solve the control problem by taking advantage of the inherent system nonlinearities.**

I. Introduction

It is well-known that a wireless network model consists of a number of nodes that interact with each other, [1], [2]. Each node (receiver) is characterized by γ_i, the ratio between the signal and the interference (SIR) with the fellow nodes:

$$\gamma_i = \frac{G_{ii} p_i}{\eta_i + \sum_{j \neq i} G_{ij} p_j} = \frac{signal}{interference} \quad (1)$$

$i, j = 1,..N$, $G_{ii}, G_{ij} > 0$, $p_i, p_j > 0$, where the signal $G_{ii} p_i$ is given by the transmitted power of the i-th link transmitter, p_i, and the i-th link channel gain, G_{ii}, and the interference

$$d_i = \eta_i + \sum_{j \neq i} G_{ij} p_j \quad (2)$$

depends on the gains of the channel between the i-th link receiver and the j-th link transmitter, G_{ij}, on the powers p_j and on the power of the additive white Gaussian noise (AWGN) that corrupts the signal at the i-th link receiver, η_i. We denoted by N the number of nodes the network should accommodate in a given time.

The power control problem comes with *constraints* on p_i and γ_i, as follows:

$$0 < p_i^m \leq p_i(t) \leq p_i^M, \quad 0 < \gamma_i^m \leq \gamma_i(t) \leq \gamma_i^M(t) \quad (3)$$

where p_i^m, p_i^M, γ_i^m and γ_i^M may be different for each node of the network. It means that the physical limitation on the power saturates the power to a maximum value and the node is non-functional for $\gamma_i < \gamma_i^m$.

Since the SIR is a measure of the link quality, performance requires the maximum possible SIR value. This comes at the expense of an increase of the power. Still, the users autonomy (i.e. the battery life time) requires to minimize the power consumption. Therefore, the *trade-off* required in wireless networks is related to the logical reasoning that if a good SIR may be reached with a lower power, there is no reason to increase the power more.

Our efforts to consider both the constraints and tradeoff in a unified manner have started in [3] yet, the disturbance rejection methodology imposes too strict limitations on the problem specifications. Our second approach [4] has other shortcomings related to the complexity of the control law. This paper continues the ideas presented in [3] and [4] aiming at finding a more simple and flexible control law without predicting or measuring the channel gain variation. We implement a bounded control as a smooth saturation control using a cubic function. We expect this control law to be easily extended to solve the admissibility problem without re-designing the control law.

The rest of the paper is organized as follows. In the next section we present a brief review of the most relevant results from the literature and the control objectives. Section III presents some issues in networks with mobile users. Section IV describes the proposed control law. Numerical simulations are shown in Section V. The paper is completed by conclusions.

Notations: The symbol $\hookrightarrow$ stands for ”requires”. We denote the equilibrium value of the variable by a bar over it, i.e. $\bar{x}$ stands for the value of $x(t)$ at the equilibrium.

II. Problem statement

There is abundant literature on the power control problem in wireless network. The old seminal papers [1], [2] are relevant for the issues of the wireless networks. It is to be noted that not all the issues were solved till now. The early control schemes approached the centralized power control focusing on proving the convergence of the power updating algorithm towards a pareto-optimal value of the power, [1], [2], [5], [6]. Still, since the interference d_i is not available to the i-th node, the centralized control schemes are not viable solutions. Thus, distributed power control schemes were proposed. They are realistic control solutions, yet they always demand a high complexity, [2], [6], [7].

Usually, for each node, the *control problem* is stated as follows: given the reference SIR, γ_i^M, implement a decentralized controller, to adjusts the power, using the actual SIR, γ_i, as feedback while coping with the interference, d_i, given the following assumption:

Assumption 1: The i-th node controller can read the values of γ_i, G_{ii}, p_i.

Remark II. 1: Since p_j and G_{ij} are not accessible to the i-th node, the interference d_i is unknown to the $i-th$ node.

The Foschini-Miljanic algorithm was the first one to provide distributed on-line power control for ad-hoc networks with user-specific SIR requirements. Most of the distributed control schemes approach the control problem from an empirical perspective. In order to reduce complexity, most of the distributed control schemes assume 1). a specified target SIR, 2). the channel gains are constant variables and 3). the number of users is assumed to be a constant. On a higher level of complexity, in [8] and [9] a wireless network with non-deterministic channel gains is studied using a probabilistic mathematical formalism. Some of the shortcomings are related to the assumption that the powers are random variables. In addition, the non-stationarity of the channels makes difficult to construct an admissions control algorithm from a stochastic approximation algorithm.

A new trend of the literature is given by the papers which approach the power control from a systems theory perspective, e.g. [10], [11], [12], [13], [14]. In [14] the problem of users mobility is taken into account through the robustness framework. Still, it does not try to maximize the SIR. In [10] a stochastic model which takes into account the users mobility is considered and the SIR is maximized using an optimization algorithm. The network dynamics is described by linear state-space models and quadratic control strategies are used to jointly control the power and SIR. The mobility of users is treated as uncertainty. Besides, an explicit formula for the taget SIR variation is given which takes into account the network congestion, as well. In [15] the problem of finding simultaneously both an optimal SIR and an optimal power, coping with mobile users was taken into account. In this paper we solve the *Problem:* Given a wireless network with mobile users ($\dot{G}_{ij} \neq 0$) with the *constraints* (3), find γ_i^M and the decentralized power control able to implement the *tradeoff*: maximize the SIR γ_i^M while the power p_i is kept low.

