H_∞ control for time-delay Takagi-Sugeno fuzzy systems with actuator saturation

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***Abstract*— This paper focuses on H_∞ control design problem for T-S fuzzy time-delay systems with actuator saturation. Based on Lyapunov-Krasovskii Functional (LKF) combining the introduction of relaxed matrices, a H_∞ controller design method is presented in Linear matrix inequality (LMI) forma. The controller designed is capable to reject the disturbance assumed known and norm bounded. A numerical example is provided to verify the effectiveness of the proposed results.**

Index-Terms– LMI, Parallel Distributed Compensation, H_∞ control, Lyapunov- Krasovskii Functional, T-S fuzzy systems, Stabilization , Time-delay, saturation constraint.

I. INTRODUCTION

These two last decades, T-S fuzzy models have been recognized as a popular and powerful tool in approximating a wide class of nonlinear systems [1]. As a consequence, T-S fuzzy models control have been extensively studied [18] -[19]. In practical situation, time delay often occurs in many dynamics systems, such as transportation systems, communication systems, chemical processing systems, environmental systems and power systems. In recent years, T-S fuzzy models have been extended to deal with nonlinear systems with time delays. In [3]-[4] and the references therein, stability analysis and synthesis based on the parallel distributed compensation scheme (PDC) were discussed. The observer based fuzzy control was treated in [5]-[7] and the references therein. In [6], the problem of H_∞ exponential stabilization was developed.
On the other hand, a main problem which is always inherent to all dynamical systems is the presence of actuator saturation. The class of systems with saturations has enjoyed great interest during the last three decades. Even for linear systems, this problem has been an active area of research for many years. Two main approaches have been developed in the literature :
- The first one, the so-called positive invariance approach, is based on the design of controllers which work inside a region of linear behavior where saturations do not occur (see [15]-[16] and the references therein). One can also cite the work of [8] where the synthesis of the controller is presented as a technique of partial eigenstructure assignment. This resolution was also associated to the constrained regulator problem. This technique has already been applied to fuzzy systems by [17] and [9]. This approach is referred to as unsaturating controller.
- The second approach allows saturations to take effect while guaranteeing asymptotic stability (see [11] and the references therein). This approach, where the control may be saturated, leads to a bounded region of stability which is ellipsoidal and symmetric. This region can be easily obtained by the resolution of a set of LMIs. In this case the approach is referred to as saturating controller. In [12], besides the saturated character of the control, additional constraints on the increment or rate are taken into account. The first works on saturated fuzzy systems without delay can be found in [13] and [14]. In this paper, we study the asymptotic stabilization of T-S fuzzy systems with state delay and actuator saturation.
The organization of this paper is as follows. Section 2 presents the description of T-S fuzzy model with time varying delay and a preliminary result. Section 3 contains the main result where a new delay dependent stabilization conditions with disturbance rejection for T-S fuzzy systems with saturation are established. In section 4, simulation example is given to demonstrate the design effectiveness. Some conclusions are made in section 5.

II. PROBLEM FORMULATION AND PRELIMINARY RESULT

Consider a nonlinear system with state-delay and actuator saturation which could be represented by a T-S fuzzy model
Plant Rule $i(i = 1, 2, \cdots, r)$: If θ_1 is μ_{i1} and $\cdots$ and θ_p is μ_{ip} THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_{\tau i} x(t-\tau(t)) + B_i sat(u(t)) \\ &\quad + B_{wi} w(t) \\ z(t) &= C_i x(t) + C_{\tau i} x(t-\tau(t)) + D_i sat(u(t)) \\ x(t) &= \psi(t), t \in [-\overline{\tau}, 0], \end{aligned} \quad (1)$$

where $\theta_j(x(t))$ and $\mu_{ij} (i = 1, ..., r, j = 1, ..., p)$ are respectively the premise variable and the fuzzy sets ; $\psi(t)$ is the initial conditions ; $x(t) \in \Re^n$ is the state ; $u(t) \in \Re^m$ is the control input ; $z(t) \in \Re^q$ is the controlled output variable ; $w(t) \in \Re^l$ is the disturbance variable ; the saturation function is assumed here to be normalized ; r is the number of IF-THEN rules ; the time delay, $\tau(t)$, is a time-varying continuous function that satisfies

