Discrete Fuzzy Polynomial Observers

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Abstract— **This paper proposes a discrete version of a previous work of the authors in which an observer design for nonlinear systems based on the fuzzy polynomial (Taylor series) representation is discussed. The methodology provides single or fuzzy observer gains which are polynomial in the output and in the estimated states, if state and error (initially) lie in some operation regions. Then, some bounds on the error can be proved.**

I. INTRODUCTION

Nowadays nonlinear systems research is focused on the use of a systematic modelling methodology which allows to prove stability/performance criteria. In this framework a class of nonlinear systems can be transformed into the so-called fuzzy Takagi-Sugeno (TS) form [1], which is a time-varying convex combination of linear models, using the sector nonlinearity methodology [2].

The above transformation is exact, meaning that the TS models are equivalent to the original non-linear models (in many cases, only locally in a compact region of interest containing the origin). In this way, nonlinear systems may be "embedded" in a linear time-varying (LTV) dynamic.

Design of state observers for nonlinear systems using TS models both, continuous and discrete, has been actively considered during last decades [2, Cap.4], [3].

Polynomial fuzzy modelling was introduced in [4], [5] where it is shown that sector nonlinearity technique can be extended to the polynomial case by assuming the equations of a nonlinear system expressed, via algebraic transformations, as $\dot{x_i} = p_i(z, x, u)$ where p_i are polynomials depending on input u, state x and a vector z of continuous functions of any relevant state, input or exogenous variable.

This technique builds a family of progressively more precise polynomial parameter-varying (PPV) models, which are based on the Taylor series approach [6], [7].

Sum-of-squares tools such as SOSTOOLS [8], [9] have been recently developed; they transform polynomial sum-of-squares problems (SOS) to semidefinite programming (SDP) problems, and then call on standard solvers which are widely used by the LMI community. Basically, a SOS solver tries to verify whether a polynomial is non-negative by the way of expressing it as a sum of squares of other polynomials.

Those tools allow doing stability analysis, stabilization and observer design for polynomial systems. Continuous observer design for polynomial systems with bounded disturbances has been introduced in [10] and, inspired on it, the metodology has been extended to a more general class of nonlinear systems (using PPV modelling approach) in [11], also using fuzzy observer gains (if premises were measurable). In this framework, the work [12] also proposes a simultaneous controller/observer design using fuzzy polynomial models but only in the particular case where separation principle holds.

In this paper the objective is to extend previous continuous observer design methodology to the discrete case. In order to verify results with previous work, an Euler discretization of the same system proposed in [11] is used for the example.

This paper is structured as follows: Section 2 presents the premises and deals with the problem of discrete fuzzy polynomial observer design, Section 3 presents the representation of the trial system as a discrete PPV model, particular conditions for the observer design and shows the simulation results. Finally, Section 4 concludes with remarks.

II. DISCRETE FUZZY POLYNOMIAL OBSERVER DESIGN

Consider an open-loop discrete nonlinear system:

$$\begin{aligned} x_{k+1} &= f(z(x_k)) + B(z(x_k))w_k \\ y_k &= C(z(x_k)) \end{aligned} \tag{1}$$

where x_k is the state at instant k, $z(x_k)$ is a vector of polynomials in the state, and B, C are polynomial functions of the augmented state. The input w_k is considered to be an unmeasurable disturbance bounded in some sense, for instance an integral bound $\sum_{k=0}^{N} w_k^T w_k \leq \beta$ or an instantaneous bound $w_k^T w_k \leq \beta^2$. In the following $z(x_k)$ will be expressed with shorthand z_k.

