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Exponentially Contractive Invariant Sets for Discrete-Time Switching Linear Systems

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Abstract— **By considering arbitrary Holder *p*-norms in the state space, the current paper studies the existence of exponentially contractive sets that are invariant with respect to the trajectories of discrete-time switching linear systems. We first provide necessary and sufficient conditions for testing if state-space sets (characterized by certain attributes such as shape, scaling factor and decreasing rate) are invariant with respect to the trajectories of a given switching system. Next, we synthesize state feedbacks so as the sets characterized by given attributes are invariant with respect to the trajectories of the closed-loop switching system. For the usual Holder p-norms, corresponding to $p \in \{1, 2, \infty\}$, we also discuss the numerical tractability of the theoretical results. A numerical example is included for practical illustration.**

I. INTRODUCTION

A. Theoretical prerequisites on switching systems and invariant sets

The investigation of *invariant sets* emerged as a research trend in mathematics at the middle of the 20-th century, as pointed out by the monograph [1]. In the 1980s, the theoretical results were applied in control engineering for developing new analysis and design techniques. The applicability of the flow-invariance concepts in the qualitative theory of dynamical systems is amply discussed by the recent monograph [2] whose reference list contains more than 300 works representing noticeable contributions to this research trend.

A *set* is (*forward*) *invariant* with respect to the trajectories of a dynamical system, if any trajectory initiated inside that set does not leave it any longer. A particular type of time-dependent invariant sets investigated by many researchers is represented by *contractive sets*, the interest focusing on sets with polyhedral shapes. Unlike constant invariant sets, the contractive ones constrain system trajectories to approach the equilibrium point. By considering exponentially contractive invariant sets defined for any Hölder *p*-norm, $1 \le p \le \infty$ paper [3] introduced a special type of stability for linear systems called *diagonally invariant exponential stability* (DIES) relative to *p*-norms.

Switching and switched systems have attracted a lot of interest in recent years since they arise in different areas of practical applications. They can be regarded as “two-level hybrid systems with the lower level governed by a set of modes described by differential and/or difference equations and the upper level a coordinator that orchestrates the switching among the modes” [4]. The commutation events can be classified into time-driven or state-driven, being autonomous (uncontrolled) or controlled [5]. For *switching systems*, the commutation is not controlled, but is determined by exogenous agents [6]. The systems for which the commutation is controlled are called *switched systems*.

In a general setting, the main aspects regarding the stability theory of switching and hybrid systems are approached in [7]. The latest survey paper [8] and the monograph [4] present up-to-date syntheses of results in the stability and stabilizability of switching linear systems, in both continuous- and discrete-time, based on extensive reference lists. Studies of invariant sets of continuous-time switched systems are presented in latest papers [9]-[11].

The current paper explores the exponentially contractive sets corresponding to the DIES of discrete-time switching linear systems. Since DIES is a stronger property than standard stability, the results on DIES for switching systems are more conservative than those referring to the stability / stabilizability of switching systems reported in the papers cited by [4] or [8]. The results presented here express necessary and sufficient conditions for the diagonally invariant exponential stability and stabilizability of discrete time SLS and are in full accordance with the continuous-time case we have addressed in [11].

The remainder of the paper is organized as follows. Section II presents the theoretical development of (i) analysis techniques for testing the existence of contractive invariant sets with given attributes and (ii) design method of a closed loop switching system with state feedback that exhibits invariant sets with given attributes. Section III is devoted to the numerical tractability of the procedures presented in Section II for the usual norms corresponding to $p \in \{1, 2, \infty\}$. Section IV illustrates the applicability of our results.

B. Notations

The following notations will be used throughout the paper:

For a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, $S(\boldsymbol{M}) = \{z \in \mathbb{C} \mid \det(z\boldsymbol{I} - \boldsymbol{M}) = 0\}$ is the spectrum of $\boldsymbol{M}$, and $\lambda_i(\boldsymbol{M}) \in S(\boldsymbol{M})$, $i = 1, \ldots, n$, denote its eigenvalues. is the spectral radius of $\boldsymbol{M}$.

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If matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is symmetric, then "$\boldsymbol{M} \succ 0$" ("$\boldsymbol{M} \preceq 0$") means "$\boldsymbol{M}$ is positive definite" ("negative semidefinite"). If $\boldsymbol{M} \succ 0$, $\lambda_{\max}(\boldsymbol{M}) = \max\{\lambda \,|\, \lambda \in S(\boldsymbol{M})\}$.

If $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times m}$, then "$\boldsymbol{X} \le \boldsymbol{Y}$", "$\boldsymbol{X} < \boldsymbol{Y}$" mean componentwise inequalities.

If $\boldsymbol{X} \in \mathbb{R}^{n\times m}$, then $|\boldsymbol{X}|$ represents the nonnegative matrix ($m \ge 2$) or vector ($m = 1$) defined by taking the absolute values of the entries of ***X***.

For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} = [x_1\ x_2\ \dots\ x_n]^T \in \mathbb{R}^n$, $\|\boldsymbol{x}\|_p$ is the Hölder vector p-norm defined by $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ if $1 \le p < \infty$, and $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$ if $p = \infty$. For a matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, $\|\boldsymbol{M}\|_p$ is the matrix norm induced by the vector norm $\|\bullet\|_p$ through $\|\boldsymbol{M}\|_p = \max_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_p = 1} \|\boldsymbol{M}\boldsymbol{x}\|_p$.