III. MOBILE USERS, $\exists i, j\ \dot{G}_{ij} \neq 0$

A. The node model

Since the gains G_{ii}, G_{ij} are related to distances between users and bases, when the users mobility is taken into account, the wireless network model transforms into a nonlinear parameter varying dynamical system, where some of the parameters are unmeasurable.

The continuous time dynamic model of a mobile node is estimated by taking the derivative of equation (1) with respect to time:

$$\begin{cases} \dot{p}_i = u_i \\ \dot{\gamma}_i = \frac{u_i}{p_i}\gamma_i + \varrho_{ii}\gamma_i - \frac{\delta_i}{p_i}\gamma_i^2 \end{cases} \tag{4}$$

where:

$$\begin{array}{l} \dot{d}_i = \dot{\eta}_i + \sum_{j \neq i} G_{ij}(\varrho_{ij}p_j + u_j) \\ \varrho_{ii} = \frac{\dot{G}_{ii}}{G_{ii}}, \quad \delta_i = \frac{\dot{d}_i}{G_{ii}}, \quad \varrho_{ij} = \frac{\dot{G}_{ij}}{G_{ij}} \end{array} \tag{5}$$

with $\dot{G}_{ii}$ and $\dot{d}_i$ unknown, and u_i is the control function to be designed. Note that since the gains are given by the inverse fourth power law, [5], i.e. $G_{ij}(t) = \epsilon(\Delta_{ij}^4(t))^{-1}$ where Δ_{ij} is the distance between the i-th base and the j-th node, it turns out that both $\dot{G}_{ii}$ and $\dot{d}_i$ depend on the velocity of the users. Therefore, it is realistic to assume some bounds

$$\varrho_{ii} \in [\nu_m, \nu_M]; \quad \delta_i \in [\delta_m, \delta_M] \tag{6}$$

on these variables, where $\nu_m, \delta_m < 0$, $\nu_M, \delta_M > 0$.

Assumption 2: A stationary AWGN is considered ($\dot{\eta}_i = 0$).

Remark III.1: By replacing $\dot{d}_i = \dot{G}_{ii} = 0$ in (4), we get:

$$\begin{array}{l} \dot{p}_i = \lim_{\tau \to 0} = \frac{p_i(t+\tau) - p_i(t)}{\tau} = \lim_{\tau \to 0} \frac{1}{\tau} \frac{\gamma_i^M(t) - \gamma_i(t)}{\gamma_i(t)} p_i(t) = \\ = \lim_{\tau \to 0} \frac{\gamma_i(t+\tau) - \gamma_i(t)}{\tau} \frac{p_i}{\gamma_i} = \frac{\dot{\gamma}_i}{\gamma_i} p_i \end{array} \tag{7}$$

In this case, since the parameters variations continuously change the network equilibrium, it makes sense to continuously update the target SIR γ_i^M in relation with the equilibria of the network, which are as follows:

$$\begin{array}{l} \overline{\gamma}_{1_i} = 0; \\ \overline{\gamma}_{2_i} = \frac{p_i}{\delta_i}(\varrho_{ii} + \frac{u_i}{p_i}) = \kappa_i p_i + \frac{1}{\delta_i} u_i, \quad \kappa_i = \varrho_{ii}/\delta_i \end{array} \tag{8}$$

Remark III.2:

a). If $\dot{p}_i = u_i = 0$ then $\overline{\gamma}_{2_i} = \kappa_i \bar{p}_i$, thus the equilibrium follows the perturbation.

b). $\overline{\gamma}_{2_i}$ refers to a family of equilibrium points. Since $\overline{\gamma}_{2_i}$ is a function of the unknown variables $\dot{G}_{ii}$ and $\dot{d}_i$ it is highly likely that $\kappa_i < 0$ may happen. The control law should cope with this critical case, too.

c).The equilibrium point, $\bar{\gamma}_{1_i} = 0$, is undesirable because it makes the wireless node non-functional. Therefore, $\bar{\gamma}_{1_i}$ should be always repulsive. The equilibrium point $\bar{\gamma}_{2_i}$ must be restricted from becoming zero or negative for the same reason. The stability of $\overline{\gamma}_{1_i}$ and $\overline{\gamma}_{2_i}$ are opposite because:

$$\frac{\partial}{\partial \gamma_i}(\dot{\gamma}_i)|_{\bar{\gamma}_i = 0} = \varrho_{ii} + \frac{u_i}{p_i} = -\frac{\partial}{\partial \gamma_i}(\dot{\gamma}_i)|_{\bar{\gamma}_i = \bar{\gamma}_{2_i}} \tag{9}$$