Applying the Taylor-series methodology in [6], the system can be expressed as:

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} \mu_i(z_k) \left(p_i(z_k) + B_i(z_k)w_k\right) \\ y_k &= C(z_k) \end{aligned} \tag{2}$$

where r denotes the number of fuzzy "rules", usually a power of 2. In the following, as the modelling methodology is exact in a region of interest Ω (defined for instance as a symmetric polytope that contains $x_k = 0$; $\Omega = x_k \backslash |a_i^T x_K| \leq \zeta \; i : 1, \ldots, n_p$), we will identify:

$$f(z_k) = \sum_{i=1}^{r} \mu_i(z_k) p_i(z_k)$$

$$B(z_k) = \sum_{i=1}^{r} \mu_i(z_k) B_i(z_k)$$

A fuzzy polynomial observer will be set up as:

$$\hat{x}_{k+1} = \sum_{i=1}^{r}(\mu_i(\hat{z}_k)p_i(\hat{z}_k) + L_i(y_k, \hat{z}_k)(y_k - C(\hat{z}_k)) \quad (3)$$

where $\hat{x}_k$ denotes the estimated state and $\hat{z}_k = z(\hat{x}_k)$, being L_i a set of polynomial functions of the measured output and the estimated state (via the expanded vector z_k). In a similar way with the model, the notation:

$$L(y_k, \hat{z}_k) = \sum_{i=1}^{r} \mu_i(z_k) L_i(y_k, \hat{z}_k)$$

will be later used.

The error dynamics $e_k = x_k - \hat{x}_k$ will be given by:

$$e_{k+1} = f(z_k) - f(\hat{z}_k) - L(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + B(z_k)w_k \quad (4)$$

A candidate Lyapunov function may be defined as

$$V(e_k) = e_k^T Q e_k$$

Then, its increment can be written as:

$$\Delta V = V_{k+1} - V_k = [f(z_k) - f(\hat{z}_k) - L(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + B(z_k)w_k]^T Q[f(z_k) - f(\hat{z}_k) - L(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + B(z_k)w_k] - (x_k - \hat{x}_k)^T Q(x_k - \hat{x}_k) \quad (5)$$

One possible objective in observer design is discussed below.

A. Invariant sets

Ensuring

$$\Delta V(x_k, \hat{x}_k, w_k) - w_k^T w_k < 0 \quad (6)$$

ensures

$$V_k < V_0 + \sum_{k=0}^{N} w_i^T w_i$$

for all $k \in \mathbb{N}$. By assumption a bound $\sum_{k=0}^{N} w_i^T w_i \leq \beta$ will be assumed available.

In polynomial SOS programming, most inequalities do not hold globally but do it locally in some regions. As in [11], the above inequality (6) will be required to hold for all x_k, $\hat{x}_k$ in a particular region defined by the sets:

$$\chi_S = \{x \in \Re^n | G_s(x_k) > 0\}$$

$$\chi_{\hat{S}} = \{\hat{x} \in \Re^n | G_{\hat{s}}(\hat{x}_k) > 0\}$$

$$\chi_E = \{e \in \Re^n | G_e(e_k) > 0\}$$

where $\chi_S \subset \Omega$, $\chi_{\hat{S}} \subset \Omega$, $\chi_E(k = 0) \subset \chi_S$ (assuming $\hat{x}_0 = 0$ as it is freely assignable) and defining:

$$G_s(x_k) = 1 - x_k^T S_X x_k$$

$$G_{\hat{s}}(\hat{x}_k) = 1 - \hat{x}_k^T S_{\hat{X}} \hat{x}_k$$

$$G_e(e_k) = 1 - e_k^T S_E e_k$$

where $S_X \succ 0$, $S_{\hat{X}} \succ 0$ and $S_E \succ 0$.

Then, if the initial error is small enough so that $V_0 \leq \alpha$, then

$$V_k = e_k^T Q e_k \leq \alpha + \beta \quad \forall\ k \quad (7)$$

The objective is to compute the smallest ellipsoid (biggest Q) such that the error will be always contained in it if the initial error is small and the disturbance has finite square integral. In this way, the course of action will be maximize a parameter γ subject to $Q > \gamma I$ and (6). Then, the maximum value ϕ so that the ellipsoid $x_k^T Q x_k \leq \phi \subset \chi_E$ will give the maximum initial state ellipsoid $x_k^T Q x_k \leq \alpha$ (equal to initial error as $\hat{x}_0 = 0$ by assumption) so that the future error will be guaranteed not to leave $G_e(e_k) > 0$, by $V(e_k) < \phi = \alpha + \beta$.