Given $\boldsymbol{D} = \operatorname{diag}\{d_1, \dots, d_n\}$ a positive definite diagonal matrix (i.e. $d_i > 0$, $i = 1, \dots n$), we define the vector norm $\|\boldsymbol{x}\|_p^{\boldsymbol{D}} = \|\boldsymbol{D}^{-1}\boldsymbol{x}\|_p$ and the induced matrix norm $\|\boldsymbol{M}\|_p^{\boldsymbol{D}} = \|\boldsymbol{D}^{-1}\boldsymbol{M}\boldsymbol{D}\|_p$ The particular cases when $p \in \{1, 2, \infty\}$ are [12, pp. 33]: $\|\boldsymbol{M}\|_1^{\boldsymbol{D}} = \max_{j=1,n} \sum_{i=1}^n \frac{d_j}{d_i} |m_{ij}|$, $\|\boldsymbol{M}\|_2^{\boldsymbol{D}} = \sqrt{\lambda_{\max}(\boldsymbol{D}\boldsymbol{M}^T(\boldsymbol{D}^{-1})^2\boldsymbol{M}\boldsymbol{D})}$, $\|\boldsymbol{M}\|_\infty^{\boldsymbol{D}} = \max_{i=1,n} \sum_{j=1}^n \frac{d_j}{d_i} |m_{ij}|$

II. THEORETICAL DEVELOPMENT

This paper focuses on discrete-time *switching linear systems* (SLSs) that are mathematically described by a collection of indexed linear difference equations:

$$\boldsymbol{\Sigma}_{\mathcal{A}}:\ \boldsymbol{x}(t+1) = \boldsymbol{A}_{\sigma(t)}\boldsymbol{x}(t),\ \ t \in \mathbb{Z}_+, \tag{1}$$

where $\mathbb{Z}_+$ stands for the set of non-negative integers and $\boldsymbol{x}(t) \in \mathbb{R}^n$ represents the continuous state variable. The logical rule that commands the switching between the subsystems is represented by the switching signal $\sigma : \mathbb{Z}_+ \to \{1, \dots, N\}$. Starting from an initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, the trajectory of SLS (1) also depends on the switching sequence σ and is denoted in the sequel by $\boldsymbol{x}(t; \boldsymbol{x}_0, \sigma)$.

At any time instant $t \in \mathbb{Z}_+$, the index $s = \sigma(t)$ determines the *active mode* of the SLS,

$$\boldsymbol{\Sigma}_{\boldsymbol{A}_s}:\ \boldsymbol{x}(t+1) = \boldsymbol{A}_s\boldsymbol{x}(t),\ \ s \in \{1, \dots, N\}, \tag{1a}$$

characterized by the $n \times n$ matrix $\boldsymbol{A}_s$.

Let $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_N\} \subset \mathbb{R}^{n\times n}$ be the set of matrices that correspond to the N modes of the SLS.

A. Characterization of Exponentially Contractive Invariant Sets

Definition 1.

Let $1 \le p \le \infty$, $\boldsymbol{D} \succ 0$ diagonal, and $0 < r < 1$. The SLS (1) is called *diagonally invariant exponentially stable relative to the p-norm and parameters* ***D****, r* (abbreviated as $\mathrm{DIES}_p^{\boldsymbol{D},r}$) under arbitrary switching if

$$\begin{aligned}&\forall \varepsilon > 0, \forall \sigma : \mathbb{Z}_+ \to \{1, \dots, N\}, \forall \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n :\\ &\|\boldsymbol{x}_0\|_p^{\boldsymbol{D}} \le \varepsilon \ \Rightarrow\ \|\boldsymbol{x}(t; \boldsymbol{x}_0, \sigma)\|_p^{\boldsymbol{D}} \le \varepsilon\, r^{(t-t_0)}, \forall t \in \mathbb{Z}_+.\end{aligned} \qquad \blacksquare\ (2)$$

Considering the *exponentially decreasing time-dependent sets*:

$$\mathcal{X}_p^{\boldsymbol{D},r}(t; \varepsilon) = \left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \|\boldsymbol{x}\|_p^{\boldsymbol{D}} \le \varepsilon r^{t-t_0}\right\}, t \in \mathbb{Z}_+,\ \varepsilon > 0, \tag{3}$$

relation (2) expresses the invariance of $\mathcal{X}_p^{\boldsymbol{D},r}(t; \varepsilon)$ with respect to the state-space-trajectories of system (1) for any switching sequence σ, i.e. $\forall\ \boldsymbol{x}_0 \in \mathcal{X}_p^{\boldsymbol{D},r}(0; \varepsilon) \Rightarrow \boldsymbol{x}(t; \boldsymbol{x}_0, \sigma) \in \mathcal{X}_p^{\boldsymbol{D},r}(t; \varepsilon)$, $\forall t \in \mathbb{Z}_+$, and $\sigma : \mathbb{Z}_+ \to \{1, \dots, N\}$.

Our study of $\mathrm{DIES}_p^{\boldsymbol{D},r}$ for discrete-time SLSs is based on Theorem 2 in [3], stated in the sequel for a time-invariant linear system:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\,\boldsymbol{x}(t),\ \ t \in \mathbb{Z}_+, \tag{4}$$

whose solution corresponding to an initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ is denoted by $\boldsymbol{x}(t; x_0)$.

Theorem 1.

Let $1 \le p \le \infty$, $\boldsymbol{D} \succ 0$ diagonal and $0 < r < 1$.

The following statements are equivalent:

(*i*) System (4) is $\mathrm{DIES}_p^{\boldsymbol{D},r}$, i.e.

$$\begin{aligned}&\forall \varepsilon > 0, \forall \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n :\\ &\qquad \|\boldsymbol{x}_0\|_p^{\boldsymbol{D}} \le \varepsilon \ \Rightarrow\ \|\boldsymbol{x}(t; x_0)\|_p^{\boldsymbol{D}} \le \varepsilon\, r^{(t-t_0)}, \forall t \in \mathbb{Z}_+.\end{aligned} \tag{5}$$

(*ii*) $\mathcal{V}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p^{\boldsymbol{D}}$ is a strong Lyapunov function for system (4), with the decreasing rate *r*, i.e.

$$\mathcal{V}(\boldsymbol{x}(t+1)) \le r\,\mathcal{V}(\boldsymbol{x}(t)), \forall t \in \mathbb{Z}_+, \tag{6}$$

along every solution $\boldsymbol{x}(t) = \boldsymbol{x}(t; x_0)$ of (4).