$$0 \le \tau(t) \le \overline{\tau}, \dot{\tau}(t) \le \beta \quad (2)$$

By using the common used center-average defuzzifier, product inference and singleton fuzzifier, the T-S fuzzy systems can be inferred as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} h_i(\theta(x(t)))[A_i x(t) + A_{\tau i} x(t-\tau(t)) \\ &\quad + B_i sat(u(t)) + B_{wi} w(t)] \\ z(t) &= \sum_{i=1}^{r} h_i(\theta(x(t)))[C_i x(t) \\ &\quad + C_{\tau i} x(t-\tau(t)) + D_i sat(u(t))] \end{aligned} \quad (3)$$

where $\theta(x(t)) = [\theta_1(x(t)), ..., \theta_p(x(t))]$ and $\nu_i(\theta(x(t)))$: $\Re^p \rightarrow [0, 1], i = 1, \cdots, r$, is the membership function of the system with respect to the ith plan rule. Denote $h_i(\theta(x(t))) = \nu_i(\theta(x(t)))/\sum_{i=1}^{r} \nu_i(\theta(x(t)))$. It is obvious that $h_i(\theta(x(t))) \geq 0$ and $\sum_{i=1}^{r} h_i(\theta(x(t))) = 1$

In the sequel, for brivety we use h_i to denote $h_i(\theta(x(t)))$. The design of state feedback stabilizing fuzzy controllers for the fuzzy system (3) is based on the Parallel Distributed Compensation.

Controller Rule $i(i = 1, 2, .., r)$: If θ_1 is μ_{i1} and $\cdots$ and θ_p is μ_{ip} THEN

$$u(t) = K_i x(t) \tag{4}$$

The overall state feedback control law is represented by

$$u(t) = \sum_{i=1}^{r} h_i K_i x(t) \tag{5}$$

Consider a linear time-invariant system with actuator saturation given by :

$$\dot{x}(t) = Ax(t) + Bsat(Kx(t)) \tag{6}$$

Define the following subsets of $\mathbb{R}^n$:

$$\varepsilon(P, \rho) = \{x \in \mathbb{R}^n | x^T P x \leq \rho\}, \tag{7}$$

$$\mathcal{L}(K) = \{x \in \mathbb{R}^n || K_l x| \leq 1, \, l = 1, \ldots, m\}, \tag{8}$$

with P a positive definite matrix, $\rho > 0$ and K_l the lth row of the matrix $K \in \mathbb{R}^{m \times n}$. Thus $\varepsilon(P, \rho)$ is an ellipsoid while $\mathcal{L}(K)$ is a polyhedral consisting of states for which the saturation does not occur.

Lemma 1: [11] For all $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$ such that $|w_l| < 1, l \in [1, m]$

$$sat(u) \in co\{E_s u + E_s^- w, s \in [1, \eta]\} \; ; \quad \eta = 2^m \tag{9}$$

where co denotes the convex hull.

Consequently, there exist $\delta_1 \geq 0, \ldots, \delta_\eta \geq 0$ with $\sum_{s=1}^{\eta} \delta_s = 1$ such that,

$$sat(u) = \sum_{s=1}^{\eta} \delta_s [E_s u + E_s^- w] \tag{10}$$

Here, E_s is an m by m diagonal matrix with elements either 1 or 0 and $E_s^- = I_m - E_s$. There are 2^m possible matrices of this type. One can also consult the work of [12] for more details and other extensions to linear systems with both constraints on the control and the increment or rate of the control.

Assume that $x(t) \in \mathcal{L}(H_i)$, using Lemma (1), the saturated feedback control (5) can be written as :

$$sat(K_i x(t)) = \sum_{s=1}^{\eta} \delta_s [E_{is} K_i + E_{is}^- H_i] x(t) \; ; \tag{11}$$

$$\delta_s \geq 0, \; \sum_{s=1}^{\eta} \delta_s = 1 \tag{12}$$

Combining (1), (5) and (11), the closed-loop saturated fuzzy system can be expressed as follows :

$$\dot{x}(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s [\widehat{A}_{ijs} x(t) + A_{\tau i} x(t - \tau(t))]$$
$$z(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s [\widehat{C}_{ijs} x(t) + C_{\tau i} x(t - \tau(t))] \tag{13}$$

with $x(t) = \psi(t)$ for $t \in [-\bar{\tau}, 0]$ and

$$\widehat{A}_{ijs} := A_i + B_i(E_{is} K_j + E_{is}^- H_j), \quad s \in [1, \eta] \tag{14}$$
$$\widehat{C}_{ijs} := C_i + D_i(E_{is} K_j + E_{is}^- H_j), \quad s \in [1, \eta] \tag{15}$$