Carrying out some operations in a similar way to [11], we can obtain the discrete fuzzy observer gains solving the following SOS problem:

$$max \quad \gamma \quad s.t.$$

$$\psi^T [Q - \gamma I] \psi \in \Sigma_\psi \quad (8)$$

$$\psi^T \begin{bmatrix} F1_{ij} & F2_{ij}^T \\ F2_{ij} & Q \end{bmatrix} \psi \in \Sigma_{\{x_k, \hat{x}_k, w_k, \psi\}} \forall\ i, j : 1, ..., r \quad (9)$$

with

$$F1_{ij} = e_k^T Q e_k + w_k^T w_k - \phi_{ij} G_s(x_k) - \theta_{ij} G_{\hat{x}}(\hat{x}_k) - \varphi_{ij} G_e(e_k)$$

$$F2_{ij} = Q(p_i(z_k) - p_j(\hat{z}_k)) - H_j(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + QB_i(z_k)w_k$$

where $\{\phi_{ij}, \theta_{ij}, \varphi_{ij}\} \in \Sigma_{\{x_k, \hat{x}_k\}}$ and $L_j(y_k, \hat{z}_k) = Q^{-1} H_j(y_k, \hat{z}_k)$.

Proof: Condition (8) means $V(e) > 0$ and, at same time, computes the smallest ellipsoid by maximizing the eigenvalues of Q.

Condition (9) means that (6) holds inside the interest state-space regions considered:

$$\Delta V(e_k) - w^T w + \phi G_s(x_k) + \theta G_{\hat{x}}(\hat{x}_k) + \varphi G_e(e_k) = e_{k+1}^T Q Q^{-1} Q e_{k+1} - e_k^T Q e_k - w^T w + \phi G_s(x_k) + \theta G_{\hat{x}}(\hat{x}_k) + \varphi G_e(e_k) < 0$$

which applying Schur complement and using PPV models, leads to (9).

Note. Σ_ν means a SOS polynomial in the variables ν.

Decay rate.

If inequality (6) is changed to:

$$V_{k+1} - \delta V_k - w_k^T w_k < 0 \quad (10)$$

and the following term is renamed as

$$V_{k+1} - \delta V_k = \mathcal{G}_k$$

then:

$$V_k = \delta^k V_0 + \sum_{i=0}^{k-1} \delta^i \mathcal{G}_{k-i-1} \quad (11)$$

Hence, as by (10), $|\mathcal{G}_k| \leq w_k^T w_k$, it implies:

$$V_k < \delta^k V_0 + \sum_{i=0}^{k-1} \delta^i w_{k-i-1}^T w_{k-i-1}$$

So considering bounded vanishing disturbances $\sum_{i=0}^{\infty} w_i^T w_i \leq \beta$:

$$\sum_{i=0}^{k-1} \delta^i w_{k-i-1}^T w_{k-i-1} \leq \beta, \quad 0 \leq \delta \leq 1 \quad (12)$$

allows bounding V_k by:

$$V_k \leq \delta^k V_0 + \beta \quad (13)$$

Hence, changing $e_k^T Q e_k$ by $\delta e_k^T Q e_k$ in the term $F1_{ij}$ allows giving the required decay-rate result.