(*iii*) The following norm inequality is satisfied

$$\|\boldsymbol{A}\|_p^{\boldsymbol{D}} \le r\,. \qquad \blacksquare\ (7)$$

The following result represents a natural extension of Theorem 1 to the case of the SLS (1).

Theorem 2.

Let $1 \le p \le \infty$, $\boldsymbol{D} \succ 0$ diagonal and $0 < r < 1$.

The following statements are equivalent:

(*i*) The SLS (1) is $\mathrm{DIES}_p^{\boldsymbol{D},r}$ under arbitrary switching.

(*ii*) $\mathcal{V}(\boldsymbol{x}) = \|\boldsymbol{x}\|_p^{\boldsymbol{D}}$ is a common strong Lyapunov function for all the subsystems of (1), with the decreasing rate *r*.

(*iii*) The following norm inequalities are satisfied

$$\|\boldsymbol{A}_s\|_p^{\boldsymbol{D}} \le r,\ \ s = 1, \dots, N\,. \tag{8}$$

Proof: (*i*) ⇒ (*ii*) ⇒ (*iii*). For every $s \in \{1,\dots,N\}$, consider the switching signals $\sigma(t) \equiv s$, then apply Theorem 1.

(*iii*) ⇒ (*i*). We construct a proof by contradiction. We assume there exist a positive $\varepsilon > 0$, a switching signal σ and an initial condition $\boldsymbol{x}_0 \in \mathcal{X}_p^{\boldsymbol{D},r}(0;\varepsilon)$ so that the set $\mathcal{X}_p^{\boldsymbol{D},r}(t;\varepsilon)$ is not invariant with respect to the solution $\boldsymbol{x}(t;\boldsymbol{x}_0,\sigma)$. This means there exists a moment t^* when the considered solution leaves the set, i.e. $\| \boldsymbol{x}(t;\boldsymbol{x}_0,\sigma) \|_p^{\boldsymbol{D}} \le \varepsilon r^{t-t_0}$, for $t_0 \le t \le t^*$, and $\| \boldsymbol{x}(t^*+1;\boldsymbol{x}_0,\sigma) \|_p^{\boldsymbol{D}} > \varepsilon r^{t^*+1}$. Let $s^* = \sigma(t^*)$ be the index of the active mode of the SLS at t^*. By taking $\boldsymbol{x}(t^*+1;\boldsymbol{x}_0,\sigma) = \boldsymbol{A}_{s^*}\boldsymbol{x}(t^*;\boldsymbol{x}_0,\sigma)$ into account, we get $\| \boldsymbol{x}(t^*+1;\boldsymbol{x}_0,\sigma) \|_p^{\boldsymbol{D}} \le \| \boldsymbol{A}_{s^*} \|_p^{\boldsymbol{D}} \; \| \boldsymbol{x}(t^*;\boldsymbol{x}_0,\sigma) \|_p^{\boldsymbol{D}} \le r \varepsilon r^{t^*-t_0}$, which contradicts the hypothesis and completes the proof. ■

Remark 1.

Theorem 2 shows that the existence of a common Lyapunov function defined by $\mathcal{V}(\boldsymbol{x}) = \| \boldsymbol{x} \|_p^{\boldsymbol{D}}$ with a positive definite diagonal matrix $\boldsymbol{D}$ is a necessary and sufficient condition for the $\mathrm{DIES}_p^{\boldsymbol{D},r}$ of SLSs under arbitrary switching signals. From the stability analysis point of view, the diagonally invariant exponentially stability property implies the uniform exponential stability of system (1) defined in [4]. As pointed out in [8], the existence of a common quadratic Lyapunov function is only sufficient for the uniform asymptotic stability of this type of systems.

We also note that since the spectral radius of a square matrix $\boldsymbol{M}$ satisfies $\rho(\boldsymbol{M}) < \|\boldsymbol{M}\|$ (for any matrix norm $\| \bullet \|$), the Schur stability of all the matrices in the set $\mathcal{A}$ (and, subsequently, the exponential stability of all the subsystems of the SLS) is necessary for the $\mathrm{DIES}_p^{\boldsymbol{D},r}$ of the discrete-time SLS (1). ■

Remark 2.

A result similar to Theorem 2 can be formulated for discrete-time polytopic systems or difference inclusions. ■

B. Allocation of Exponentially Contractive Invariant Sets through State-Feedback

Consider a switching linear system that commutes between $N \in \mathbb{N}$ subsystems:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}_{\sigma(t)}\boldsymbol{x}(t) + \boldsymbol{B}_{\sigma(t)}\boldsymbol{u}(t),\ t \in \mathbb{Z}_+, \tag{9}$$

governed by the switching signal $\sigma : \mathbb{Z}_+ \to \{1,\dots,N\}$. Let $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_N\} \subset \mathbb{R}^{n\times n}$, $\mathcal{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2, \dots, \boldsymbol{B}_N\} \subset \mathbb{R}^{n\times m}$ with $m \le n$.

Choose a state-feedback built with a unique gain matrix:

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t),\ \boldsymbol{K} \in \mathbb{R}^{m\times n}, \tag{10}$$

and define the closed-loop switching system:

$$\hat{\boldsymbol{x}}(t+1) = (\boldsymbol{A}_{\sigma(t)} - \boldsymbol{B}_{\sigma(t)}\boldsymbol{K})\hat{\boldsymbol{x}}(t),\ t \in \mathbb{Z}_+. \tag{11}$$

Definition 2.

Let $1 \le p \le \infty$, $\boldsymbol{D} \succ 0$ diagonal and $0 < r < 1$.

If there exists a state-feedback control (10) so that the closed-loop switching system (11) is $\mathrm{DIES}_p^{\boldsymbol{D},r}$ under arbitrary switching, then the switching system (9) is called $\mathrm{DIES}_p^{\boldsymbol{D},r}$*-stabilizable*. ■

Theorem 3.

