

Chaotic synchronization of regular complex networks with fractional-order oscillators

Angulo-Guzmán S. Y., Posadas-Castillo C.*, Díaz-Romero D. A., López-Gutiérrez R. M. and Cruz-Hernández C.

Abstract—In this paper, the synchronization of N-coupled fractional-order chaotic systems with regular connection topology is presented. Synchronization of coupled Lorenz, Rössler and Chen fractional-order systems with unidirectional coupling is achieved based on a coupling matrix from complex systems theory. Synchronization of this systems is achieved considering a coupling without master node. The interactions in the complex networks are defined by coupling only one state of each fractional-order system. Numerical simulations are provided to verify the effectiveness of this method.

I. INTRODUCTION

The concept of fractional derivative stems from 1695 when L'Hopital asked to Leibniz what meaning take $D^n f$ if n were a fraction. Since then, several mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, among others, have made contributions to the theory of fractional calculus [1]. For many years fractional derivatives were not used and the main reason was the absence of method to solve such differential equations.

The theory of fractional calculus is a 300-year-old mathematical topic but in the last 20 years it has attracted much attention in physics and mathematics because some events in nature can be modeled with fractional derivatives. They are observed in many biological systems, for example, fractional derivatives can determine the interbeat interval in human heartbeats and the variations in human strides, they describe the branching of trees and they are also seen in snowflake growth. The applications of this theory are found in many fields of science and engineering, for example mechanics, physics, electronics, medicine, ecology or economy. Some of these applications are viscoelasticity, dielectric polarization, quantitative finance, seismology and processing and image manipulation. The advantages of fractional derivatives becomes apparent in modeling mechanical and electrical

Posadas-Castillo C., Díaz-Romero D.A. and Angulo-Guzmán S.Y. are with Engineering Mechanic and Electric Faculty of Nuevo Leon Autonomous University (UANL), Pedro de Alba, S.N., Cd. Universitaria, San Nicolás de los Garza, NL, México corneliiov@gmail.com, ddiaz.uanl@gmail.com, sarayaelag@gmail.com

Cruz-Hernández C. is with Electronics and Telecommunications Department, Scientific Research and Advanced Studies of Ensenada (CICESE), Km. 107, Carretera Tijuana-Ensenada, 22860 Ensenada, BC, México, ccruz@cicese.mx

López-Gutiérrez R.M. is with Engineering Faculty, Baja California Autonomous University (UABC), Km. 103, Carretera Tijuana-Ensenada, 22860 Ensenada, BC, México, roslopez@uabc.edu.mx

*Corresponding author: Posadas-Castillo C., Av. Pedro de Alba s/n, Cd. Universitaria. C.P. 66451, Apartado Postal 076, Suc. F, San Nicolás de los Garza, N.L. Tel. +01(81)13404020 ext. 5773, corneliiov@gmail.com

properties of real materials, as well as in the description of rheological properties of rocks [2], [3].

Chaotic synchronization has received increasing attention from researchers in the last 10 years. Since Pecora and Carroll synchronizing two identical chaotic systems with different initial conditions, chaotic synchronization was intensively studied in the integer-order systems and recently in the fractional-order systems. Synchronization is investigated due to its potential applications in secure communications [4], [5], process control, chemical and biological systems description. One of the advantages of fractional chaotic systems are its adjustable variables, chaos can exist for different system parameters with different order in the derivatives. This allows applications such as data encryption where the key space can be enlarged [6].

Many methods have been proposed to achieve synchronization between two fractional-order chaotic systems, i.e. Pecora-Carroll (PC) method [7], via linear control [8], active control [9], backstepping design [10] and recently a method to synchronize N-coupled fractional-order chaotic systems in ring connection was proposed [11]. These methods are effective but sometimes the design procedure is complex and the law of control is present in every state of the system. In this paper we synchronize N-coupled fractional-order systems applying a law of control only in one state.

This paper is organized as follow: Section II some necessary definitions and notations of fractional calculus, in Section III a brief review on synchronization of complex dynamical networks. In Section IV, the problem of synchronization in N-coupled fractional-order chaotic systems in regular network is exposed as well as the fractional-order model of Lorenz, Rössler and Chen systems which will be used as fundamental nodes to compose the regular networks. The corresponding simulation results are provided in section IV. Finally, some conclusions are given in Section V.

II. PRELIMINARIES

A. Fractional-order derivatives

Fractional calculus is a generalization of integration and differentiation to non-integer-order fundamental operator ${}_a D_t^\alpha$, where a and t are the bounds of the operation and $\alpha \in \mathbb{R}$. The continuous integro-differential operator defined as

$${}_a D_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0 \end{cases} \quad (1)$$

There are three common definitions of a fractional-order derivative: the Grünwald-Letnikov (GL) definition, the Caputo definition and the Riemann-Liouville (RL) definition [1], [3], [12]. The GL, Caputo y RL definitions are equivalent under some conditions. First, we define the GL definition in non-integer differentiation given by following

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{j} f(t-jh), \quad (2)$$

where

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, \quad (3)$$

is the relation between Euler's *Gamma* function and factorial.