B. Stability of equilibria and limitations of the control

In the following, we show how much the control variable may influence the dynamics of the SIR variable. The dynamical system (4) has two distinct classes of equilibrium points, i.e. $(\overline{p}_i, 0)$ and $(\overline{p}_i, \kappa_i \bar{p}_i)$, see (8), where $\bar{p}_i$ are the solutions of the equation $\dot{p}_i = 0$. The stability of the equilibrium points may be read from the sum S and product P of the eigenvalues of the Jacobian of (4), which is as follows:

$$J|_{u_i=0} = \begin{bmatrix} \frac{\partial u_i}{\partial p_i} & \frac{\partial u_i}{\partial \gamma_i} \\ \frac{\gamma_i}{p_i}(\frac{\gamma_i}{p_i}\delta_i + \frac{\partial u_i}{\partial p_i}) & \varrho_{ii} + \frac{\gamma_i}{p_i}(\frac{\partial u_i}{\partial \gamma_i} - 2\delta_i) \end{bmatrix} \tag{10}$$

$$\begin{array}{ll} tr(J) = \frac{\partial u_i}{\partial p_i} + \varrho_{ii} + \frac{\gamma_i}{p_i}(\frac{\partial u_i}{\partial \gamma_i} - 2\delta_i) & (trace) \\ det(J) = (\varrho_{ii} - 2\delta_i \frac{\gamma_i}{p_i})\frac{\partial u_i}{\partial p_i} - \delta_i (\frac{\gamma_i}{p_i})^2 \frac{\partial u_i}{\partial \gamma_i} & (determinant) \end{array} \tag{11}$$

For a second order system the eigenvalues λ_k satisfy:

$$\lambda_k^2 - S\lambda_k + P = 0, \quad \lambda_{1,2} = \frac{S \pm \sqrt{\Delta}}{2} \quad \Delta = S^2 - 4P \tag{12}$$

The link between (11) and (12) is, [16]: $S = tr(J)|_{(\bar{p}_i, \bar{\gamma}_i)}$, $P = det(J)|_{(\bar{p}_i, \bar{\gamma}_i)}$.

1. For the *equilibrium points* $(\overline{p}_i, 0)$ from (11) we infer:

$$S = \frac{\partial u_i}{\partial p_i}(\overline{p}_i, 0) + \varrho_{ii}; \quad P = \varrho_{ii}\frac{\partial u_i}{\partial p_i}(\overline{p}_i, 0) \tag{13}$$

Since (12) and (13) yield $P = \varrho_{ii}(S - \varrho_{ii})$, $\Delta = S^2 - 4P = (S - 2\varrho_{ii})^2 > 0$, the eigenvalues of (4) about $(\overline{p}_i, 0)$ are:

$$\lambda_1 = S - \varrho_{ii} = \frac{\partial u_i}{\partial p_i}(\overline{p}_i, 0), \quad \lambda_2 = \varrho_{ii} \tag{14}$$

Note that λ_2 is unknown and uncontrollable. Since $\gamma_i = 0$ is undesirable, we should constrain the equilibria $(\overline{p}_i, 0)$ be unstable, either as saddles ($P < 0$) when $\varrho_{ii} < 0$ or sources ($P > 0$ & $S > 0$) when $\varrho_{ii} > 0$, by the constraint:

$$\frac{\partial u_i}{\partial p_i}(\bar{p}_i, 0) \geq 0 \tag{15}$$

2. For the *equilibrium points* $(\overline{p}_i, \kappa_i \bar{p}_i)$ from (11) we infer:

$$\begin{aligned} S &= \left(\frac{\partial u_i}{\partial p_i} + \kappa_i \frac{\partial u_i}{\partial \gamma_i}\right)_{(\overline{p}_i, \kappa_i \bar{p}_i)} - \varrho_{ii}; \\ P &= -\varrho_{ii}\left(\frac{\partial u_i}{\partial p_i} + \kappa_i \frac{\partial u_i}{\partial \gamma_i}\right)_{(\overline{p}_i, \kappa_i \bar{p}_i)} \end{aligned} \tag{16}$$

Then, (12) and (16) yield $P = -\varrho_{ii}(S + \varrho_{ii})$, $\Delta = S^2 - 4P = (S + 2\varrho_{ii})^2 > 0$ Thus, the eigenvalues of (4) about the equilibria $(\overline{p}_i, \kappa_i \bar{p}_i)$

$$\lambda_1 = S + \varrho_{ii} = \left(\frac{\partial u_i}{\partial p_i} + \kappa_i \frac{\partial u_i}{\partial \gamma_i}\right)_{(\overline{p}_i, \kappa_i \bar{p}_i)}, \quad \lambda_2 = -\varrho_{ii} \tag{17}$$

show that no matter the form of the decentralized control law u_i, if $\varrho_{ii} < 0$ the system (4) has a positive eigenvalue in the vicinity of the equilibrium $(\overline{p}_i, \kappa_i \bar{p}_i)$. Still, if $\varrho_{ii} > 0$ the equilibrium point may be a stable node if the control variable is chosen to fulfill the following inequality:

$$\frac{\partial u_i}{\partial p_i}(\overline{p}_i, \kappa_i \bar{p}_i) + \kappa_i \frac{\partial u_i}{\partial \gamma_i}(\overline{p}_i, \kappa_i \bar{p}_i) \leq 0 \tag{18}$$

Since κ_i is unknown, it is more appropriate to choose u_i so that $\frac{\partial u_i}{\partial \gamma_i}(\overline{p}_i, \kappa_i \bar{p}_i) = 0$.