Let $1 \le p \le \infty$, $\boldsymbol{D} \succ 0$ diagonal and $0 < r < 1$.

The switching system (13) is $\mathrm{DIES}_p^{\boldsymbol{D},r}$-stabilizable through the state-feedback (14) if and only if matrix $\boldsymbol{K}$ is a solution to the inequalities

$$\| \boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K} \|_p^{\boldsymbol{D}} \le r,\quad s = 1,\dots,N. \tag{12}$$

Proof: We apply Theorem 2 to the closed-loop switching system (11). ■

Remark 3.

Any $\mathrm{DIES}_p^{\boldsymbol{D},r}$- feedback of the switching system (9) ensures the assignment of the closed-loop eigenvalues in the Schur-stability region of the complex plane, $\mathbb{C}_S = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$, for each subsystem of the closed loop system (11). ■

III. NUMERICAL TRACTABILITY

In this section we are interested in pointing out the numerical approach to the analysis and allocation of diagonally invariant sets for the switching linear system (1) in the particular cases of the usual Hölder norms corresponding to $p \in \{1, 2, \infty\}$.

For the usual norms corresponding to $p \in \{1,\ 2,\ \infty\}$, the sets defined by (3) have well-known geometric shapes (i.e. hyper-diamonds for $p = 1$, ellipses for $p = 2$, and rectangles for $p = \infty$) scaled in accordance with the diagonal entries of matrix $\boldsymbol{D}$. The constant $0 < r < 1$ expresses the contraction rate of the considered sets.

Corollary 1.

(*i*) For $p = 1$, inequalities (8) are equivalent to

$$| \boldsymbol{A}_s |^T \, \boldsymbol{\delta} \le r\boldsymbol{\delta},\ s = 1,\dots,N, \tag{13}$$

where

$$\boldsymbol{\delta} = [\delta_1 \dots \delta_n]^T \in \mathbb{R}^n,\ \delta_i = 1/d_i,\ i = 1,\dots,n, \tag{14}$$

is a positive vector formed with the inverses of the diagonal entries of matrix $\boldsymbol{D}$.

(*ii*) For $p = 2$, inequalities (8) are equivalent to

$$\boldsymbol{A}_s^T \boldsymbol{P} \boldsymbol{A}_s - r^2 \boldsymbol{P} \preceq 0,\ s = 1,\dots,N, \tag{15}$$

where $\boldsymbol{P} = (\boldsymbol{D}^{-1})^2$.

(*iii*) For $p = \infty$, inequalities (8) are equivalent to

$$| \boldsymbol{A}_s | \, \boldsymbol{d} \le r\boldsymbol{d},\ s = 1,\dots,N, \tag{16}$$

where $\boldsymbol{d} = [d_1 \dots d_n]^T \in \mathbb{R}^n$ is a positive vector formed with the diagonal entries of matrix $\boldsymbol{D}$.

Proof: For a simultaneous handling of all matrices $\boldsymbol{A}_s$, $s = 1,\dots,N$, involved in (13), (15) and (16), let us consider a generic notation $\boldsymbol{M} = \left[m_{ij}\right],\ i, j = 1,\dots,n$, when referring to a real square matrix.

(*i*) For $p = 1$ we have

$$\|\boldsymbol{M}\|_1^{\boldsymbol{D}} \le r \quad \Leftrightarrow \quad \max_{1\le j\le n}\left(|m_{ii}| + \sum_{i=1, i\ne j}^{n} |m_{ij}| \frac{d_j}{d_i} \right) \le r \quad \Leftrightarrow$$

$$\max_{1\le j\le n}\left(|m_{ii}| + \sum_{i=1, i\ne j}^{n} |m_{ij}| \frac{\delta_i}{\delta_j} \right) \le r \Leftrightarrow |m_{jj}|\delta_j + \sum_{i=1, i\ne j}^{n} |m_{ij}|\delta_i \le$$

$r\delta_j,\quad j = 1,\dots,n \Leftrightarrow |\boldsymbol{M}|^T\ \boldsymbol{\delta} \le r\boldsymbol{\delta}$, where vector $\boldsymbol{\delta}$ is defined by (14). Therefore, for $p = 1$ inequalities (8) are equivalent to (13).

(*ii*) For $p = 2$ inequalities (8) become $\|\boldsymbol{M}\|_2^{\boldsymbol{D}} \le r \Leftrightarrow \max_{i=1,n}\left\{\lambda_i(\boldsymbol{D}\boldsymbol{M}^T(\boldsymbol{D}^{-1})^2\boldsymbol{M}\boldsymbol{D})\right\} \le r^2$, which is equivalent to $\boldsymbol{D}\boldsymbol{M}^T(\boldsymbol{D}^{-1})^2\boldsymbol{M}\boldsymbol{D} - r^2\boldsymbol{I} \preceq 0$. By left and right multiplication by the positive definite diagonal matrix $\boldsymbol{D}^{-1}$, we get the Stein matrix inequality $\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M} - r^2\boldsymbol{P} \preceq 0$ written for $\boldsymbol{P} = (\boldsymbol{D}^{-1})^2$, which completes the proof.

(*iii*) For $p = \infty$ we have

$$\|\boldsymbol{M}\|_\infty^{D} \le r \quad \Leftrightarrow \quad \max_{1\le i\le n}\left(|m_{ii}| + \sum_{j=1, j\ne i}^{n} |m_{ij}| \frac{d_j}{d_i} \right) \le r \quad \Leftrightarrow$$

$$|m_{ii}|d_i + \sum_{j=1,\ j\ne i}^{n} |m_{ij}|d_j \le rd_i, \qquad j = 1,\dots,n \qquad \Leftrightarrow$$

$|\boldsymbol{M}|\,\boldsymbol{d} \le r\boldsymbol{d}$. This proves that for $p = \infty$ inequalities (8) are equivalent to (16). ■

Remark 4.