For numerical solution of fractional-order differentiation we can use the relation derived from GL definition given by the following expression [1], [3]

$${}_{k-Lm/h} D_{t_k}^q f(t) \approx h^{-q} \sum_{j=0}^k (-1)^j \binom{\alpha}{j} f(t_k - j). \quad (4)$$

The general numerical solution of (4) is

$${}_a D_t^q y(t) = f(y(t), t), \quad (5)$$

and can be expressed as

$$y(t_k) = f(y(t_k), t_k) h^q - \sum_{j=v}^k c_j^{(q)} y(t_k - j), \quad (6)$$

where the calculation of binomial coefficients is given by:

$$c_0^{(q)} = 1, c_j^{(q)} = \left(1 - \frac{1+q}{j}\right) c_{j-1}^{(q)}. \quad (7)$$

III. COMPLEX NETWORKS

A complex network is defined as an interconnected set of nodes (two or more), where each node is a fundamental unit, with its dynamic depending of the nature of the network. Each node is defined as follows

$$\begin{cases} {}_a D_t^\alpha x_i(t) = f_i(x_i, t) + u_{i1}, \\ x_i(0) = c_i, i = 0, 1, 2, \dots, n, \end{cases} \quad (8)$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbb{R}^n$ are the state variables of the node i and u_i establishes the synchronization between two or more systems and is defined as follows [12], [13], [17], [19]

$$u_{i1} = c \sum_{j=1}^n a_{ij} \Gamma x_j, i = 1, 2, \dots, n, \quad (9)$$

the constant c positive defined represents the coupling strength and Γ is a constant matrix linking coupled state variables. In this matrix, two nodes are linked through their i th state variables. Assume that $\Gamma = \text{diag}(r_1, r_2, \dots, r_n)$ is a diagonal matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $j \neq i$.

The matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is the coupling matrix which shows a connection between node i and j , then $a_{ij} = 1$,

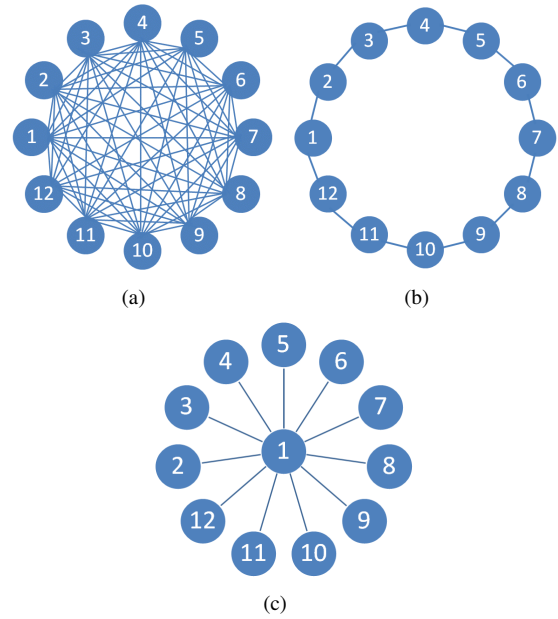


Fig. 1. Regular network topologies: (a) Globally coupled network, (b) Ring coupled network, (c) Star coupled network.

otherwise $a_{ij} = 0$ for $i \neq j$. The diagonal elements of A are defined as

$$a_{ii} = - \sum_{j=1, j \neq i}^n a_{ij} = - \sum_{j=1, j \neq i}^n a_{ji}, i = 1, 2, \dots, n. \quad (10)$$

The dynamical network (8) and (9) is said to achieve synchronization if $x_1(t) = x_2(t) = \dots = x_n(t)$, as $t \rightarrow \infty$.

The complexity of the network refers to the characteristics of the nodes or network topology. We consider a network with N identical nodes with a fractional-order chaotic dynamic. The topology is defined as a connection among each node, with a regular or irregular pattern. We consider only the regular topologies. In Fig. 1 shows the three types of regular networks: global connection, ring connection and star connection.

IV. SYNCHRONIZATION OF N-COUPLED FRACTIONAL-ORDER SYSTEMS VIA COUPLING MATRIX

Regular coupled networks are one of the coupling configurations studied in synchronization of complex networks (globally coupled, ring coupled and star coupled networks). Synchronization is achieved using different systems, in fractional-order chaotic Lorenz system globally coupled, fractional-order chaotic Rössler system using ring connection and fractional-order chaotic Chen system using star connection.