Measurable premise variables

The "fuzzy" gain above is useless here because any L_j must make the expression hold for all i. In the particular case where $\mu(z_k)$ depends only on measurable components, such as $\mu(y_k)$, then we have a single-sum

$$\sum_{i=0}^{r} \mu_i(z_k) \pi_i(x_k, \hat{x}_k, w_k, \psi) \geq 0 \quad (14)$$

with:

$$\pi_i = \psi^T \begin{bmatrix} F1_i & F2_i^T \\ F2_i & Q \end{bmatrix} \psi$$

being

$$\begin{aligned} F1_i = {} & \delta e_k^T Q e_k + w_k^T w_k \\ & - \phi_i G_s(x_k) - \theta_i G_{\hat{x}}(\hat{x}_k) - \varphi_i G_e(e_k) \end{aligned}$$

$$\begin{aligned} F2_i = {} & Q(p_i(z_k) - p_i(\hat{z}_k)) \\ & - H_i(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + QB_i(z_k) w_k \end{aligned}$$

Furthermore, even under the assumption of measurable premises, a fuzzy output $y_k = \sum_i \mu_i(z_k) C_i(z_k)$ will give rise to a double sum too:

$$\sum_{i=0}^{r} \sum_{j=0}^{r} \mu_i(z_k) \mu_j(z_k) \pi_{ij}(x_k, \hat{x}_k, w_k, \psi) \geq 0 \quad (15)$$

with:

$$\pi_{ij} = \psi^T \begin{bmatrix} F1_{ij} & F2_{ij}^T \\ F2_{ij} & Q \end{bmatrix} \psi$$

being

$$\begin{aligned} F1_{ij} = {} & \delta e_k^T Q e_k + w_k^T w_k \\ & - \phi_{ij} G_s(x_k) - \theta_{ij} G_{\hat{x}}(\hat{x}_k) - \varphi_{ij} G_e(e_k) \end{aligned}$$

$$\begin{aligned} F2_{ij} = {} & Q(p_i(z_k) - p_i(\hat{z}_k)) \\ & - H_i(y_k, \hat{z}_k)(C_j(z_k) - C_j(\hat{z}_k)) + QB_i(z_k) w_k \end{aligned}$$

hence, with the change $\mu_i = \sigma_i^2$ we get:

$$\sum_{i=0}^{r} \sum_{j=0}^{r} \sigma_i^2 \sigma_j^2 \pi_{ij}(x_k, \hat{x}_k, w_k, \psi) \geq 0 \quad (16)$$

which is a polynomial in variables $x_k, \hat{x}_k, w_k, \psi, \sigma$ which may be tested for SOS.

B. Reachable sets

Consider the problem of observer design, assuming that the state and its estimate lie in $G_s(x_k) > 0; G_{\hat{s}}(\hat{x}_k) > 0$. The objective now is to prove that for large errors (but state in the working zone) and in presence of peak disturbances, the error decreases and finally lies in a small zone near equilibrium.

That amounts to prove, for *fixed* S, that if $e_k^T S e_k > 1$, $w_k^T w_k \leq \beta^2$ and the state and its estimate are inside the working region, then $\Delta V \leq 0$.

The goal is to minimise the volume of the region $\{e^T S e < 1\}$ and this will be done by solving the following semidefinite programming problem:

$$\begin{aligned} & max \quad logdet(S) \quad s.t. \\ & \psi^T \begin{bmatrix} F1_{ij} & F2_{ij}^T \\ F2_{ij} & Q \end{bmatrix} \psi \in \Sigma_{\{x_k, \hat{x}_k, w_k, \psi\}} \forall\ i, j : 1, ..., r \end{aligned} \quad (17)$$

with

$$\begin{aligned} F1_{ij} = {} & \delta e_k^T Q e_k - \eta_{ij}(\beta^2 - w_k^T w_k) \\ & - \phi_{ij} G_s(x_k) - \theta_{ij} G_{\hat{x}}(\hat{x}_k) - \varphi_{ij}(e_k^T S e_k - 1) \end{aligned}$$

$$\begin{aligned} F2_{ij} = {} & Q(p_i(z_k) - p_j(\hat{z}_k)) \\ & - H_j(y_k, \hat{z}_k)(C(z_k) - C(\hat{z}_k)) + QB_i(z_k) w_k \end{aligned}$$

where $0 < \delta \leq 1$, $\{\eta_{ij}, \phi_{ij}, \theta_{ij}, \varphi_{ij}\} \in \Sigma_{\{x_k, \hat{x}_k\}}$ and $L_j(y_k, \hat{z}_k) = Q^{-1} H_j(y_k, \hat{z}_k)$.