Since the 1-norm and the ∞-norm are dual in the sense that $\|\boldsymbol{M}\|_1 = \|\boldsymbol{M}^T\|_\infty$ for any square matrix $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, the $\boldsymbol{D}$-weighted norms satisfy $\|\boldsymbol{M}\|_1^{\boldsymbol{D}} = \|\boldsymbol{M}^T\|_\infty^{\boldsymbol{D}^{-1}}$ for an arbitrary positive definite diagonal matrix $\boldsymbol{D} \succ 0$. The dual system of SLS $\boldsymbol{\Sigma}_{\mathcal{A}}$ (1) is described by

$$\boldsymbol{\Sigma}_{\mathcal{A}^T}:\ \boldsymbol{y}(t+1) = \boldsymbol{A}_{\sigma(t)}^{T}\boldsymbol{y}(t),\quad t \in \mathbb{Z}_+.$$

The SLS $\boldsymbol{\Sigma}_{\mathcal{A}}$ (1) is $\mathrm{DIES}_1^{\boldsymbol{D},r}$ if and only if its dual system $\boldsymbol{\Sigma}_{\mathcal{A}^T}$ is $\mathrm{DIES}_\infty^{\boldsymbol{D}^{-1},r}$. ■

The next result represents the discrete time extension of the Corollary 41 from [13] referring to a continuous time positive SLS with 2 subsystems.

Proposition 1.

If there exist $\boldsymbol{D}_1, \boldsymbol{D}_\infty \succ 0$ diagonal and $0 < r_1, r_\infty < 1$, so that the SLS (1) is $\mathrm{DIES}_1^{\boldsymbol{D}_1,r_1}$ and $\mathrm{DIES}_\infty^{\boldsymbol{D}_\infty,r_\infty}$, then SLS (1) is also $\mathrm{DIES}_2^{\boldsymbol{D}_2,r_2}$, where $\boldsymbol{D}_2 \succ 0$ and $r_2 > 0$ satisfy $\boldsymbol{D}_2^2 = \boldsymbol{D}_1\boldsymbol{D}_\infty$, $r_2^2 = r_1 r_\infty$.

Proof: The proof is based on Lemma 1 presented in the Appendix. ■

Corollary 2.

(*i*) For $p = 1$, inequalities (12) are equivalent to the linear inequalities:

$$\begin{aligned}
&-\boldsymbol{B}_s\boldsymbol{K} - \boldsymbol{G}_s \le -\boldsymbol{A}_s,\\
&\boldsymbol{B}_s\boldsymbol{K} - \boldsymbol{G}_s \le \boldsymbol{A}_s,\\
&\boldsymbol{G}_s^T\boldsymbol{\delta} \le r\boldsymbol{\delta},\qquad s = 1,\dots,N,
\end{aligned} \tag{17}$$

where the vector $\boldsymbol{\delta} = [\delta_1 \dots \delta_n]^T \in \mathbb{R}^n$ has the same meaning as in Corollary 1 (*i*) and $\boldsymbol{K} \in \mathbb{R}^{m\times n}$, $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, $s = 1,\dots,N$, are unknown matrices.

(*ii*) For $p = 2$, inequalities (12) are equivalent to the linear matrix inequalities

$$(\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K})^T\boldsymbol{P}(\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}) - r^2\boldsymbol{P} \preceq 0,\ s = 1,\dots,N, \tag{18}$$

where $\boldsymbol{P} = (\boldsymbol{D}^{-1})^2$ and $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ is an unknown matrix.

(*iii*) For $p = \infty$, inequalities (12) are equivalent to the linear inequalities

$$\begin{aligned}
&-\boldsymbol{B}_s\boldsymbol{K} - \boldsymbol{G}_s \le -\boldsymbol{A}_s,\\
&\boldsymbol{B}_s\boldsymbol{K} - \boldsymbol{G}_s \le \boldsymbol{A}_s,\\
&\boldsymbol{G}_s\boldsymbol{d} \le r\boldsymbol{d},\quad s = 1,\dots,N,
\end{aligned} \tag{19}$$

where the vector $\boldsymbol{d} = [d_1 \cdots d_n]^T \in \mathbb{R}^n$ has the same meaning as in Corollary 1 (*iii*), and $\boldsymbol{K} \in \mathbb{R}^{m\times n}$, $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, $s = 1,\dots,N$, are unknown matrices.

Proof: (*i*) Corollary 1 (*i*) written for $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ instead of $\boldsymbol{A}_s$ shows that inequalities (16) with $p = 1$ are equivalent to the existence of a matrix $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ such that

$$|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|^T\ \boldsymbol{\delta} \le r\boldsymbol{\delta},\ s = 1,\dots,N. \tag{20}$$

First we prove the implication "(20) $\Rightarrow$ (17)". If there exists $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ such that (20) is satisfied, then matrices $\boldsymbol{G}_s = |\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|$, $s = 1,\dots,N$, satisfy $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K} \le \boldsymbol{G}_s$ and $-\boldsymbol{G}_s \le \boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$. Therefore, matrices $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, $s = 1,\dots,N$, satisfy inequalities (17).

Now we prove the converse statement "(17) $\Rightarrow$ (20)". If inequalities (17) are true for $\boldsymbol{K} \in \mathbb{R}^{m\times n}$, $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, $s = 1,\dots,N$, then $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K} \le \boldsymbol{G}_s$ and $-\boldsymbol{G}_s \le \boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ hold, implying $|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}| \le \boldsymbol{G}_s$. Consequently we can write the inequalities $|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|^T\ \boldsymbol{\delta} \le (\boldsymbol{G}_s)^T\boldsymbol{\delta} \le r\boldsymbol{\delta}$, $s = 1,\dots,N$, from which we conclude that inequalities (20) have solutions.

(*ii*) Corollary 1 (*ii*) written for $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ instead of $\boldsymbol{A}_s$ shows that inequalities (16) with $p = 2$ are equivalent to the existence of $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ such that inequalities (18) are satisfied.