A. Globally coupled network

First, we synchronize complex networks of identical fractional-order nodes globally coupled where any two nodes in the network are connected directly. This globally coupled network topology is illustrated in Fig. 1(a). Edward N. Lorenz from the equations that predict weather patterns found a chaotic attractor in 1963 and was named Lorenz

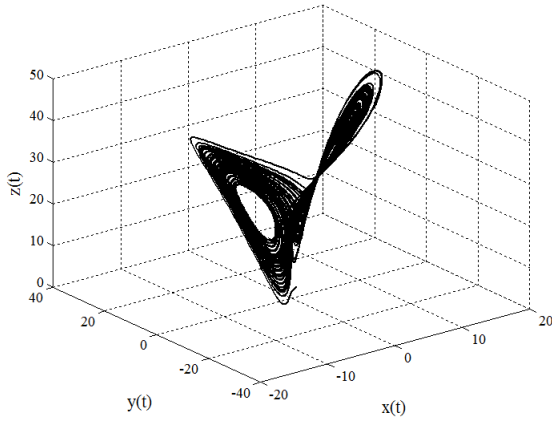


Fig. 2. Attractor of a fractional-order Lorenz system with $\sigma = 10$, $\rho = 29$, $\beta = 8/3$ and $q_1 = q_2 = q_3 = 0.995$.

attractor. Time later was found chaotic behavior with different fractional-orders in the derivatives of the system. We use fractional-order Lorenz system in each node which is defined as follows

$$\begin{aligned} {}_0D_t^{q_1} x_i(t) &= \sigma(y_i(t) - x_i(t)), \\ {}_0D_t^{q_2} y_i(t) &= x_i(t)(\rho - z_i(t)) - y_i(t), \\ {}_0D_t^{q_3} z_i(t) &= x_i(t)y_i(t) - \beta z_i(t), \end{aligned} \quad (11)$$

where $\sigma = 10$, $\rho = 29$, $\beta = 8/3$ and the fractional-order when $q \in [0, 1]$ is given by $q_1 = q_2 = q_3 = 0.995$ the system is chaotic [13]. The attractor is shown in Fig. 2. Considering a synchronization scheme N-coupled fractional-order chaotic Lorenz systems, with global connection the first node N_1 we define

$$\begin{aligned} {}_0D_t^{q_1} x_{i1}(t) &= \sigma(y_{i2}(t) - x_{i1}(t)) + u_{i1}, \\ {}_0D_t^{q_2} y_{i2}(t) &= x_{i1}(t)(\rho - z_{i3}(t)) - y_{i2}(t), \\ {}_0D_t^{q_3} z_{i3}(t) &= x_{i1}(t)y_{i2}(t) - \beta z_{i3}(t). \end{aligned} \quad (12)$$

The coupling matrix for this configuration is given by

$$A_{gc} = \begin{bmatrix} -N+1 & 1 & 1 & \dots & 1 \\ 1 & -N+1 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & -N+1 \end{bmatrix}. \quad (13)$$

This matrix has a single eigenvalue at 0 and all other equal to $-N$. For this case, the coupling matrix is defined with $N = 12$ is as follows

$$A_{gc} = \begin{bmatrix} -11 & 1 & 1 & \dots & 1 \\ 1 & 11 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & -11 \end{bmatrix}. \quad (14)$$

The Gamma matrix is defined as $\Gamma = \text{diag}(1, 0, 0)$ that means the synchronization is achieved by the first nodes. According to (9), the first law of control is given by the A_{gc} matrix $u_{11} = c(-11x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} +$

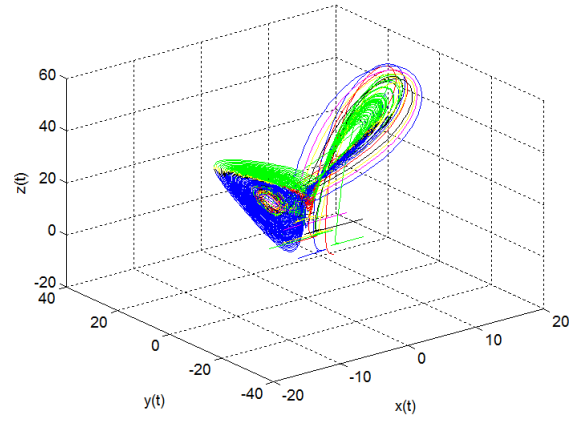


Fig. 3. Fractional-order Lorenz attractor of the synchronization.

$x_{11} + x_{12}$).