Denote

$$\widehat{A}(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s \widehat{A}_{ijs}, A_\tau(t) = \sum_{i=1}^{r} h_i A_{\tau i},$$
$$B_w(t) = \sum_{i=1}^{r} h_i B_{wi}, \widehat{C}(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s \widehat{C}_{ijs}$$
$$C_\tau(t) = \sum_{i=1}^{r} h_i C_{\tau i}$$

we get

$$\begin{array}{l} \dot{x}(t) = \widehat{A}(t)x(t) + A_\tau(t)x(t - \tau(t)) + B_w(t)w(t) \\ z(t) = \widehat{C}(t)x(t) + C_\tau(t)x(t - \tau(t)) \end{array} \tag{16}$$

For a prescribe scalar $\gamma > 0$, the performance index J is defined as
$J = \int_0^\infty (z(s)^T z(s) - \gamma^2 w(s)^T w(s)) ds$
the purpose of this work is to design a H_∞ fuzzy control law (5) such that the following requirement are satisfied

- The closed-loop fuzzy system (16) with $w(t) = 0$ is asymptotically stable.
- Under the zero initial condition, system (16) satisfies $\|z(t)\|_\infty < \gamma \|w(t)\|_\infty$ for any non-zero $w(t)$.

Lemma 2: [19] Consider a negative definite matrix $\Pi < 0$. Given a matrix X of appropriate dimension such that $X^T \Pi X < 0$, then, $\exists \lambda \in \Re^+$ such that
$X^T \Pi X \leq -2\lambda X - \lambda^2 \Pi^{-1}$

III. MAIN RESULTS

A. Time-delay dependent stabilization conditions with saturating controller

Theorem 1: System (16) is asymptotically stable $\forall x \in \varepsilon(P, \rho)$, if there exist some matrices $P > 0$, $S > 0$, $Z > 0$, Y_i, T_i, H_i and X_{ij} with X_{ii} symetrical, $i, j = 1, 2, .., r$ and $i \leq j$, $s = 1, \ldots, \eta$, such that the following conditions hold :

$$\Xi_{ijs} + \Xi_{jis} \leq X_{ij} + X_{ij}^T \tag{17}$$

$$\begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r}^T & \cdots & X_{rr} \end{bmatrix} \leq 0 \tag{18}$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(H_i) \tag{19}$$

where

$$\Xi_{ijs} = \begin{bmatrix} \Phi_{ijs} & \Theta_{ijs}^T Z & W_i^T & U_{ijs}^T \\ & -\frac{1}{\overline{\tau}} Z & 0 & 0 \\ & * & -\frac{1}{\overline{\tau}} Z & 0 \\ & * & * & -I \end{bmatrix} \tag{20}$$

in which Φ_{ijs} is shown in (31) at the bottom of the page.

$$\Theta_{ijs} = \begin{bmatrix} \widehat{A}_{ijs} & A_{\tau i} & B_{wi} \end{bmatrix} \tag{21}$$

$$W_i = \begin{bmatrix} Y_i^T & T_i^T & 0 \end{bmatrix} \tag{22}$$

$$U_{ijs} = \begin{bmatrix} \widehat{C}_{ijs} & C_{\tau i} & 0 \end{bmatrix} \tag{23}$$

Proof : Assume that $x(t) \in \varepsilon(P, \rho)$. According to condition (19), $x(t) \in \mathcal{L}(H_i)$. In this case, using Lemma (1), saturated feedback control (5) can be used to write system (3) as (13). The objective is then to guarantee the local asymptotic stability of this system inside the level set $\varepsilon(P, \rho)$.

