

Self-Triggered Linear Quadratic Networked Control

Matheus Souza, Grace S. Deaecto, José C. Geromel and Jamal Daafouz

Abstract—This paper deals with Networked Control Systems (NCS) design, under the constraint of limited bandwidth on the communication channel. A linear quadratic regulator for a fixed sampling period is solved and this result is used for the development of \mathcal{H}_2 and \mathcal{H}_∞ performance indexes, yielding to the statement and solution of \mathcal{H}_2 and \mathcal{H}_∞ optimal control problems. Finally, a self-triggered controller is designed with a switched system approach in order to improve performance. Examples are presented to illustrate the validity of the theory.

Index Terms—Networked control systems, Switched linear systems, LMI.

I. INTRODUCTION

Networks have become widespread in present days and this fact is causing the integration of various areas of Engineering to a networked environment. This is due to the flexibility in the data traffic between components of the system and to the reduced cost required to its implementation [17]. In this context, a Networked Control System (NCS) is a spatially distributed system in which all the signals travel through a communication network [9] and this structure differs from the classical control system one. Moreover, NCS exhibits some limitations which may compromise the overall performance [9], [17]. In this paper, we focus our study on the so called direct structure, which requires more careful design techniques, see [17].

Several results may be found involving networked control systems analysis. In [10] we can find meaningful results on stabilization of linear systems with minimum *bit-rate* and in [15] some models of networked control systems with limited bandwidth are shown and necessary and sufficient stability conditions are discussed for state-feedback and output-feedback design. Furthermore, [3], [19] provide stability conditions on NCSs involving minimum information rate, data delay and packet dropout.

Focusing on closed-loop performance, [11] analyze the performance of NCSs with a Quality of Service (QoS) and Quality of Performance (QoP) point of view. An interesting approach to the optimal sampled-data control problem is done by [4], where the LQR problem is solved for a periodically sampled control signal, and this result will be used afterwards. Switching strategies are also applied for networked control systems by [5] and [6]. Both papers use

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This research was partially supported by “Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP”, “Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES” and “Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq”, Brazil.

a switched approach to allocate network access in order to prevent collision in the shared medium.

Another approach to NCSs design satisfying some performance index is done with *self-triggering* and *event-triggering* strategies [1], [13], [14], [16]. Both techniques try to reduce network resource consumption by the control system, but they differ in some aspects. The first one uses the current sampled state to determine the control signal and the next sampling time, involving a *scheduling* procedure. The second one uses an *event detector*, that is located next to the plant and uses a function of the current state to determine when the next sampling time must occur.

Analyzing all these results, one may conclude that there are several results on stability of NCSs, but one may notice that there are few results on design of NCSs that satisfy some performance index. Optimal control results are obtained for fixed sampling time and this may not be well adapted for shared networks. Furthermore, we will show that better results are obtained if we allow a dynamically chosen sampling period. Another important remark is that event-triggering may consume network resources excessively in order to detect when the expected event occurs.

Therefore, we conclude that a self-triggering approach to the design of a NCS is well adapted to a networked environment, since it allows a sampled-data control that does not demand network resources excessively. In this context, our goal is to provide results on NCSs optimal design for a fixed sampling period, based on [4], considering the well established \mathcal{H}_2 and \mathcal{H}_∞ performance indexes. Later on we relax this constraint and we design a switched control system, where the switching function determines the next sampling time, yielding a self-triggering controller. In this paper, stability results and guaranteed costs for discrete time switched systems obtained by [7] will be extensively used.

The notation throughout is standard. For square matrices $\text{Tr}(\cdot)$ denotes the trace function and $\sigma_{max}(\cdot)$ is its maximum singular value. For real matrices or vectors (\prime) indicates transpose. For symmetric matrices, the symbol (\bullet) denotes each of its symmetric blocks. The set \mathcal{M} is composed by all Metzler matrices with nonnegative elements $\pi_{ji} \geq 0, \forall i, j$ such that $\sum_j \pi_{ji} = 1, \forall i$. The set of real and natural numbers are denoted as \mathbb{R} and \mathbb{N} whereas $\mathbb{K} = \{1, 2, \dots, N\}$. The squared norm of a trajectory $\xi(t)$ defined for all $t \geq 0$, denoted by $\|\xi\|_2^2$, is equal to $\|\xi\|_2^2 = \int_0^\infty \xi(t)' \xi(t) dt$. All trajectories with finite norm, that is $\|\xi\|_2 < \infty$, constitute the set \mathcal{L}_2 . With a little abuse of notation, the symbol \mathcal{L}_2 also denotes the set of all trajectories with finite norm in discrete-time.