We defined the next nodes as follows ($N = 2$)

$$\begin{aligned} {}_0D_t^{q_1} x_{21}(t) &= \sigma(y_{22}(t) - x_{21}(t)) + u_{21}, \\ {}_0D_t^{q_2} y_{22}(t) &= x_{21}(t)(\rho - z_{23}(t)) - y_{22}(t), \\ {}_0D_t^{q_3} z_{23}(t) &= x_{21}(t)y_{22}(t) - \beta z_{23}(t), \end{aligned} \quad (15)$$

and the second control law is $u_{21} = c(x_{11} - 11x_{21} + x_{31} + x_{41} + x_{51} + x_{61} + x_{71} + x_{81} + x_{91} + x_{101} + x_{111} + x_{121})$, until the last node ($N = 12$)

$$\begin{aligned} {}_0D_t^{q_1} x_{121}(t) &= \sigma(y_{122}(t) - x_{121}(t)) + u_{121}, \\ {}_0D_t^{q_2} y_{122}(t) &= x_{121}(t)(\rho - z_{123}(t)) - y_{122}(t), \\ {}_0D_t^{q_3} z_{123}(t) &= x_{121}(t)y_{122}(t) - \beta z_{123}(t), \end{aligned} \quad (16)$$

$u_{12} = c(x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} + x_{71} + x_{81} + x_{91} + x_{101} + x_{111} - 11x_{121})$.

The initial conditions for this simulation are

$$\begin{aligned} (x_{11}, y_{12}, z_{13}) &= (-5.1, 0.2, 2.5), \\ (x_{21}, y_{22}, z_{23}) &= (0.8, 12, -1), \\ (x_{31}, y_{32}, z_{33}) &= (2.3, -5, -2.6), \\ (x_{41}, y_{42}, z_{43}) &= (10, 10, 1), \\ (x_{51}, y_{52}, z_{53}) &= (6.5, 9, 3), \\ (x_{61}, y_{62}, z_{63}) &= (9, 1.2, 3.57), \\ (x_{71}, y_{72}, z_{73}) &= (-3.7, -1, 4), \\ (x_{81}, y_{82}, z_{83}) &= (4.5, 8, 6), \\ (x_{91}, y_{92}, z_{93}) &= (-1, 0.01, -4), \\ (x_{101}, y_{102}, z_{103}) &= (5.7, -7.9, 3.1), \\ (x_{111}, y_{112}, z_{113}) &= (0.1, 3, 0.3), \\ (x_{121}, y_{122}, z_{123}) &= (0.19, 4, 7.5). \end{aligned}$$

The coupling strength used for this case is $c = 20$. The Fig. 3 shows the chaotic attractor of fractional-order Lorenz system with 12 nodes. In Fig. 4, the phase portrait of the synchronization of the first states where each state is compared with each other is shown and we can see that each state synchronize with the other first states.

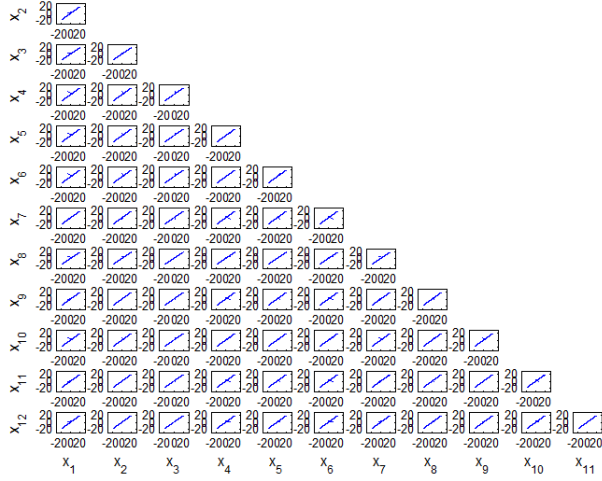


Fig. 4. Synchronization of first states (x_i , where $i = 1, 2, \dots, 12$) of the twelve Lorenz systems.

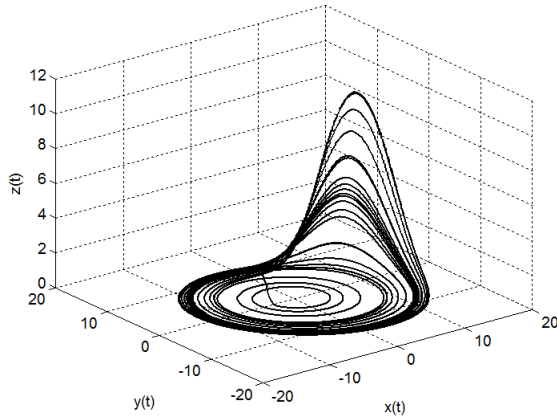


Fig. 5. Fractional-order Rössler attractor with $a = 0.4, b = 0.2, c = 10$ and $q_1 = 0.9, q_2 = 0.9, q_3 = 0.9$.