(*iii*) Corollary 1 (*iii*) written for $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ instead of $\boldsymbol{A}_s$ shows that inequalities (16) with $p = \infty$ are equivalent to the existence of $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ such that

$$|\boldsymbol{A}_s - \boldsymbol{B}_s \boldsymbol{K}|\,\boldsymbol{d} \le r\boldsymbol{d}\,,\ s = 1,\ldots,N\,. \tag{21}$$

First we prove the implication "(21) $\Rightarrow$ (19)". If inequalities (21) are true, there exists $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ such that $|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|\,\boldsymbol{d} \le r\boldsymbol{d}$, $s = 1,\ldots,N$. By taking $\boldsymbol{G}_s = |\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|$, $s = 1,\ldots,N$, we get $-\boldsymbol{G}_s \le \boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ and $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K} \le \boldsymbol{G}_s$. Therefore matrices $\boldsymbol{K} \in \mathbb{R}^{m\times n}$, $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, $s = 1,\ldots,N$, satisfy inequalities (19).

Next we prove the converse statement "(19) $\Rightarrow$ (21)". If inequalities (19) are true for $\boldsymbol{K} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{G}_s \in \mathbb{R}^{n\times n}$, then $-\boldsymbol{G}_s \le \boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}$ and $\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K} \le \boldsymbol{G}_s$ hold, which imply $|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}| \le \boldsymbol{G}_s$, for all $s = 1,\ldots,N$. Consequently we can write $|\boldsymbol{A}_s - \boldsymbol{B}_s\boldsymbol{K}|\boldsymbol{d} \le \boldsymbol{G}_s\boldsymbol{d} \le r\boldsymbol{d}$, $s = 1,\ldots,N$, which proves that inequalities (21) have solutions. ■

Remark 5.
The analysis and allocation of diagonally invariant sets for the usual matrix norms corresponding to $p \in \{1,2,\infty\}$ appeal to simple numerical tools. Thus, for DIES analysis in the cases when $p \in \{1,\infty\}$, testing the inequalities (13) and (16) requires only matrix (vectors) multiplications, whereas for $p = 2$ inequality (11) involves eigenvalue computation. For DIES synthesis, if $p \in \{1,\infty\}$, the resolution of inequalities (17) or (19) can be seen as a linear programming problem; if $p = 2$, the linear matrix inequalities (12) can be solved along the guidelines in [14]. ■

IV. NUMERICAL EXAMPLE

A. *Invariant Set Analysis in the Discrete Time Dynamics*

Let us consider a switching linear system described by (1) for $N = 2$ and $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2\}$ with:

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.50 & -0.25 \\ -0.25 & 0.40 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} 0.20 & 0.30 \\ 0.10 & -0.30 \end{bmatrix}. \tag{22}$$

In $\mathbb{R}^2$ we consider exponentially contractive sets $\mathcal{X}_p^{\boldsymbol{D},r}(t;\varepsilon)$ defined by (3) for $p \in \{1,2,\infty\}$ with

$$\boldsymbol{D} = \mathrm{diag}\{1.25,\ 1\}\,,\ r = 0.80\,. \tag{23}$$

In order to test the invariance of these sets with respect to system (1)&(22) we apply Theorem 2 and Corollary 1.

For $p = 1$, simple calculations show that inequalities (13) written for $\boldsymbol{\delta} = [0.80 \quad 1]^T$ are not true. Consequently, the contractive sets $\mathcal{X}_1^{\boldsymbol{D},r}(t;\varepsilon)$ are not invariant with respect to SLS (1)&(22) and, equivalently, the SLS is not $\mathrm{DIES}_1^{\boldsymbol{D},r}$.

For $p = 2$, we compute the matrices $\boldsymbol{P} = (\boldsymbol{D}^{-1})^2$ and $\boldsymbol{M}_s = \boldsymbol{A}_s^T\boldsymbol{P}\boldsymbol{A}_s - r^2\boldsymbol{P}$, $s = 1,2$. Simple tests show that inequalities (15) are satisfied. Subsequently, the contractive sets $\mathcal{X}_2^{\boldsymbol{D},r}(t;\varepsilon)$ are invariant with respect to SLS (1)&(22) and, equivalently, the considered system is $\mathrm{DIES}_2^{\boldsymbol{D},r}$.

For $p = \infty$, inequalities (16) written for $\boldsymbol{d} = [1.25 \quad 1]^T$ are true, Therefore, the contractive sets $\mathcal{X}_\infty^{\boldsymbol{D},r}(t;\varepsilon)$ are invariant with respect to SLS (1)&(22) and, equivalently, the considered SLS is $\mathrm{DIES}_\infty^{\boldsymbol{D},r}$.

B. *Invariant Set Allocation by Feedback Synthesis*

Let us consider a switching linear system of form (9) for $N = 2$, $\mathcal{A} = \{\boldsymbol{A}_1, \boldsymbol{A}_2\}$ with the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ defined by (22), and $\mathcal{B} = \{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ with:

$$\boldsymbol{B}_1 = \begin{bmatrix} -2.25 \\ 1.59 \end{bmatrix},\ \boldsymbol{B}_2 = \begin{bmatrix} -0.53 \\ -0.35 \end{bmatrix} \tag{24}$$

For $p \in \{1,2,\infty\}$ and the attributes in (23), we apply Corollary 1 in order to study the $\mathrm{DIES}_p^{\boldsymbol{D},r}$-stabilizability under arbitrary switching through state-feedback for the switching system (9)&(22)&(24).