For the rest of proof, we choose the Lyapunov-Krasovskii functional as

$$\begin{aligned} V(x(t)) = {} & x(t)^T P x(t) + \int_{t-\tau(t)}^{t} x(\alpha)^T S x(\alpha) d\alpha \\ & + \int_{-\overline{\tau}}^{0} \int_{t+\sigma}^{t} \dot{x}(s)^T Z \dot{x}(s) ds d\sigma \end{aligned} \tag{24}$$

Then, we have

$$\begin{aligned} \dot{V}(x(t)) \quad & + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \\ & = 2x(t)^T P \dot{x}(t) + x(t)^T S x(t) \\ & -(1 - \dot{\tau}(t)) x(t - \tau(t))^T S x(t - \tau(t)) \\ & + \overline{\tau} \dot{x}(t)^T Z \dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}^T(s) Z \dot{x}(s) ds \\ & + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \end{aligned} \tag{25}$$

taking into account the Newton-Leibniz formula
$x(t - \tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)^T ds$
we obtain equation (32) as shown at the bottom of the page.
where $Y(t) = \sum_{i=1}^{r} h_i Y_i$, $T(t) = \sum_{i=1}^{r} h_i T_i$
let $\eta(t)^T = [x(t)^T, x(t - \tau(t))^T, w(t)^T]^T$, we obtain

$$\begin{aligned} & \dot{V}(x(t)) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \\ & \leq \eta(t)^T [\tilde{\Phi}(t) + U(t)^T U(t) + \overline{\tau} W(t)^T Z^{-1} W(t)] \eta(t) \\ & - \int_{t-\tau(t)}^{t} [\eta(t)^T W(t)^T + \dot{x}(s)^T Z] Z^{-1} \\ & \times [\eta(t)^T W(t)^T + \dot{x}(s)^T Z]^T ds \end{aligned} \tag{26}$$

where $\tilde{\Phi}(t)$ is shown in (33) at the bottom of the page.

$$W(t) = \begin{bmatrix} Y(t)^T & T(t)^T & 0 \end{bmatrix} \tag{27}$$

$$U(t) = \begin{bmatrix} \widehat{C}(t) & C_\tau(t) & 0 \end{bmatrix} \tag{28}$$

Denote

$$\Xi(t) = \begin{bmatrix} \Phi(t) & \Theta(t)^T Z & W(t)^T & U(t)^T \\ & -\frac{1}{\overline{\tau}} Z & 0 & 0 \\ & * & -\frac{1}{\overline{\tau}} Z & 0 \\ & * & * & -I \end{bmatrix} \tag{29}$$

where $\Phi(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s \Phi_{ijs}$, $\Theta(t) = \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s \Theta_{ijs}$

By applying Schur complement, $\tilde{\Phi}(t) + U(t)^T U(t) + \overline{\tau} W(t)^T Z^{-1} W(t) < 0$ is equivalent to :

$$\begin{aligned} \Xi(t) &= \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s \Xi_{ijs} \\ &= \frac{1}{2} \sum_{s=1}^{\eta} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \delta_s (\Xi_{ijs} + \Xi_{jis}) \\ &\leq \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j (X_{ij} + X_{ij}^T) \\ &= [h_1 I, \cdots, h_r I] \begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r}^T & \cdots & X_{rr} \end{bmatrix} \begin{bmatrix} h_1 I \\ \vdots \\ h_r I] \end{bmatrix} \leq 0 \end{aligned}$$

This implies that

$$\dot{V}(x(t)) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) < 0 \tag{30}$$

$$\Phi_{ijs} = \begin{bmatrix} P\widehat{A}_{ijs} + \widehat{A}_{ijs}^T P + S + Y_i + Y_i^T & PA_{\tau i} - Y_i + T_i^T & PB_{wi} \\ & -(1-\beta)S - T_i - T_i^T & 0 \\ & * & -\gamma^2 I \end{bmatrix} \tag{31}$$