B. Ring coupled network

In this type of coupling, the connections are between only nearest neighbors and the last node is connected to the first node. In this case, we synchronized a ring coupled Rössler system with twelve nodes ($N = 12$). This topology is illustrated in Fig. 1(b) and each node is a fractional-order Rössler system. In 1976, Otto Rössler proposed Rössler system with a strange attractor. This equations are useful to model equilibrium in chemical reactions. The fractional-order Rössler system is defined as follow

$$\begin{aligned} {}_0D_t^{q_1} x_i(t) &= -y_i(t) - z_i(t), \\ {}_0D_t^{q_2} y_i(t) &= x_i(t) + ay_i(t), \\ {}_0D_t^{q_3} z_i(t) &= b + z_i(t)(x_i(t) - c). \end{aligned} \quad (17)$$

Under the parameters $a = 0.4, b = 0.2, c = 10$ and a fractional-order equal to $q = 2.7$, where $q_1 = 0.9, q_2 = 0.9, q_3 = 0.9$ the system is chaotic [16]. The attractor is presented in Fig. 5. For this case, the nodes of the dynamical

network are arranged as follows:

$$\begin{aligned} {}_0D_t^{q_1} x_{i1}(t) &= -y_{i2}(t) - z_{i3}(t), \\ {}_0D_t^{q_2} y_{i2}(t) &= x_{i1}(t) + ay_{i2}(t) + u_{i2}, \\ {}_0D_t^{q_3} z_{i3}(t) &= b + z_{i3}(t)(x_{i1}(t) - c), \end{aligned} \quad (18)$$

where $i = 1, 2, \dots, 12$. The coupling matrix for ring connection is given by

$$A_{nc} = \begin{bmatrix} -k & 1 & 0 & \dots & 1 \\ 1 & -k & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 & -k \end{bmatrix}, \quad (19)$$

where $k = 2$ since each node is adjacent to the neighboring nodes. The Gamma matrix for this case is defined as $\Gamma = \text{diag}(0, 1, 0)$ that means the synchronization is achieved by the second nodes. The synchronization is given by the following coupling signals and each u_{i2} is applied to each node (N_i).

$$\begin{aligned} u_{21} &= c(-2y_{12} + y_{22} + y_{122}), \\ u_{22} &= c(y_{12} - 2y_{22} + y_{32}), \\ u_{23} &= c(y_{22} - 2y_{32} + y_{42}), \\ &\vdots \\ u_{210} &= c(y_{92} - 2y_{102} + y_{112}), \\ u_{211} &= c(y_{102} - 2y_{112} + y_{122}), \\ u_{212} &= c(y_{12} + y_{112} - 2y_{122}). \end{aligned} \quad (20)$$

The initial conditions for the ring connection are

$$\begin{aligned} (x_{11}, y_{12}, z_{13}) &= (0.2, -0.1, 0.1), \\ (x_{21}, y_{22}, z_{23}) &= (8, -4, -9), \\ (x_{31}, y_{32}, z_{33}) &= (4, 0.1, 2), \\ (x_{41}, y_{42}, z_{43}) &= (-7, 5, 5.21), \\ (x_{51}, y_{52}, z_{53}) &= (5, 3.2, -3), \\ (x_{61}, y_{62}, z_{63}) &= (-4, -2.9, 0.22), \\ (x_{71}, y_{72}, z_{73}) &= (3.4, -2, -4.3), \\ (x_{81}, y_{82}, z_{83}) &= (6.1, 7.7, -5.2), \\ (x_{91}, y_{92}, z_{93}) &= (4.5, 8.1, 1.1), \\ (x_{101}, y_{102}, z_{103}) &= (2.1, -1.1, 0.65), \\ (x_{111}, y_{112}, z_{113}) &= (7.2, 3, 8), \\ (x_{121}, y_{122}, z_{123}) &= (10.1, -6.1, 3). \end{aligned}$$

The attractor of the twelve fractional-order Rössler systems is show in Fig. 6, Fig. 7 shows the twelve states of the Rössler systems for different initial conditions. Synchronization is achieved using a coupling strength $c = 3$ and the phase portrait reflects the appearance of a line 45° to the center of the graph shows in Fig. 8. For this case, the synchronization is achieved with a small coupling strength compared with the Lorenz systems synchronization where the coupling strength is better if it is large.

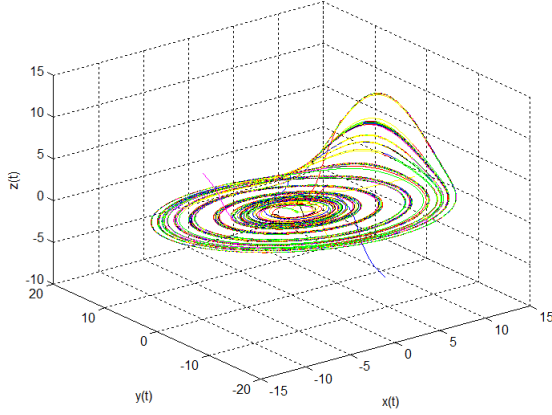


Fig. 6. Twelve chaotic attractor of fractional-order Rössler systems.