For $p = 1$, the design of a $\mathrm{DIES}_1^{\boldsymbol{D},r}$ feedback for the switching system relies on solving the linear inequalities (17) with $\boldsymbol{\delta} = [0.80 \quad 1]^T$ by considering the elements of matrices $\boldsymbol{K} \in \mathbb{R}^{2\times 1}$, $\boldsymbol{G}_1, \boldsymbol{G}_2 \mathbb{R}^{2\times 2}$ as unknowns. Numerically, the solution to inequalities (17) can be approached as a linear programming problem minimizing the cost function:

$$J(\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{K}, \lambda) = \lambda \tag{25}$$

under the constraints:

$$\begin{aligned} &-\boldsymbol{B}_1\boldsymbol{K} - \boldsymbol{G}_1 \le -\boldsymbol{A}_1,\ \boldsymbol{B}_1\boldsymbol{K} - \boldsymbol{G}_1 \le \boldsymbol{A}_1,\ \boldsymbol{G}_1^T\boldsymbol{\delta} \le \lambda\boldsymbol{\delta}, \\ &-\boldsymbol{B}_2\boldsymbol{K} - \boldsymbol{G}_2 \le -\boldsymbol{A}_2,\ \boldsymbol{B}_2\boldsymbol{K} - \boldsymbol{G}_2 \le \boldsymbol{A}_2,\ \boldsymbol{G}_2^T\boldsymbol{\delta} \le \lambda\boldsymbol{\delta},\ \lambda \le r. \end{aligned} \tag{26}$$

This problem is solved by calling the **linprog** function from the Matlab Optimization Toolbox, that provides the following results:

$$\begin{gathered} \lambda = 0.3632\,,\ \boldsymbol{K} = [-0.1914 \quad 0.2720], \\ \boldsymbol{G}_1 = \begin{bmatrix} 0.1637 & 0.2940 \\ 0.1073 & 0.1280 \end{bmatrix},\ \boldsymbol{G}_2 = \begin{bmatrix} 0.2961 & 0.1640 \\ 0.0534 & 0.2320 \end{bmatrix}. \end{gathered} \tag{27}$$

This solution is also a solution to inequalities (17) since the $\lambda = 0.3632 < r = 0.80$ implies $\boldsymbol{G}_1\boldsymbol{\delta} \le \lambda\boldsymbol{\delta} < r\boldsymbol{\delta}$ and $\boldsymbol{G}_2\boldsymbol{\delta} \le \lambda\boldsymbol{\delta} < r\boldsymbol{\delta}$. Therefore, the state feedback (10) constructed with $\boldsymbol{K} = [-0.1914 \quad 0.2720]$ ensures not only the invariance of the contractive sets $\mathcal{X}_1^{\boldsymbol{D},r}(t;\varepsilon)$ with respect to the closed-loop SLS (11)&(22)&(24), but also the invariance of the sets $\mathcal{X}_1^{\boldsymbol{D},\lambda}(t;\varepsilon)$ (which decrease faster than $\mathcal{X}_1^{\boldsymbol{D},r}(t;\varepsilon)$). Moreover, the designed feedback ensures the allocation in the region $\{z \in \mathbb{C} |\, |z| < \lambda\}$ of the eigenvalues for both subsystems defining the dynamics of the closed-loop switching system with $S(\boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{K}) = \{0.1226 \pm 0.1311i\}$ and $S(\boldsymbol{A}_2 - \boldsymbol{B}_2\boldsymbol{K}) = \{0.3115,\ -0.2477\}$.

For $p = 2$, the design of a $\mathrm{DIES}_2^{\boldsymbol{D},r}$ feedback for the switching system relies on Theorem 2 and Corollary 2 (*ii*). For $\boldsymbol{P} = (\boldsymbol{D}^{-1})^2$, we have to solve the following inequalities with respect to the unknown matrix $\boldsymbol{K}$:

$$\begin{aligned}&(\boldsymbol{A}_1-\boldsymbol{B}_1\boldsymbol{K})^T\boldsymbol{P}(\boldsymbol{A}_1-\boldsymbol{B}_1\boldsymbol{K})-r^2\boldsymbol{P}\prec=0,\\&(\boldsymbol{A}_2-\boldsymbol{B}_2\boldsymbol{K})^T\boldsymbol{P}(\boldsymbol{A}_2-\boldsymbol{B}_2\boldsymbol{K})-r^2\boldsymbol{P}\prec=0.\end{aligned}\tag{28}$$

Since matrix $\boldsymbol{P}$ is positive definite, $\boldsymbol{P}\succ 0$, we write (28) into LMI form using the Schur complements:

$$\begin{aligned}&\begin{bmatrix} r^2\boldsymbol{P} & (\boldsymbol{A}_1-\boldsymbol{B}_1\boldsymbol{K})^T\\ \boldsymbol{A}_1-\boldsymbol{B}_1\boldsymbol{K} & \boldsymbol{P}^{-1}\end{bmatrix}\succeq 0\\&\begin{bmatrix} r^2\boldsymbol{P} & (\boldsymbol{A}_2-\boldsymbol{B}_2\boldsymbol{K})^T\\ \boldsymbol{A}_2-\boldsymbol{B}_2\boldsymbol{K} & \boldsymbol{P}^{-1}\end{bmatrix}\succeq 0\end{aligned}\tag{29}$$

For solving problem (29) we used the Multi Parametric Toolbox (MPT) for MATLAB [15] that is freely available. The solution to (29) is $\boldsymbol{K}=[-0.2190 \quad 0.2425]$. The designed feedback ensures the invariance of the contractive sets $\mathcal{X}_2^{\boldsymbol{D},r}(t;\varepsilon)$ with respect to the SLS (11)&(22)&(24) and the allocation in the region $\{z\in\mathbb{C} \mid |z|<r\}$ of the eigenvalues for both subsystems defining the dynamics of the closed-loop SLS since $S(\boldsymbol{A}_1-\boldsymbol{B}_1\boldsymbol{K})=\{0.1097\pm 0.0706i\}$ and $S(\boldsymbol{A}_2-\boldsymbol{B}_2\boldsymbol{K})=\{0.3239,-0.2537\}$.