Remark III.3: We conclude this section by emphasizing three facts: 1). one of the eigenvalues of (4) ($\lambda_2 = \pm\varrho_{ii}$) is uncontrollable, 2). for the mobile node (4) the control variable u_i should be able to make the equilibria $(\overline{p}_i, 0)$ repulsive and 3). the control function should be chosen to fulfill $\partial u_i / \partial \gamma_i = 0$ because κ_i in (18) is unmeasurable.

C. The control objective

Since $\dot{G}_{ii}$ and $\dot{d}_i$ are exogeneous, unmeasurable variables, they are unknown to the i-th node. We denote by h_i the unmeasurable part in the SIR dynamics:

$$h_i = \varrho_{ii} - \delta_i \frac{\gamma_i}{p_i} \quad \hookrightarrow \quad \dot{\gamma}_i = (\frac{u_i}{p_i} + h_i)\gamma_i \tag{19}$$

Thus, each node should implement a decentralized control u_i able to either reject or observe these variables. Obviously, if $|u_i| > |h_i| p_i$ the control u_i is able to implement the disturbance rejection paradigm. Yet because of the constraint (3) on p_i, the variable u_i cannot always reject the disturbance. Instead, it acts more like a perturbation in the vicinity of the equilibria. However, for a perturbed nonlinear dynamical system, by a proper calibration of the perturbation near an equilibrium point, remarkable orbits can be generated.

Here, we propose a control approach based on the equilibrium point alteration. This approach does not require $|u_i| > |h_i| p_i$ but only $p_i \in [p_m, p_M]$. To cope with the uncertainty on the equilibrium point and $\kappa_i < 0$ (Remark III.2) we design the decentralized control system to work in the vicinity of a time-varying equilibrium $\gamma_i^M(t) > 0$, assumed to be close to the operating point γ_i. Given γ_i^M, the control should minimize the error e_i

$$e_i = \gamma_i^M - \gamma_i \tag{20}$$

Note that its derivative $\dot{e}_i$

$$\dot{e}_i = \dot{\gamma}_i^M - \dot{\gamma}_i = \dot{\gamma}_i^M - \tfrac{u_i}{p_i}\gamma_i - h_i \gamma_i \tag{21}$$

cannot be computed because h_i is unmeasurable. Therefore, we require the controller to be based on a variable $\hat{\gamma}_i$:

$$\hat{\gamma}_i = \int_0^\infty \dot{\hat{\gamma}}_i dt, \qquad \dot{\hat{\gamma}}_i = \tfrac{u_i}{p_i}\gamma_i + \hat{h}_i \gamma_i \tag{22}$$

where $\hat{h}_i$ is chosen to guarantee the convergence $e_i \to 0$.

We define the offset $\tilde{\gamma}_i$ and its dynamics, as follows:

$$\tilde{\gamma}_i = \gamma_i - \hat{\gamma}_i, \qquad \dot{\tilde{\gamma}}_i = \dot{\gamma}_i - \dot{\hat{\gamma}}_i = (h_i - \hat{h}_i)\gamma_i \tag{23}$$

Proposition 1: Given (6), the convergence of the error towards zero $e_i \to 0$ is guaranteed by:

$$\hat{h}_i = \begin{cases} \nu_m - \delta_M \frac{\gamma_i}{p_i} = \hat{h}_i^- < 0 & if \quad e_i > 0 \\ 0 & if \quad e_i = 0 \\ \nu_M - \delta_m \frac{\gamma_i}{p_i} = \hat{h}_i^+ > 0 & if \quad e_i < 0 \end{cases} \tag{24}$$

Proof: The convergence $e_i \to 0$ implies $e_i \dot{e}_i < 0$, or equivalently $\dot{e}_i = -\xi_1' e_i$, $\xi_1' > 0$. Thus, (23) yields $\dot{e}_i = -\frac{\xi_1'}{\xi_1}\dot{\tilde{\gamma}}_i = -\frac{\xi_1'}{\xi_1}(h_i - \hat{h}_i)\gamma_i$. Convergence requires $e_i(h_i - \hat{h}_i) = e_i(\varrho_{ii} - \delta_i \frac{\gamma_i}{p_i} - \hat{h}_i) > 0$, therefore (24).

Remark III.4: In order for the SIR variable to fulfill (3), the control law should be designed for a translation of (4) by $\gamma_i := \gamma_i + \gamma_m$. This translation may be done into the controller at the end.

IV. The proposed control system

In our control design we use cubic functions with varying coefficients because of their remarkable features.

A. Cubic differential equations - transformations, bifurcations

Cubic differential equations with varying coefficients may change the number of equilibrium points from one to two or three, see [17], [18]. They also allow to implement a continuous time relay structure. There are three important ideas to take into account.

a). The cubic differential equation:

$$\dot{x} = x^3 - ax + b, \qquad a > 0 \tag{25}$$

undergoes three distinct structures: A: $b \in (-\frac{2a}{3}\sqrt{\frac{a}{3}}, \frac{2a}{3}\sqrt{\frac{a}{3}})$ - 3 equilibrium points; B: $b \in (-\infty, -\frac{2a}{3}\sqrt{\frac{a}{3}})$ - 1 equilibrium point; C: $b \in (\frac{2a}{3}\sqrt{\frac{a}{3}}, \infty)$ - 1 equilibrium point. For $b \in \{-\frac{2a}{3}\sqrt{\frac{a}{3}}, \frac{2a}{3}\sqrt{\frac{a}{3}}\}$ (2 equilibrium points) the cubic dynamics undergoes bifurcations.