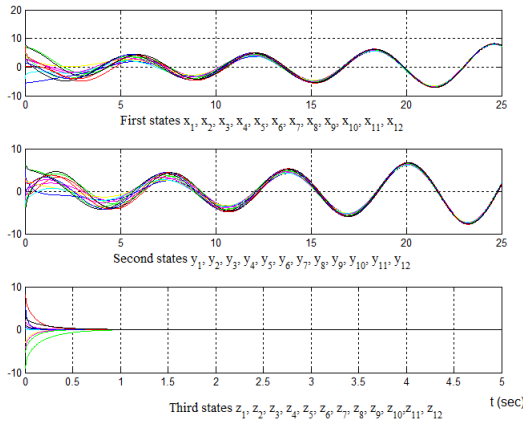


Fig. 7. States (y_i , where $i = 1, 2, \dots, 12$) of the Rössler systems synchronization.

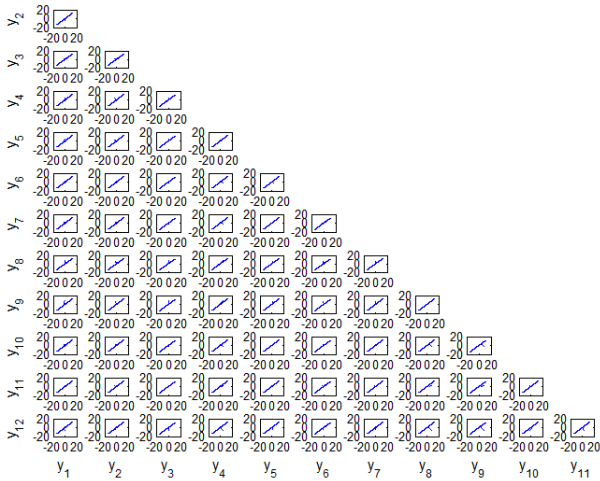


Fig. 8. Phase portrait of the second states (y_i , where $i = 1, 2, \dots, 12$) of Rössler systems.

C. Star coupled network

Finally, the last regular network that we use is a star connection where there is a central node to which other nodes are connected, as is shown in Fig. 1(c). We use a fractional-order Chen system in each node. The fractional-order representation of this model is the following

$$\begin{aligned} {}_0D_t^{q_1} x_{i1}(t) &= a(y_{i2}, -x_{i1}), \\ {}_0D_t^{q_2} y_{i2}(t) &= -dx_{i1} - x_{i1}z_{i3} + cy_{i2} + u_{i2}, \\ {}_0D_t^{q_3} y_{i3}(t) &= x_{i1}y_{i2} - bz_{i3}, \end{aligned} \quad (21)$$

where the parameters $a = 35$, $b = 3$, $c = 28$, $d = -7$ and the fractional orders $q_1 = 0.9$, $q_2 = q_3 = 0.95$ make the system behave chaotically [17]. Fig. 9 shows the fractional-order attractor of Chen system. The coupling signal used for synchronization of the twelve nodes are defined as follow matrix

$$A_{st} = \begin{bmatrix} -N+1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & 0 & -1 \end{bmatrix}. \quad (22)$$

For this case, the Γ matrix is defined as $\Gamma = \text{diag}(0, 1, 0)$ as in the ring connection because the synchronization is achieved by the second nodes. Each law of synchronization for each node is given by

$$\begin{aligned} u_{12} &= c(-11y_1 + y_2 + y_3 + y_4 + y_5 \\ &\quad + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}), \\ u_{22} &= c(y_1 - y_2), \\ u_{32} &= c(y_1 - y_3), \\ &\vdots \\ u_{102} &= c(y_1 - y_{10}), \\ u_{112} &= c(y_1 - y_{11}), \\ u_{122} &= c(y_1 - y_{12}). \end{aligned} \quad (23)$$

The initial conditions for this case are

$$\begin{aligned} (x_{11}, y_{12}, z_{13}) &= (1.5, 2, 0.5), \\ (x_{21}, y_{22}, z_{23}) &= (8, -4, -9), \\ (x_{31}, y_{32}, z_{33}) &= (-3, -1, 6), \\ (x_{41}, y_{42}, z_{43}) &= (-7, -5, 9), \\ (x_{51}, y_{52}, z_{53}) &= (5, 9, -3), \\ (x_{61}, y_{62}, z_{63}) &= (1, 3, -0.5), \\ (x_{71}, y_{72}, z_{73}) &= (3.2, 8, -6.5), \\ (x_{81}, y_{82}, z_{83}) &= (-10.3, 7.2, -1.25), \\ (x_{91}, y_{92}, z_{93}) &= (0.01, 3.4, -1.6), \\ (x_{101}, y_{102}, z_{103}) &= (7.2, 4.3, 3.2), \\ (x_{111}, y_{112}, z_{113}) &= (-5, -3, 2.2), \\ (x_{121}, y_{122}, z_{123}) &= (11.1, 0, 5). \end{aligned}$$