Proof: It's easy to achieve by applying Schur complement in a similar way to objective II-A.

III. EXAMPLE

Now the example of observer design following the objective II-A is provided.

Let us consider the Euler discretization of the two-rule PPV model (2) in [11] at sampling period $T = 0.01$ sec. with membership functions and its corresponding polynomial functions $f_i(x_k)$ and $B(x_k)$ given by:

$$\mu_1 = 1 - e^{-(0.1 y_k)^2}; \ \mu_2 = 1 - \mu_1;$$

$$f_1(x_k) = \begin{bmatrix} x_{1k} + 0.012 x_{2k} \\ -0.01 x_{1k} + 1.012 x_{2k} - 0.01 x_{1k}^2 x_{2k} \end{bmatrix};$$

$$f_2(x_k) = \begin{bmatrix} x_{1k} + 0.0098 x_{2k} \\ -0.01 x_{1k} + 1.0098 x_{2k} - 0.01 x_{1k}^2 x_{2k} \end{bmatrix};$$

$$B(x_k) = \begin{bmatrix} 0 \\ 0.005 x_{1k} x_{2k} \end{bmatrix};$$

and the output equation is:

$$y_k = x_{1k}^3 + 0.5x_{2k}$$

Note. The results are only approximate as the Euler discretization is not exact; they are valid for small enough sampling period. As larger period more inaccurate is the solution found but not because of the proposed SOS methodology.

The region of study χ_S, $\chi_{\hat{S}}$, and χ_E for the states, estimated states and error are specified by the matrices:

$$S_X = S_{\hat{X}} = S_E = 10^{-2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e., a sphere of radius 10 is assumed to be enough to contain the trajectories of the state and estimated state over the course of simulations. And the highest degree of the fuzzy-polynomial observer $L(y_k, \hat{x}_k)$ is fixed at 2.

With those assumptions, we are also in the case of measurable premises so that an actual fuzzy observer can be sought.

One solution found for the problem is:

$$\gamma = 0.0269;\ Q = \begin{bmatrix} 0.37536 & 0.005358 \\ 0.005358 & 0.054277 \end{bmatrix};$$

and the fuzzy observer is:

$$L = \mu_1 L_1 + \mu_2 L_2 = \mu_1 \begin{pmatrix} L_{11} \\ L_{12} \end{pmatrix} + \mu_2 \begin{pmatrix} L_{21} \\ L_{22} \end{pmatrix}$$

being the computed polynomial observer gains L_1, L_2 given by:

$$\begin{aligned} L_1(y_k, \hat{x}_k) = 10^{-3}\Bigg(&\begin{bmatrix} 20.11 \\ 31.55 \end{bmatrix} + \begin{bmatrix} -0.0338 \\ -0.0957 \end{bmatrix}\hat{x}_{1k}^2 \\ &+ \begin{bmatrix} -0.004 \\ 0.0069 \end{bmatrix}\hat{x}_{2k}^2 + \begin{bmatrix} -0.0017 \\ -0.0066 \end{bmatrix} y_k\hat{x}_{1k} \\ &+ \begin{bmatrix} 0 \\ -0.002 \end{bmatrix} y_k\hat{x}_{2k} \begin{bmatrix} 0.0001 \\ -0.0023 \end{bmatrix}\hat{x}_{1k}\hat{x}_{2k}\Bigg); \end{aligned}$$

$$\begin{aligned} L_2(y_k, \hat{x}_k) = 10^{-3}\Bigg(&\begin{bmatrix} 16.345 \\ 27.95 \end{bmatrix} + \begin{bmatrix} -0.0212 \\ -0.07 \end{bmatrix}\hat{x}_{1k}^2 \\ &+ \begin{bmatrix} -0.0054 \\ -0.00625 \end{bmatrix}\hat{x}_{2k}^2 + \begin{bmatrix} -0.0008 \\ -0.0035 \end{bmatrix} y_k\hat{x}_{1k} \\ &+ \begin{bmatrix} 0.00002 \\ -0.002 \end{bmatrix} y_k\hat{x}_{2k} \begin{bmatrix} 0.00027 \\ 0.0007 \end{bmatrix}\hat{x}_{1k}\hat{x}_{2k}\Bigg); \end{aligned}$$