b). Given the cubic differential equation $\dot{x} = \pm x^3 + ax + b = F(a, b, x)$ depending on the real parameters a, b, the bifurcation values of the parameters fulfill:

$$F(a, b, x) = \pm x^3 + ax + b = 0; \quad \tfrac{\partial F}{\partial x} = \pm 3x^2 + a = 0 \tag{26}$$

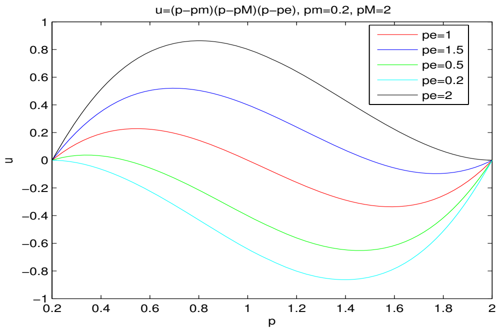

Fig. 1. Control function

Therefore, the bifurcation is defined by the equations: $a = \mp 3x^2$ and $b = \pm 2x^3$ and the equation of a cusp:

$$4a^3 = \mp 27b^2 \tag{27}$$

A *relay structure* (switching between two equilibrium points) occurs when (26) is fulfilled.

c). The quadratic term from the general cubic equation:

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0 \tag{28}$$

can be eliminated by the substitution $x := y - b/3a$. The transformed equation becomes

$$y^3 + py + q = 0, \quad p = \frac{3ac-b^2}{3a^2}, \quad q = \frac{2b^3-9abc+27a^2d}{27a^3} \tag{29}$$

B. The decentralized control law

In order to keep the value of the variable p_i bounded between p_m and p_M, see (3), we choose to implement the control law as a scalar cubic dynamical system, Fig. 1 :

$$u_i(t) = (p_i(t) - p_i^e(t))(p_i(t) - p_i^m)(p_i(t) - p_i^M) \tag{30}$$

where p_i^m and p_i^M are the given problem specifications from (3) and $p_i^e(\cdot)$ should be designed to appropriately control the magnitude and the sign of the power variation.

The equilibria of (30) are $\overline{p}_i \in \{p_i^m,\ p_i^M,\ p_i^e\}$. The stability of the equilibrium points is given by the sign of $\frac{\partial u_i}{\partial p_i}|_{\bar{p}_i}$ where:

$$\begin{array}{l} \frac{\partial u_i}{\partial p_i} = (p_i - p_i^m)(p_i - p_i^M) + \\ +(p_i - p_i^e)(p_i - p_i^M) + (p_i - p_i^e)(p_i - p_i^m) \end{array} \tag{31}$$

Remark IV.1: For (30) p_i^e is a time-varying equilibrium point.

Assumption 4: We assume $p_i^e(t)$ depends only indirectly on p_i and γ_i, ($\frac{\partial p_i^e}{\partial p_i} = 0$, $\frac{\partial p_i^e}{\partial \gamma_i} = 0$). See Remark III.3.

Proposition 2: Given (30) and $p_i^e(t) \in [p_i^m, p_i^M]$, if $p_i(0) \in [p_i^m, p_i^M]$ then $p_i(t) \in [p_i^m, p_i^M]$.

Proof: See Appendix VII-A.

Given (30) the node dynamics (4) yields six equilibrium points, as follows:

$$\begin{array}{lll} (p_i^m, 0), & (p_i^e, 0), & (p_i^M, 0) \\ (p_i^m, \kappa_i p_i^m), & (p_i^M, \kappa_i p_i^M), & (p_i^e, \kappa_i p_i^e) \end{array} \tag{32}$$

The eigenvalues of (10) for all six equilibrium points are computed in Appendix VII-B. Their analysis shows that the equilibrium points $(p_i^m, 0)$ and $(p_i^M, 0)$ are saddles when $\varrho_{ii} < 0$ and unstable nodes when $\varrho_{ii} > 0$, (since $\lambda_1 > 0$ and $\lambda_2 = \varrho_{ii}$). The equilibrium point $(p_i^e, 0)$ is a stable node if $\varrho_{ii} < 0$ and a saddle if $\varrho_{ii} > 0$ (since $\lambda_1 < 0$ and $\lambda_2 = \varrho_{ii}$). In addition, the equilibrium points $(p_i^m, \kappa_i p_i^m)$ and $(p_i^M, \kappa_i p_M)$ are unstable nodes if $\varrho_{ii} < 0$ and saddles if $\varrho_{ii} > 0$ The equilibrium point $(p_e, \kappa_i p_e)$ is a saddle if $\varrho_{ii} < 0$ and a stable node if $\varrho_{ii} > 0$. We denote by

$$e_i^p = p_i^e - p_i \tag{33}$$

the offset between the power reference $p_i^e(t)$ and the power variable $p_i(t)$. We also make the following notations:

$$\hat{P} = (p_i - p_i^m)(p_i - p_i^M), \quad P = (p_i^e - p_i^m)(p_i^e - p_i^M) \tag{34}$$

Note that $-\hat{P} > 0$ and $-P > 0$, if $p_i, p_i^e \in [p_i^m, p_i^M]$.