Fig. 10 shows the twelve fractional-order Chen attractors and the phase portrait is shown in Fig. 11. The synchronization

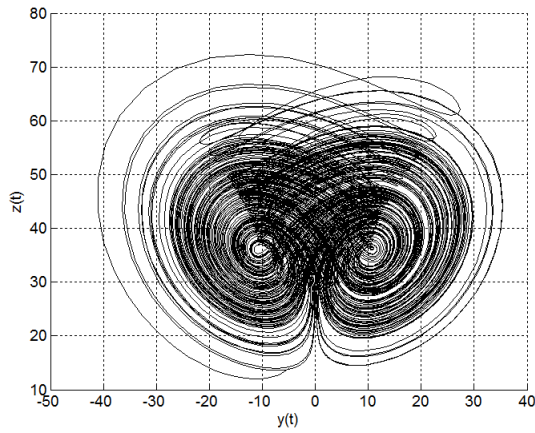


Fig. 9. Attractor of the fractional-order Chen system with $a = 35, b = 3, c = 28, d = -7$ and $q_1 = 0.985, q_2 = 0.99, q_3 = 0.98$.

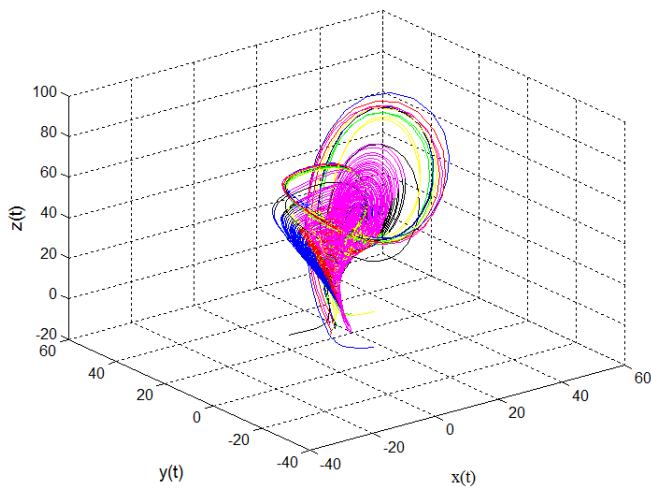


Fig. 10. Twelve chaotic attractor of fractional-order Chen systems.

is achieved with the coupling strength $c = 33$ and the convergence for each state is shown in Fig. 12.

V. CONCLUSIONS AND FUTURE WORKS

In this paper the synchronization of N-coupled regular network is achieved using the complex network theory. We use a coupling signal only in one state of the system and adjusting the coupling strength c the synchronization is achieved in fractional-order Lorenz, Rössler and Chen systems. The proposed synchronization law has the advantage that is simply to develop and the network can be synchronized applying this law only in one state of the system. We can observe the convergence in each state of the systems and the error system decay towards zero as $t \rightarrow \infty$. Simulations show the effectiveness of the proposed synchronization scheme. We use three different systems to illustrate the synchronization for regular network topology, however, the three systems synchronize for each type of coupling, global, star and ring. We explore only the regular complex networks with fractional-order nodes but we think that exploring the irregular network synchronization is important for future work.

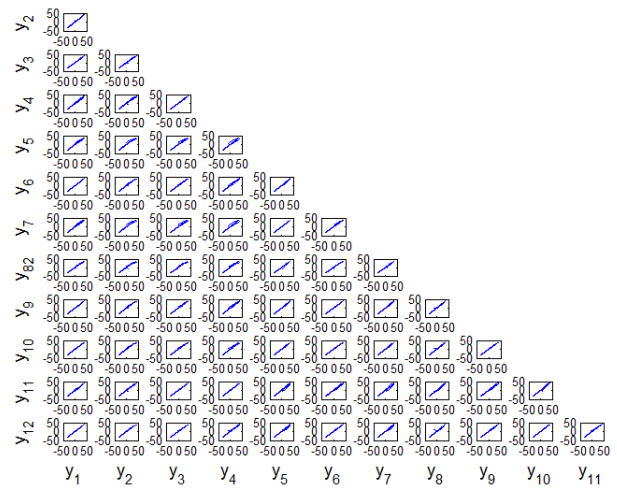


Fig. 11. Phase portrait of the second states (y_i , where $i = 1, 2, \dots, 12$) of the twelve fractional-order Chen systems.