$$\begin{aligned} & \dot{V}(x(t)) + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \leq 2x(t)^T P\widehat{A}(t) x(t) + 2x(t)^T PA_\tau(t) x(t - \tau(t)) + 2x(t)^T PB_w(t) w(t) \\ & + x(t)^T S x(t) - (1 - \beta) x(t - \tau(t))^T S x(t - \tau(t)) + \overline{\tau} \dot{x}(t)^T Z \dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)^T Z \dot{x}(s) ds \\ & + 2[x(t)^T Y(t) + x(t - \tau(t))^T T(t)] \times [x(t) - x(t - \tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s) ds] + z(t)^T z(t) - \gamma^2 w(t)^T w(t) \end{aligned} \tag{32}$$

$$\tilde{\Phi}(t) = \begin{bmatrix} \begin{matrix} P\widehat{A}(t) + \widehat{A}(t)^T P + S \\ + Y(t) + Y(t)^T + \overline{\tau} \widehat{A}(t)^T Z \widehat{A}(t) \end{matrix} & \begin{matrix} PA_\tau(t) - Y(t) + T(t)^T \\ + \overline{\tau} \widehat{A}(t)^T Z A_\tau(t) \end{matrix} & \begin{matrix} PB_w(t) \\ + \overline{\tau} \widehat{A}(t)^T Z B_w(t) \end{matrix} \\ & \begin{matrix} -(1-\beta)S - T(t) - T(t)^T \\ + \overline{\tau} A_\tau(t)^T Z A_\tau(t) \end{matrix} & \overline{\tau} A_\tau(t)^T Z B_w(t) \\ & * & -\gamma^2 I + \overline{\tau} B_w(t)^T Z B_w(t) \end{bmatrix} \tag{33}$$

when $w(t)=0$, (30) means $\dot{V}(x(t))<0$, therfore system (16) is asymtotically stable in the case of $w(t)=0$. When $w(t)\neq 0$, integrating both sides of (30) from 0 to t yields

$$\begin{array}{l} V(x(t))-V(0)+\int_0^t z(s)^T z(s)ds \\ -\int_0^t \gamma^2 w(s)^T w(s)ds<0 \end{array} \tag{34}$$

letting $t\to\infty$ and under zero initial condition, we can show from (34) that

$$\int_0^{\infty} z(s)^T z(s)ds<\int_0^{\infty}\gamma^2 w(s)^T w(s)ds \tag{35}$$

that is, $\|z(t)\|_2<\gamma\|w(t)\|_2$, therfore, $J<0$, the proof is compluted.

Our objective is to transform the conditions in Theorem 1 in LMI terms which can be easily solved using existing solvers such as LMI TOOLBOX in Matlab software.

Theorem 2: For a given positives scalars λ and ρ. System (16) is asymptotically stable $\forall x\in\varepsilon(X^{-1},\rho)$, if there exist some matrices $X>0$, $\widetilde{S}>0$, $\widetilde{Z}>0$, $\widetilde{Y}_i$, $\widetilde{T}_i$, N_i, M_i and $\widetilde{X}_{ijs}$ with $\widetilde{X}_{ii}$ symetrical, $i,j=1,2,\cdots,r, i\leq j$, $l=1,\ldots,m$ such that the following LMIs hold :

$$\widetilde{\widetilde{\Xi}}_{ijs}+\widetilde{\widetilde{\Xi}}_{jis}\leq\widetilde{X}_{ijs}+\widetilde{X}_{ijs}^T \tag{36}$$

$$\begin{bmatrix} \widetilde{X}_{11s} & \cdots & \widetilde{X}_{1rs} \\ \vdots & \ddots & \vdots \\ \widetilde{X}_{1rs}^T & \cdots & \widetilde{X}_{rrs} \end{bmatrix}\leq 0 \tag{37}$$

$$\begin{bmatrix} \frac{1}{\rho} & M_{il} \\ & X \end{bmatrix}>0, \tag{38}$$

where

$$\widetilde{\widetilde{\Xi}}_{ijs}=\begin{bmatrix} \widetilde{\Phi}_{ijs} & \widetilde{\Theta}_{ijs}^T & -\widetilde{W}_i^T & \widetilde{U}_{ijs}^T \\ & \frac{1}{\bar{\tau}}(-2\lambda X+\lambda^2\widetilde{Z}) & 0 & 0 \\ & * & -\frac{1}{\bar{\tau}}\widetilde{Z} & 0 \\ & * & * & -I \end{bmatrix} \tag{39}$$

where $\widetilde{\Phi}_{ijs}$ is shown in (46) at the bottom of the page.