For $p=\infty$, the numerical approach to the design of a $\mathrm{DIES}_\infty^{\boldsymbol{D},r}$ state-feedback for the SLS (9)&(22)&(24) is similar to the case when $p=1$, and consists in solving the linear programming problem minimizing the cost function (25) under the constraints:

$$\begin{aligned}&-\boldsymbol{B}_1\boldsymbol{K}-\boldsymbol{G}_1\le-\boldsymbol{A}_1,\ \boldsymbol{B}_1\boldsymbol{K}-\boldsymbol{G}_1\le\boldsymbol{A}_1,\ \boldsymbol{G}_1\boldsymbol{d}\le\lambda\boldsymbol{d},\\&-\boldsymbol{B}_2\boldsymbol{K}-\boldsymbol{G}_2\le-\boldsymbol{A}_2,\ \boldsymbol{B}_2\boldsymbol{K}-\boldsymbol{G}_2\le\boldsymbol{A}_2,\ \boldsymbol{G}_2\boldsymbol{d}\le\lambda\boldsymbol{d},\ \lambda\le r.\end{aligned}\tag{29}$$

The solution provided by the call to the **linprog** function is:

$$\begin{aligned}&\lambda=0.4120,\ \boldsymbol{K}=[-0.2193 \quad 0.3441],\\&\boldsymbol{G}_1=\begin{bmatrix}0.0615 & 0.4281\\0.1073 & 0.1251\end{bmatrix},\ \boldsymbol{G}_2=\begin{bmatrix}0.3096 & 0.1280\\0.0777 & 0.2377\end{bmatrix}.\end{aligned}\tag{30}$$

Therefore, the state feedback (10) constructed with $\boldsymbol{K}=[-0.2193 \quad 0.3441]$ ensures the invariance of the sets $\mathcal{X}_\infty^{\boldsymbol{D},r}(t;\varepsilon)$ and $\mathcal{X}_\infty^{\boldsymbol{D},\lambda}(t;\varepsilon)$ with respect to the closed-loop SLS (11)&(22)&(24).

V. CONCLUSIONS

This paper provides theoretical and numerical instruments for the usage of invariant sets in analysis and synthesis of switching linear systems. For analysis we have developed procedures that test the invariance of sets with given attributes. For synthesis we have proposed techniques that construct feedback laws ensuring the invariance of sets with pre-assigned attributes. Our theoretical results cover all shapes of sets defined by p-norms, with $1\le p\le\infty$. The numerical tractability is discussed by considering the usual p-norms ($p\in\{1,2,\infty\}$). A numerical example is included for practical illustration.

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APPENDIX

Lemma 1.

Let $\boldsymbol{M}\in\mathbb{R}^{n\times n}$ be a nonnegative matrix. If there exist the positive definite diagonal matrices $\boldsymbol{D}_1,\boldsymbol{D}_\infty\succ 0$ and the scalars $0<r_1,r_\infty<1$, so that

$$\|\boldsymbol{M}\|_1^{\boldsymbol{D}_1}\le r_1,\ \|\boldsymbol{M}\|_\infty^{\boldsymbol{D}_\infty}\le r_\infty,\tag{A1}$$

then

$$\|\boldsymbol{M}\|_2^{\boldsymbol{D}_2}\le r_2,\tag{A2}$$

with $\boldsymbol{D}_2\succ 0$ and $r_2>0$, satisfying $\boldsymbol{D}_2^2=\boldsymbol{D}_1\boldsymbol{D}_\infty$, $r_2^2=r_1r_\infty$.

Proof: Let $\boldsymbol{d}_1=\left[d_1^1\dots d_n^1\right]^T$ and $\boldsymbol{d}_\infty=\left[d_1^\infty\dots d_n^\infty\right]^T$ be the positive vectors formed with the diagonal entries of matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_\infty$. Inequalities (A1) are equivalent to $\boldsymbol{M}^T\boldsymbol{\delta}_1\le r_1\boldsymbol{\delta}_1$, and $\boldsymbol{M}\boldsymbol{d}_\infty\le r_\infty\boldsymbol{d}_\infty$. with $\boldsymbol{\delta}_1=[\delta_1^1\ldots\delta_n^1]^T\in\mathbb{R}^n$, $\delta_i^1=1/d_i^1$, $i=1,\ldots,n$, Let $\boldsymbol{P}=(\boldsymbol{D}_2{}^{-1})^2=(\boldsymbol{D}_1\boldsymbol{D}_\infty)^{-1}$, $\boldsymbol{F}=\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M}-r_2^2\boldsymbol{P}$. Compute $\boldsymbol{F}\boldsymbol{d}_\infty=\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M}\boldsymbol{d}_\infty-r_2^2\boldsymbol{P}\boldsymbol{d}_\infty\le r_\infty\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{d}_\infty-r_2^2\boldsymbol{P}\boldsymbol{d}_\infty$. Since $\boldsymbol{P}\boldsymbol{d}_\infty=\boldsymbol{D}_1^{-1}\boldsymbol{D}_\infty^{-1}\boldsymbol{d}_\infty=\boldsymbol{\delta}_1$, we get $\boldsymbol{F}\boldsymbol{d}_\infty\le r_\infty\boldsymbol{M}^T\boldsymbol{\delta}_1-r_2^2\boldsymbol{\delta}_1\le r_\infty r_1\boldsymbol{\delta}_1-r_2^2\boldsymbol{\delta}_1$. Thus, $(\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M}-r_2^2\boldsymbol{P})\boldsymbol{d}_\infty\le 0 \Leftrightarrow \boldsymbol{D}_2\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M}\boldsymbol{D}_2\boldsymbol{d}_\infty\le r_2^2\boldsymbol{d}_\infty$ implies $\lambda_{\max}(\boldsymbol{D}_2\boldsymbol{M}^T\boldsymbol{P}\boldsymbol{M}\boldsymbol{D}_2)\le r_2^2 \Leftrightarrow \|\boldsymbol{M}\|_2^{\boldsymbol{D}_2}\le r_2$. ■