II. PRELIMINARIES

This paper begins by the calculation of two performance indexes that we think are specially well adapted to NCS where the effect of limited bandwidth in measurement and control channels are taking into account. This is the basis for the solution of what we call Linear Quadratic Networked Control Problem - LQN, see [4] for similar results.

A. Sample-data system

Let us consider a continuous-time linear system with minimal state space realization and piecewise constant control of the standard form

$$\dot{x}(t) = Ax(t) + Bu_k(t), \quad x(0) = \xi \quad (1)$$

$$z(t) = Cx(t) + Du_k(t) \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^q$ is the controlled output and $u_k(t) \in \mathbb{R}^m$ is the control signal transmitted through a limited bandwidth channel. Following [12], let us suppose that the communication channel is a perfect noiseless channel, with no time delay, and that $\mu > 0$ is the maximum allowed bit-rate. Then, for a given $r > 0$, we consider the class \mathcal{S}_r of such channels with any sampling period $T > 0$ satisfying

$$\frac{\mu}{T} \leq r, \quad (3)$$

which models bandwidth limitations. Hence, this constraint is equivalent to $T \geq T_* \equiv \mu/r > 0$, where T_* is the minimum sampling period allowed for transmission. With this in mind, supposing that quantization effects can be ignored, the control input is modeled as a piecewise constant signal of the form

$$u_k(t) = u_k, \quad \forall t \in [t_k, t_{k+1}) \quad (4)$$

for all $k \in \mathbb{N}$ where, for the moment, the time duration between two successive sampling instants is constant, that is

$$t_{k+1} - t_k = T, \quad \forall k \in \mathbb{N} \quad (5)$$

This assumption will be relaxed afterwards to cope with non-uniform samplings defined as $T_k = t_{k+1} - t_k, \forall k \in \mathbb{N}$, as an additional decision variable. This is a control system of data-rate-limited type with limited bandwidth equal to $1/T$. We start by introducing the following notation associated to the continuous-time system (1)-(2), namely

$$A = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)} \quad (6)$$

$$C = \begin{bmatrix} C & D \end{bmatrix} \in \mathbb{R}^{q \times (n+m)} \quad (7)$$

and stating a central result to be extensively used in the sequel. Although a similar result can be found in [4], it has been obtained in a different context and has been used to different purposes. It is included here for completeness.

Lemma 1: Consider the system (1)-(2), define matrices of compatible dimensions (A_d, B_d, C_d, D_d) such that

$$e^{AT} = \begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} \quad (8)$$

$$\int_0^T e^{A't} C' C e^{At} dt = \begin{bmatrix} C'_d \\ D'_d \end{bmatrix} \begin{bmatrix} C'_d \\ D'_d \end{bmatrix}' \quad (9)$$

and the discrete-time invariant linear system

$$x_{k+1} = A_d x_k + B_d u_k, \quad x_0 = \xi \quad (10)$$

$$z_k = C_d x_k + D_d u_k \quad (11)$$

then the following equality holds

$$\int_0^\infty z(t)' z(t) dt = \sum_{k=0}^\infty z'_k z_k \quad (12)$$

Proof: Taking into account (4), simple integration enables us to verify that, for all $t \in [t_k, t_{k+1})$,

$$x(t) = e^{A(t-t_k)} x_k + \left(\int_0^{(t-t_k)} e^{A\tau} d\tau \right) B u_k, \quad (13)$$

where $x_k = x(t_k)$ for all $k \in \mathbb{N}$. Hence, at $t = t_{k+1}$ this equality together with (5) and (8) provide (10). Moreover,

$$\begin{aligned} z(t) &= C \left(e^{A(t-t_k)} x_k + \left(\int_0^{(t-t_k)} e^{A\tau} d\tau \right) B u_k \right) + D u_k \\ &= C e^{A(t-t_k)} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \end{aligned} \quad (14)$$