To constrain $p_i, p_i^e \in [p_m, p_M]$ we should require

$$-(p_i^M - p_i^m) \leq e_i^p \leq (p_i^M - p_i^m) \tag{35}$$

Empirical rules may be established as follows:

$$\begin{array}{l} e_i < 0 \hookrightarrow \dot{p}_i < 0 \hookrightarrow if\ e_i^p > 0 \hookrightarrow \dot{p}_i^e < 0 \\ e_i > 0 \hookrightarrow \dot{p}_i > 0 \hookrightarrow if\ e_i^p < 0 \hookrightarrow \dot{p}_i^e > 0 \\ e_i < 0 \hookrightarrow \dot{p}_i < 0 \hookrightarrow if\ e_i^p < 0 \hookrightarrow \dot{p}_i^e \diamond \\ e_i > 0 \hookrightarrow \dot{p}_i > 0 \hookrightarrow if\ e_i^p > 0 \hookrightarrow \dot{p}_i^e \diamond \end{array} \tag{36}$$

where an optimization is possible for the last two cases ($\dot{p}_i^e \diamond$). Thus, $\dot{p}_i^e$ should be so that:

$$sign(\dot{p}_i^e) = \begin{cases} sign(e_i) & if \quad e_i e_i^p < 0 \\ \{-1, 0, 1\} & if \quad e_i e_i^p > 0 \end{cases} \tag{37}$$

C. The target dynamics for p_i^e and γ_i^M

The local controller tries to guess γ_i^M and p_i^e by solving the convergence problem:

$$e_i \rightarrow 0, \quad e_i^p \rightarrow 0 \tag{38}$$

with the minimization of the index:

$$I = \int_0^\infty (p_i^e(t) + \lambda \frac{1}{\gamma_i^M(t)})dt \quad when \quad e_i e_i^p > 0 \tag{39}$$

where $\lambda > 0$ and the constraints (35) and:

$$e_i \dot{p}_i^e > 0 \quad when \quad e_i e_i^p < 0 \tag{40}$$

We make the following assumptions:

Assumption 5: We assume

$$\dot{e}_i = -\mu sign(e_i), \quad \dot{e}_i^p = -\chi sign(e_i^p) \tag{41}$$

where $\chi > 0$. From (41):

$$\begin{array}{l} \dot{\gamma}_i^M = -\mu sign(e_i) + \dot{\hat{\gamma}}_i + \dot{\tilde{\gamma}}_i \\ \dot{p}_i^e = sign(e_i^p)((-\hat{P})|e_i^p| - \chi) \end{array} \tag{42}$$

Note that the overall error decreases if for $\epsilon_1, \epsilon_2, \epsilon_3 > 0$:

$$\begin{array}{l} \dot{e}_i^p sign(e_i^p) + \epsilon_1 \dot{e}_i sign(e_i) + \epsilon_2 \dot{e}_i^p sign(e_i) + \epsilon_3 \dot{e}_i sign(e_i^p) = \\ = -\chi - \epsilon_1 \mu - \epsilon_2 \chi sign(e_i e_i^p) - \epsilon_3 \mu sign(e_i e_i^p) < 0 \end{array} \tag{43}$$

Proposition 3: The following parameters χ, μ:

$$\begin{array}{l} \chi = (-\hat{P})|e_i^p| + \kappa(-P)\frac{4}{(p_i^M - p_i^m)}, \ \kappa \in [-1, 1](-\hat{P})\frac{|e_i^p|}{p_i^M - p_i^m} \\ \mu(\epsilon_1 + \epsilon_3 sign(e_i e_i^p)) > -\chi(1 + \epsilon_2 sign(e_i e_i^p)) \end{array} \tag{44}$$

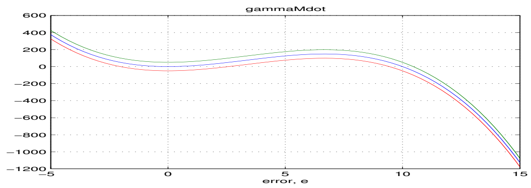

Fig. 2. The target variation of $\dot{\gamma}_i^M$

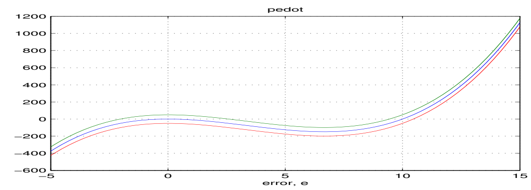

Fig. 3. The target variation of $\dot{p}_i^e$

solve the convergence problem (43).

Proof: See Appendix VII-C.