$$\widetilde{\Theta}_{ijs}=\begin{bmatrix} A_iX+B_iE_{is}N_j+B_iE_{is}^-M_j & A_{\tau i}X & B_{wi} \end{bmatrix} \tag{40}$$

$$\widetilde{W}_i=\begin{bmatrix} \widetilde{Y}_i^T & \widetilde{T}_i^T & 0 \end{bmatrix} \tag{41}$$

$$\widetilde{U}_{ijs}=\begin{bmatrix} C_iX+D_iE_{is}N_j+D_iE_{is}^-M_j & C_{\tau i}X & 0 \end{bmatrix} \tag{42}$$

If this is the case, the K_i and H_i gains are given by

$$K_i=N_iX^{-1},\ \ H_i=M_iX^{-1}i=1,2,..,r \tag{43}$$

Proof : let $X=P^{-1}$. Multiplying Ξ_{ijs} on both sides by $diag\begin{bmatrix} X & X & I & X & X & I \end{bmatrix}$ and let $N_j=K_jX, M_j=H_jX, \widetilde{S}=XSX, \widetilde{Z}=XZX, \widetilde{T}_i=XT_iX, \widetilde{Y}_i=XY_iX$, we get

$$\begin{bmatrix} \widetilde{\Phi}_{ijs} & \widetilde{\Theta}_{ijs}^T X^{-1}\widetilde{Z} & \widetilde{W}_i^T & \widetilde{U}_{ijs}^T \\ & -\frac{1}{\tau}\widetilde{Z} & 0 & 0 \\ & * & -\frac{1}{\bar{\tau}}\widetilde{Z} & 0 \\ & * & * & -I \end{bmatrix} \tag{44}$$

Pre and post-multiplying the previous matrices by $diag\begin{bmatrix} I & I & I & X\widetilde{Z}^{-1} & I & I \end{bmatrix}$ and its transpose, we get

$$\begin{bmatrix} \widetilde{\Phi}_{ijs} & \widetilde{\Theta}_{ijs}^T & \widetilde{W}_i^T & \widetilde{U}_{ijs}^T \\ & -\frac{1}{\tau}X\widetilde{Z}^{-1}X & 0 & 0 \\ & * & -\frac{1}{\tau}\widetilde{Z} & 0 \\ & * & * & -I \end{bmatrix} \tag{45}$$

It follows from lemma (2) that there exist a scalar $\lambda>0$ such that
$-X\widetilde{Z}^{-1}X\leq -2\lambda X+\lambda^2\widetilde{Z}$.

setting
$\widetilde{P}=diag\begin{bmatrix} I & I & I & X\widetilde{Z}^{-1} & I & I \end{bmatrix}$
$\times\, diag\begin{bmatrix} X & X & I & X & X & I \end{bmatrix}$
and $\widetilde{X}_{ij}=\widetilde{P}X_{ij}\widetilde{P}^T$
we obtain (56).
it also result in (37)

$$\begin{bmatrix} \widetilde{X}_{11s} & \cdots & \widetilde{X}_{1rs} \\ \vdots & \ddots & \vdots \\ \widetilde{X}_{1rs}^T & \cdots & \widetilde{X}_{rrs} \end{bmatrix}=\begin{bmatrix} \widetilde{P} & & \\ & \ddots & \\ & & \widetilde{P} \end{bmatrix}$$
$$\times\begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r}^T & \cdots & X_{rr} \end{bmatrix}\begin{bmatrix} \widetilde{P}^T & & \\ & \ddots & \\ & & \widetilde{P}^T \end{bmatrix}\leq 0$$

due to (18).
the inclusion condition $\varepsilon(P,\rho)\subset\mathcal{L}(H_i)\quad\forall i=1,\ldots,r$ holds if $1/\rho-H_{il}X_iH_{il}^T>0,\forall l\in[1,m]$ [10], which is equivalent to,
$1/\rho-(H_iX)_l(X^{-1})(H_iX)_l^T>0$. That is, by virtue of (43) $1/\rho-(M_{il})(X^{-1})(M_{il})^T>0$. By Schur complement, LMI (38) is obtained. this complute the proof.