With this fuzzy discrete observer the results show that $\phi = \alpha + \beta = 5.42$, so we have proved that starting with any initial conditions inside $V_0 < \alpha = 4.22$ the error dynamics is stable in the sense that it will not abandon the interest zone χ_E.

Once this observer is in place, the decay rate can be proved to be $\delta = 0.9995$, so it is actually proved that under vanishing disturbances, the error will reach zero.

For simulation, consider two different scenarios; first starting from initial conditions $x(0) = [-2\ 7]$ and considering that there is a disturbance $w_k = 0.4775\ 0.9^k$ applied at time $t = 0.3$ seconds, and then another scenario starting from $x(0) = [-1.5\ \ -7.8]$ and considering a disturbance $w_k = \sqrt{1.2}$ only if $k = 1$.

As it can be seen easily, in the two cases the disturbances verify $\sum_{k=0}^{\infty} w_k^T w_k = 1.2$, so $\beta = 1.2$ can be set up as the square integral disturbance bound.

On next figures we can see the trajectory of the states and estimated states corresponding to the first scenario.

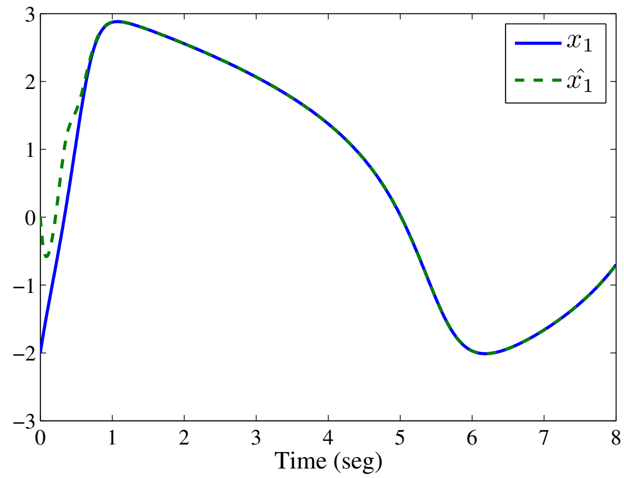

Fig. 1. Evolution of x_1 and $\hat{x_1}$

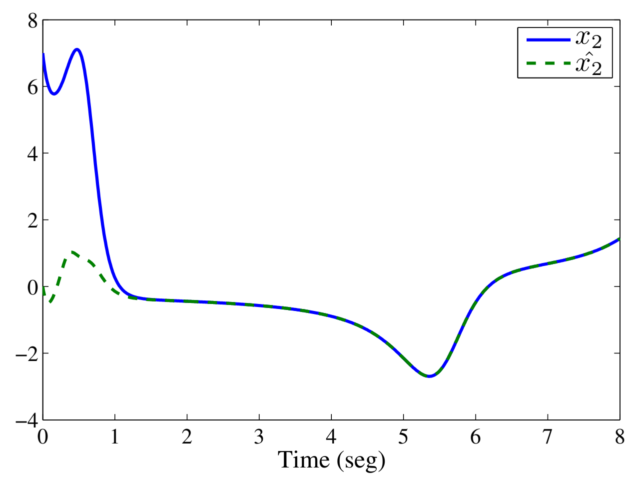

Fig. 2. Evolution of x_2 and $\hat{x_2}$

In figures 1 and 2 the states x_1 and x_2 are rearched in about one second by the estimated ones instead of the disturbance. It can be checked too in Figure 3 where trajectories of the system and the observed system are shown.