hold whenever $t \in [t_k, t_{k+1})$. We may calculate

$$\begin{aligned} \int_0^\infty z(t)' z(t) dt &= \sum_{k=0}^\infty \int_0^{t_{k+1}-t_k} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' e^{A'\tau} C' C e^{A\tau} \begin{bmatrix} x_k \\ u_k \end{bmatrix} d\tau \\ &= \sum_{k=0}^\infty \begin{bmatrix} x_k \\ u_k \end{bmatrix}' \begin{bmatrix} C'_d \\ D'_d \end{bmatrix} \begin{bmatrix} C'_d \\ D'_d \end{bmatrix}' \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ &= \sum_{k=0}^\infty z'_k z_k \end{aligned} \quad (15)$$

where we have used equations (9) and (11) to get the second and third equalities. The proof is completed. ■

The proof of this lemma makes clear that, generally, the dimensions of $z(t)$ and z_k are not the same. It may occur that the left hand side of (9) be a positive definite matrix which imposes $z_k \in \mathbb{R}^{n+m}$ even though $z(t) \in \mathbb{R}^q$ with $q \leq n < n+m$. Moreover, given the state space representation of the continuous-time system (A, B, C, D) and $T > 0$, it is a simple matter to calculate the state space minimal representation of the discrete-time system (A_d, B_d, C_d, D_d) such that the equality (12) holds. This is important in the context of NCS since it provides the way to calculate a discrete-time linear system such that the \mathcal{L}_2 norm of $z_k, \forall k \in \mathbb{N}$, equals the \mathcal{L}_2 norm of $z(t), \forall t \geq 0$, of the continuous-time system. This fact, expressed by the equality (12), does not involve any kind of approximation.

For any $T > 0$ the result of Lemma 1 can be used to determine the exact value of this quantity whenever the control signal $u_k(t)$ is transmitted through the limited bandwidth channel satisfying (4). Hence, problems formulated in continuous-time are exactly solved in discrete-time.

B. Performance indexes

Based on the result of Lemma 1 we develop two performance indexes associated to the networked system under consideration. They are closely related to the concept of \mathcal{H}_2 and \mathcal{H}_∞ norms of LTI systems, respectively. Consider an open-loop continuous-time system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Ew(t), \quad x(0) = 0 \\ z(t) &= Cx(t) + Fw(t)\end{aligned}\quad (16)$$

where, as before, $x(t) \in \mathbb{R}^n$ is the state, $z(t) \in \mathbb{R}^q$ is the output and $w(t) \in \mathbb{R}^r$ is the exogenous perturbation. In this section, it is assumed that matrix A is Hurwitz which implies that matrix A_d is Schur for all $T > 0$. First, as usually done in the definition of \mathcal{H}_2 norm we assume that $F = 0$ and the perturbation is such that $w(t) = e_i \delta(t)$ where $e_i \in \mathbb{R}^m, i = 1, \dots, m$, are the columns of the identity matrix. Hence, denoting by $z^i(t)$ the output trajectory, the performance index is expressed simply as

$$J_2 = \sum_{i=1}^m \int_0^\infty z^i(t)' z^i(t) dt \quad (18)$$

and, remembering that the impulse at $t = 0$ induces a discontinuity on the initial condition corresponding to move it instantaneously from $x(0) = 0$, Lemma 1 applied to $\dot{x}(t) = Ax(t)$, $z(t) = Cx(t)$ for each initial condition $x(0) = Ee_i, i = 1, \dots, m$ yields

$$J_2 = \sum_{i=1}^m \sum_{k=0}^\infty (z_k^i)' (z_k^i) \quad (19)$$

where $x_{k+1} = A_d x_k$, $x_0 = Ee_i$ and $z_k^i = C_d x_k$ for each $i = 1, \dots, m$ and

$$e^{AT} = A_d, \quad \int_0^T e^{A't} C' C e^{At} dt = C_d' C_d \quad (20)$$

From this calculation it follows that

$$\begin{aligned}J_2 &= \text{Tr} \left(E' \sum_{k=0}^\infty A_d^{tk} C_d' C_d A_d^k E \right) \\ &= \|C_d(zI - A_d)^{-1} E\|_2^2\end{aligned}\quad (21)$$

It is interesting to observe that the pair of matrices (A_d, C_d) depends on $T > 0$ but the index J_2 does not. We conclude that

$$\|C(sI - A)^{-1} E\|_2^2 = \|C_d(zI - A_d)^{-1} E\|_2^2, \quad \forall T > 0 \quad (22)$$

which means that this is just an alternative and equivalent way to calculate the performance index in terms of the \mathcal{H}_2 norm of the associated discrete-time system transfer function. As it will be clear in the sequel, this is an useful result to dealing with networked control systems.