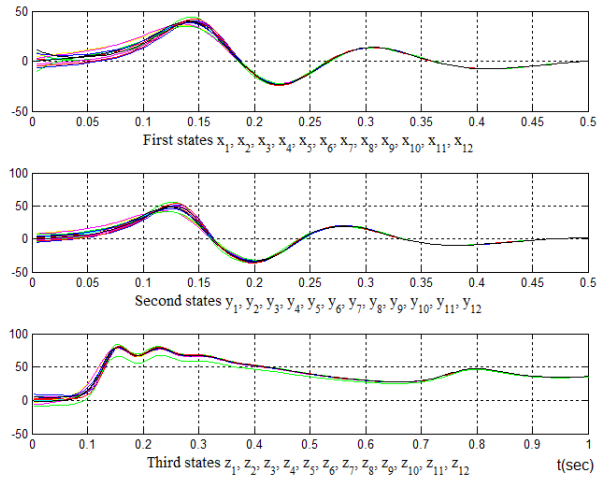


Fig. 12. Convergence in states (y_i , where $i = 1, 2, \dots, 12$) of the twelve fractional-order Chen systems.

VI. ACKNOWLEDGMENTS

This work was supported by CONACYT, México under Research Grant No. 166654 and by FIME-UANL.

REFERENCES

- [1] I. Podlubny, *Fractional Differential Equations*, Academic Press, SD, 1999.
- [2] Kilbas, Srivastava and Trujillo. *Theory and application of fractional differential equations*. North Holland mathematics studies, Elsevier, vol. 207, 2006.
- [3] I. Petrás, *Fractional-Order Nonlinear Systems*, Springer, 2011.
- [4] H. Serrano-Guerrero, C. Cruz-Hernández, R. M. López-Gutiérrez, C. Posadas-Castillo and E. Inzunza-Gonzalez, *Chaotic Synchronization in Star Coupled Networks of 3D CNNs and its Application in Communications*, *International Journal of Nonlinear Sciences and Numerical Simulation*, ISSN Number: 1565-1339, 2010, pp 571-580.
- [5] A. Y. Aguilar-Bustos, C. Cruz-Hernández, R. M. López-Gutiérrez and C. Posadas-Castillo, *Synchronization of Different Hyperchaotic Maps for Encryption*, *Advances in Chaotic Dynamics and Applications (Stability, Oscillations and Optimization of Systems)*, ISBN 978-1-904868-57-6, 2006, pp 403-422.

- [6] A. Kiani-B, K. Fallahi, N. Pariz and H. Leung, A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter, *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, 2009, pp 863 - 879.
- [7] L.M. Pecora and T.L. Carroll, Synchronization in chaotic systems, *Physical Review Letters*, vol. 64, 1990, pp 821-824.
- [8] Z. M. Odibat, N. Corson, M. Aziz-Alaoui, C. Bertelle, Synchronization of chaotic fractional-order systems via linear control, *International Journal of Bifurcation and Chaos*, vol. 20, 2010, pp 1-15.
- [9] A. Razminia, V. J. Majd and D. Baleanu, SChaotic incommensurate fractional order Rssler system: active control and synchronization, *Advances in Difference Equations*, vol. 1, 2011, pp 479-487.
- [10] E. Naseri, A. Ranjbar and S.H. HosseinNia, "Backstepping Control of Fractional Order Chen System", in *ASME 2009 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, San Diego, CA, 2009.
- [11] Y. Tang and J. Fang, Synchronization of N-coupled fractional-order chaotic systems with ring connection, *Commun Nonlinear Sci Numer Simulat*, vol. 15, 2010, pp 401412.
- [12] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, *Dover Books on Mathematics*, Mineola, NY, 2006.
- [13] X.F. Wang and G. Chen, Synchronization in Small-World Dynamical Networks, *International Journal of Bifurcation and Chaos*, vol. 12, 2002, pp 187-192.
- [14] C. Posadas-Castillo, C. Cruz-Hernández and R.M. López-Gutiérrez, Experimental realization of synchronization in complex networks with Chuas circuits like nodes, *Chaos, Solitons and Fractals*, vol. 40, 2009, pp 19631975.
- [15] I. Grigorenko and E. Grigorenko, Chaotic Dynamics of the Fractional Lorenz System, *Physical Review Letters*, vol. 91, 2003.
- [16] C. Li and G. Chen, Chaos and hyperchaos in fractional order Rössler equations, *Physica A: Statistical Mechanics and its Applications*, vol. 341, 2004, pp 5561.
- [17] J. Wang, X. Xiong and Y. Zhang, Extending synchronization scheme to chaotic fractional-order Chen systems, *Physica A: Statistical Mechanics and its Applications*, vol. 370, 2006, pp 279285.
- [18] C. Posadas-Castillo, C. Cruz-Hernández and R. M. López-Gutiérrez, Synchronization of Chaotic Neural networks with Delay in Irregular Networks, *Applied Mathematics and Computation*, vol. 205, 2008, pp 487-496.
- [19] C. Posadas-Castillo, E. Garza-González, C. Cruz-Hernández, E. Alcorta-García and D. A. Díaz-Romero, Chaotic synchronization of complex networks with Rössler oscillators in Hamiltonian form like nodes, *The 4th Chaotic modeling and simulation international conference*, 31 May to 3 June, 2011.