Now in figure 4 the evolution of the Lyapunov function of the error is represented for the two scenarios considered. It can be checked that the Lyapunov function decreases except when the disturbance is applied but, instead of this, finally it tends to zero.

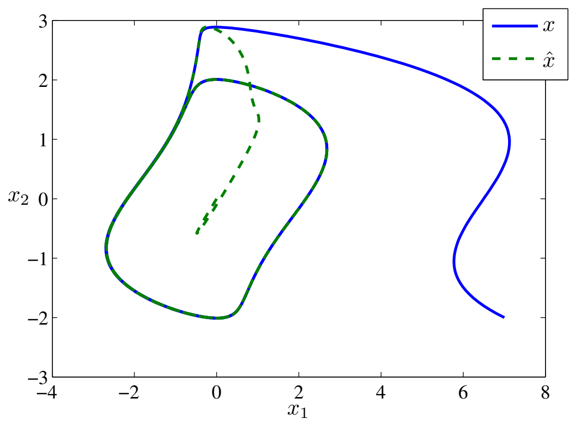

Fig. 3. Trajectory of x and $\hat{x}$

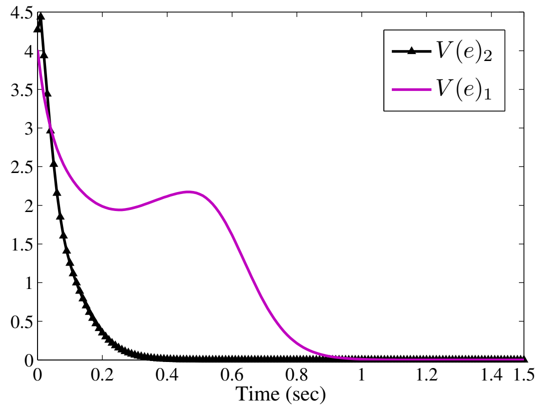

Fig. 4. Lyapunov function V(e)

Finally Figure 5 shows the trajectory of the errors starting from the initial conditions (inside the ellipsoid $V(e) < \alpha$). The two trajectories don't abandon region χ_E and go to zero instead of the presence of the disturbances. In fact, the size of initial error region obtained with the same disturbance bound in the continuous case [11] is a bit small, as it is expected due to conservativeness in the discretization method.

IV. Conclusions

This paper shows how to design a stable discrete fuzzy-polynomial observer for a nonlinear system under some assumptions (bounded disturbances, initial estimated state equal to the equilibrium point, measurable premises). The result is a fuzzy observer gain which is polynomial on the output and estimated states at k period. If premises were unmeasurable a non-fuzzy single-gain observer would be the only option, at least with the proposed method. The proposed discrete methodology has been verified using a discrete model from a continuous one of a previous work.

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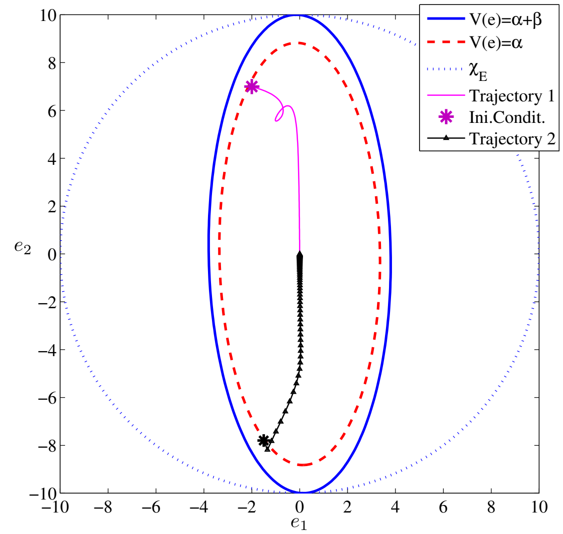

Fig. 5. Error trajectory

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