Note that (44) makes

$$\dot{p}_i^e = -\kappa(-P)\frac{4}{(p_i^M - p_i^m)} sign(e_i^p) \tag{45}$$

Assumption 6: We assume:

$$\begin{array}{l} \dot{\gamma}_i^M = -e_i^2(e_i - \alpha(t)) + \xi_2(t), \\ \dot{p}_i^e = (-P)(e_i^2(e_i - \alpha(t)) + \xi_3(t)) \end{array} \tag{46}$$

where P is as in (34) and $\alpha > 0$. From (46):

$$\dot{p}_i^e = (-P)(\xi_2 + \xi_3 - \dot{\gamma}_i^M) \tag{47}$$

Remark IV.2: If $\xi_2 = 0$ $\dot{\gamma}_i^M$ is positive for $e_i \in (-\infty, \alpha)$, negative only for $e_i \in (\alpha, \infty)$ and $\dot{\gamma}_i^M = 0$ when $e_i = \{0, \alpha\}$. Since $\frac{\partial}{\partial \gamma_i^M}\dot{\gamma}_i^M|_{e_i=\alpha} < 0$ the equilibrium point $e_i = \alpha$ is stable and $e_i = 0$ is unstable. Thus we always try to force an increase in the value of γ_i^M. See Figure 2. Similarly, p_i^e is kept low as much as possible. See Figure 3. With $\xi_2 \neq 0$ and / or $\xi_3 \neq 0$ it is still possible to implement a relay structure.

Theorem 1: The controller (30), (46) where:

$$\begin{array}{l} \xi_2 = -2(\frac{\alpha}{3})^3 + \dot{\hat{\gamma}}_i - \mu sign(e_i) \\ \xi_3 = 2(\frac{\alpha}{3})^3 + e_i^p(-\hat{P})\frac{\gamma_i}{p_i} + \hat{h}_i\gamma_i - \kappa\frac{4}{(p_i^M - p_i^m)} sign(e_i^p) \end{array} \tag{48}$$

$$\alpha = 3\sqrt[3]{\frac{1}{2}\left|(-\hat{P})\frac{\gamma_i}{p_i}e_i^p sign(e_i) - |\hat{h}_i|\gamma_i\right|} \tag{49}$$

$$\begin{array}{lll} sign(\kappa) = sign(e_i) & if & e_i e_i^p > 0 \\ \kappa > 0 & if & e_i e_i^p < 0 \end{array} \tag{50}$$

$$\mu = \iota_1\chi + \iota_2(\frac{\alpha}{3})^3 \tag{51}$$

with μ, χ as in (44), solve the optimization problem (38)-(40).

Proof: See Appendix VII-D

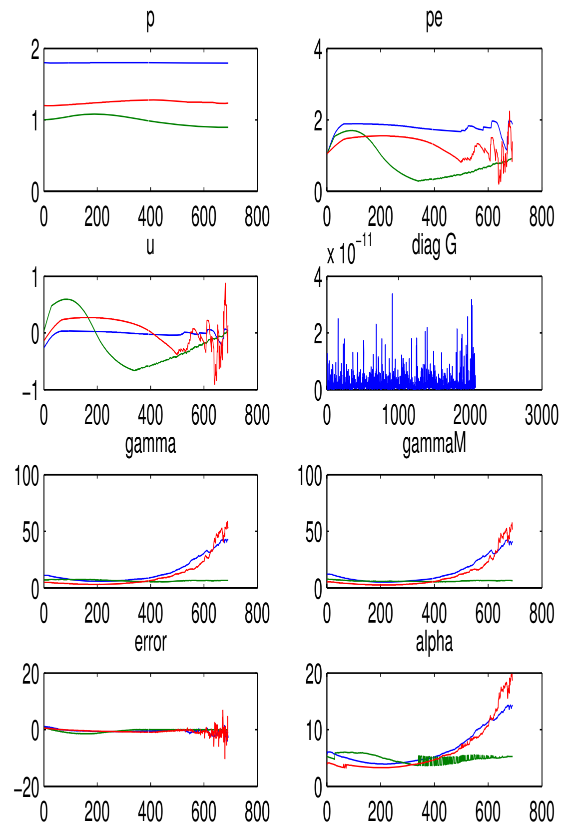

Fig. 4. The simulation results

V. Numerical simulations

Numerical simulations were performed for a "fixed assignment" scenario [2], with a single base and three users, using the Euler integration algorithm. In our simulations we used the following values for the parameters: $\epsilon_1 = 0.9$, $\epsilon_2 = 0.3$, $\epsilon_3 = 0.6$ and many different values of ι_2. A snapshot of numerical simulations is presented in Figure 4. Note that the power is decreasing slowly while the SIR γ_i for the users are slowly approaching close values. Still, simulations are very sensitive to the ι parameter and an optimal value is to be estimated in our future work.

VI. Conclusions

The paper presents a control system for adjusting the power in wireless networks. The controller computes a virtual equilibrium and constrain the system to approach it. Cubic functions are used in design because of their remarkable features. Further work should be done to estimate the optimal values of the parameters κ, ι_1 and ι_2 involved by the control law.