$$\widetilde{\Phi}_{ijs}=\begin{bmatrix} A_iX+B_iE_{is}N_j+B_iE_{is}^-M_j+XA_i^T+N_j^TE_{is}^TB_i^T+\widetilde{S}+\widetilde{Y}_i+\widetilde{Y}_i^T & A_{\tau i}X-\widetilde{Y}_i+\widetilde{T}_i^T & B_{wi} \\ & -(1-\beta)\widetilde{S}-\widetilde{T}_i-\widetilde{T}_i^T & 0 \\ & * & -\gamma^2 I \end{bmatrix} \tag{46}$$

B. Time-delay dependent stabilization conditions with unsaturating controller

In this section, the state feedback control gain is noted F_i. Assume that state $x(t) \in \mathcal{L}(F_i), \forall t$. In this case, $sat(F_i x(t)) = F_i x(t)$. This controller is called unsaturating since saturation does not occur inside polyhedral set $\mathcal{L}(F_i)$. The induced system in closed-loop is then given by :

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j [\widehat{A}_{ij} x(t) + A_{\tau i} x(t-\tau(t))] \quad (47)$$

$$z(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j [\widehat{C}_{ij} x(t) + C_{\tau i} x(t-\tau(t))] \quad (48)$$

with $x(t) = \psi(t)$ for $t \in [-\bar{\tau}, 0]$ and

$$\begin{aligned} \widehat{A}_{ij} : &= A_i + B_i F_j \\ \widehat{C}_{ij} : &= C_i + D_i F_j \end{aligned}$$

Theorem 3: System (47) is asymptotically stable $\forall x \in \varepsilon(P, \rho)$, if there exist some matrices $P > 0$, $S > 0$, $Z > 0$, Y_i, T_i, F_i and X_{ij} with X_{ii} symetrical, $i, j = 1, 2, .., r$ and $i \leq j$, $s = 1, \ldots, \eta$, such that the following conditions hold :

$$\Xi_{ij} + \Xi_{ji} \leq X_{ij} + X_{ij}^T \quad (49)$$

$$\begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{1r}^T & \cdots & X_{rr} \end{bmatrix} \leq 0 \quad (50)$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(F_i) \quad (51)$$

where

$$\Xi_{ij} = \begin{bmatrix} \Phi_{ij} & \Theta_{ij}^T Z & W_i^T & U_{ij}^T \\ & -\frac{1}{\bar{\tau}} Z & 0 & 0 \\ & * & -\frac{1}{\bar{\tau}} Z & 0 \\ & * & * & -I \end{bmatrix} \quad (52)$$

in which Φ_{ij} is shown in (64) at the bottom of the page.

$$\Theta_{ij} = \begin{bmatrix} \widehat{A}_{ij} & A_{\tau i} & B_{wi} \end{bmatrix} \quad (53)$$

$$W_i = \begin{bmatrix} Y_i^T & T_i^T & 0 \end{bmatrix} \quad (54)$$

$$U_{ij} = \begin{bmatrix} \widehat{C}_{ij} & C_{\tau i} & 0 \end{bmatrix} \quad (55)$$

Proof : The proof is a particular case of the one of Theorem 1 with $E_{is} = I, E_{is}^- = 0 \forall i, s$. This idea was already used in [2].

The objective is now to transform the conditions in Theorem 3 in LMI terms.

Theorem 4: For a given positives scalars λ and ρ. System (47) is asymptotically stable $\forall x \in \varepsilon(X^{-1}, \rho)$, if there exist some matrices $X > 0$, $\widetilde{S} > 0$, $\widetilde{Z} > 0$, $\widetilde{Y}_i$, $\widetilde{T}_i$ and $\widetilde{X}_{ij}$ with $\widetilde{X}_{ii}$ symmetrical, $i, j = 1, 2, \cdots, r, i \leq j$ such that the following LMIs hold :

$$\widetilde{\Xi}_{ij} + \widetilde{\Xi}_{ji} \leq \widetilde{X}_{ij} + \widetilde{X}_{ij}^T \quad (56)$$