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VII. Appendix

A. Proof of Proposition 2

The equilibrium points $\overline{p}_i = p_m$ and $\overline{p}_i = p_M$ are repulsive if $p_i^e \in [p_m, p_M]$ since:

$$\begin{array}{l} \frac{\partial u_i}{\partial p_i}|_{p_i=p_m} = (p_m - p_i^e)(p_m - p_M) > 0; \\ \frac{\partial u_i}{\partial p_i}|_{p_i=p_M} = (p_M - p_i^e)(p_M - p_m) > 0 \end{array} \quad (52)$$

The equilibrium point $\overline{p}_i = p_i^e$ is stable if $p_i^e \in [p_m, p_M]$ since:

$$\frac{\partial u_i}{\partial p_i}|_{p_i=p_i^e} = (p_i^e - p_m)(p_i^e - p_M) < 0 \quad (53)$$

In addition, if $p_i^e = p_m$, $\dot{p}_i = u_i = (p_i - p_m)^2(p_i - p_M)$ and

$$\frac{\partial u_i}{\partial p_i}|_{p_i=p_M} = (p_M - p_m)^2 > 0; \quad \frac{\partial u_i}{\partial p_i}|_{p_i=p_m} = 0,$$

and since $\dot{p}_i|_{p_i \cong p_m} < 0$, p_M is unstable and p_m is stable. If $p_i^e = p_M$, $\dot{p}_i = u_i = (p_i - p_M)^2(p_i - p_m)$

$$\frac{\partial u_i}{\partial p_i}|_{p_i=p_m} = (p_m - p_M)^2 > 0; \quad \frac{\partial u_i}{\partial p_i}|_{p_i=p_M} = 0;$$

and since $\dot{p}_i|_{p_i \cong p_M} > 0$, p_m is unstable and p_M is stable.

B. Eigenvalues of the system Jacobian

From (13) and (30) for $\bar{\gamma}_{i_1} = 0$, if $p_i^e \in (p_m, p_M)$:

$$\begin{array}{l} \bar{p}_i = p_m \ \rightarrow \ \lambda_1 = (p_m - p_i^e)(p_m - p_M) > 0, \ \lambda_2 = \varrho_i \\ \bar{p}_i = p_M \ \rightarrow \ \lambda_1 = (p_M - p_i^e)(p_M - p_m) > 0, \ \lambda_2 = \varrho_i \\ \bar{p}_i = p_e \ \rightarrow \ \lambda_1 = (p_i^e - p_m)(p_i^e - p_M) < 0, \ \lambda_2 = \varrho_i \end{array}$$

Similarly, from (16) and (30), for $\bar{\gamma}_{i_2} = \kappa_i \bar{p}_i$:

$$\begin{array}{l} \bar{p}_i = p_m \ \rightarrow \ \lambda_1 = (p_m - p_i^e)(p_m - p_M) > 0, \ \lambda_2 = -\varrho_i \\ \bar{p}_i = p_M \ \rightarrow \ \lambda_1 = (p_M - p_i^e)(p_M - p_m) > 0, \ \lambda_2 = -\varrho_i \\ \bar{p}_i = p_e \ \rightarrow \ \lambda_1 = (p_i^e - p_m)(p_i^e - p_M) < 0, \ \lambda_2 = -\varrho_i \end{array}$$

C. Proof of Proposition 4

Since $max(-P) = (p_i^M - p_i^m)^2/4$ a possible choice for χ is

$$\chi = (-\hat{P})|e_i^p| + \kappa(-P)(p_i^M - p_i^m)$$

or, equivalently (44) . The second relation is straightforward from (43).

D. Proof of Theorem 1

From (45), (46), (48) it follows that:

$$\begin{array}{l} -e_i^3 + \alpha e_i^2 + \xi_2 = -\mu sign(e_i) + \dot{\hat{\gamma}}_i + \hat{h}_i \gamma_i + e_i^p(-\hat{P})\frac{\gamma_i}{p_i} \\ e_i^3 - \alpha e_i^2 + \xi_3 = -\kappa \frac{4}{p_i^M - p_i^m} sign(e_i^p) \end{array} \quad (54)$$

We choose ξ_2 and ξ_3 as in (48) and rewrite (54) as follows:

$$e_i^3 - \alpha e_i^2 + 2(\frac{\alpha}{3})^3 + sign(e_i)(e_i^p sign(e_i)(-\hat{P})\frac{\gamma_i}{p_i} - |\hat{h}_i|\gamma_i) = 0 \quad (55)$$

By eliminating the square term from the second equation of (55) it becomes:

$$\begin{array}{l} y^3 + Ay + B = 0, \quad y = e_i - \frac{\alpha}{3}, \quad A = -\frac{\alpha^2}{3} \\ B = sign(e_i)(e_i^p sign(e_i)(-\hat{P})\frac{\gamma_i}{p_i} - |\hat{h}_i|\gamma_i) \end{array} \quad (56)$$

The cusp is as follows:

$$((\frac{\alpha}{3})^2)^3 = (\frac{1}{2} sign(e_i)(e_i^p sign(e_i)(-\hat{P})\frac{\gamma_i}{p_i} - |\hat{h}_i|\gamma_i)^2 \quad (57)$$

therefore it turns out (49). From (45), if $e_i e_i^p < 0$ the constraint (40) $\dot{p}_i^e sign(e_i) > 0$ requires $\kappa > 0$. In order to deal with (39) we study the inequality:

$$\dot{p}_i^e - \frac{\lambda}{(\gamma_i^M)^2}(-\mu - |\hat{h}_i|\gamma_i + |e_i^p|(-\hat{P})\frac{\gamma_i}{p_i})sign(e_i) < 0$$

Obviously, if $\lambda << 1$ we should require $\dot{p}_i^e < 0$ when $e_i e_i^p > 0$. Therefore, we could make a conservative choice of κ as in (50). Taking into account (57) the parameters ι_1 and ι_2 are chosen to make $\dot{\gamma}_i^M$ positive as much as possible.