$$\begin{bmatrix} \widetilde{X}_{11} & \cdots & \widetilde{X}_{1r} \\ \vdots & \ddots & \vdots \\ \widetilde{X}_{1r}^T & \cdots & \widetilde{X}_{rr} \end{bmatrix} \leq 0 \quad (57)$$

$$\begin{bmatrix} \frac{1}{\rho} & N_{il} \\ & X \end{bmatrix} > 0, \quad (58)$$

where

$$\widetilde{\Xi}_{ij} = \begin{bmatrix} \widetilde{\Phi}_{ij} & \widetilde{\Theta}_{ij}^T & \widetilde{W}_i^T & \widetilde{U}_{ij}^T \\ & \frac{1}{\bar{\tau}}(-2\lambda X + \lambda^2 \widetilde{Z}) & 0 & 0 \\ & * & -\frac{1}{\bar{\tau}} \widetilde{Z} & 0 \\ & * & * & -I \end{bmatrix} \quad (59)$$

where $\widetilde{\Phi}_{ij}$ is shown in (65) at the bottom of the page.

$$\widetilde{\Theta}_{ij} = \begin{bmatrix} A_i X + B_i N_j & A_{\tau i} X & B_{wi} \end{bmatrix} \quad (60)$$

$$\widetilde{W}_i = \begin{bmatrix} \widetilde{Y}_i^T & \widetilde{T}_i^T & 0 \end{bmatrix} \quad (61)$$

$$\widetilde{U}_{ij} = \begin{bmatrix} C_i X + D_i N_j & C_{\tau i} X & 0 \end{bmatrix} \quad (62)$$

If this is the case, the F_i gains are given by

$$F_i = N_i X^{-1}, i = 1, 2, .., r \quad (63)$$

Proof : The proof is a particular case of the one of Theorem 3 with $E_{is} = I, E_{is}^- = 0, \forall (i, s)$.

$$\Phi_{ij} = \begin{bmatrix} P\widehat{A}_{ij} + \widehat{A}_{ij}^T P + S + Y_i + Y_i^T & PA_{\tau i} - Y_i + T_i^T & PB_{wi} \\ & -(1-\beta)S - T_i - T_i^T & 0 \\ & * & -\gamma^2 I \end{bmatrix} \quad (64)$$

$$\widetilde{\Phi}_{ij} = \begin{bmatrix} A_i X + B_i N_j + X A_i^T + N_j^T B_i^T + \widetilde{S} + \widetilde{Y}_i + \widetilde{Y}_i^T & A_{\tau i} X - \widetilde{Y}_i + \widetilde{T}_i^T & B_{wi} \\ & -(1-\beta)\widetilde{S} - \widetilde{T}_i - \widetilde{T}_i^T & 0 \\ & * & -\gamma^2 I \end{bmatrix} \quad (65)$$

IV. Numerical example

Consider the following example

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i[(1-s)A_i x(t) + sA_i x(t-\tau(t)) \\ &\quad + B_i u(t) + B_{wi} w(t)] \\ z(t) &= \sum_{i=1}^{r} h_i[(1-s)C_i x(t) + sC_i x(t-\tau(t)) \\ &\quad + D_i u(t)] \end{aligned} \quad (66)$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ 17 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 4.5 & 0 \end{bmatrix}$
$B_1 = \begin{bmatrix} 0 \\ -176.5 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ -11 \end{bmatrix}$
$B_{w1} = B_{w2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$
$D1 = 0.008, D2 = 0.006$

Let $s = 0.1$ and $\tau = 0.2$ and $\gamma = 0.9$

Figure 1 shows the inclusion of the ellipsoid $\varepsilon(P, \rho)$ set inside the sets of saturation $\mathcal{L}(H_i)$ by using theorem 2.

Figure 2 shows the inclusion of the ellipsoid $\varepsilon(P, \rho)$ set inside the polyhedral sets of linear behavior $\mathcal{L}(F_i)$ by using theorem 4.

V. Conclusion

New delay dependent method H_∞ control for T-S fuzzy systems with time varying delay and actuator saturations has been studied in this paper. the stabilization conditions are presented in terms of Linear Matrix Inequalities. An illustrative example is given to demonstrate the effectiveness of the proposed result.

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