Now, let us move our attention to the \mathcal{H}_∞ performance index. To this end, we assume that $w(t) \neq 0$ is an arbitrary perturbation in \mathcal{L}_2 with finite norm transmitted through the channel. In other words, it is modelled as an additive perturbation that acts in the control channel. After transmission,

denoting again $z(t)$ the corresponding output signal given by (16)-(17), we define the performance index

$$J_\infty = \sup_{w_k(t) \neq 0 \in \mathcal{L}_2} \frac{\int_0^\infty z(t)' z(t) dt}{\int_0^\infty w_k(t)' w_k(t) dt} \quad (23)$$

It is important to give some interpretation about the index we have just introduced that is strongly related to the \mathcal{H}_∞ norm of the continuous-time. Indeed, we immediately have

$$\begin{aligned}J_\infty &\leq \sup_{w(t) \neq 0 \in \mathcal{L}_2} \frac{\int_0^\infty z(t)' z(t) dt}{\int_0^\infty w(t)' w(t) dt} \\ &\leq \|C(sI - A)^{-1} E + F\|_\infty^2\end{aligned}\quad (24)$$

Although the upper bound does not depend on $T > 0$, the value of the index J_∞ does. However, it can be calculated by applying once again the result of Lemma 1 which yields

$$J_\infty = \sup_{w_k \neq 0 \in \mathcal{L}_2} \frac{\sum_{k=0}^\infty z_k' z_k}{T \sum_{k=0}^\infty w_k' w_k} \quad (25)$$

where $x_{k+1} = A_d x_k + E_d w_k$, $z_k = C_d x_k + F_d w_k$. This is now a classical \mathcal{H}_∞ norm evaluation in the discrete-time domain, which immediately gives

$$J_\infty = \left\| \frac{C_d(zI - A_d)^{-1} E_d + F_d}{\sqrt{T}} \right\|_\infty^2 \quad (26)$$

which means that, for $T > 0$ given, the proposed index is proportional to the \mathcal{H}_∞ squared norm of the transfer function from the input w_k to the output z_k of the previously given discrete-time system. Finally, denoting $H(s) = C(sI - A)^{-1} E + F$ and taking $T > 0$ arbitrarily small we have

$$\begin{aligned}J_\infty &= (1/T) \|C_d(zI - A_d)^{-1} E_d + F_d\|_\infty^2 \\ &\approx \|C((zI - I)/T - A)^{-1} E + F\|_\infty^2 \\ &\approx \|H(s)\|_\infty^2\end{aligned}\quad (27)$$

where the last approximation follows from the relation

$$j\omega \approx \frac{e^{j\omega T} - 1}{T} \quad (28)$$

valid for all $\omega \in \mathbb{R}$ and $T > 0$ arbitrarily small. As expected, the index J_∞ tends to the square \mathcal{H}_∞ norm of the continuous-time system under consideration. It can also be shown that $J_\infty \approx \sigma_{max}^2(F - CA^{-1}E)$ for large values of $T > 0$. In our opinion, both indexes are valid for all values of $T > 0$ whenever the performance deterioration due to bandwidth limitations has to be measured.

The next section is devoted to state and solve the \mathcal{H}_2 and \mathcal{H}_∞ optimal control problems for networked systems taking into account the limited bandwidth constraint. The rationale to be adopted is to convert the continuous-time model to the discrete-time one as indicated in Lemma 1.

III. LINEAR QUADRATIC NETWORKED CONTROL

Let the control system with bandwidth limitation be given

$$\dot{x}(t) = Ax(t) + Bu_k(t) + Ew(t) \quad (29)$$

$$z(t) = Cx(t) + Du_k(t) + Fw(t) \quad (30)$$

with zero initial condition and where $w(t)$ is the exogenous input. The main goal is to determine a state feedback gain $L \in \mathbb{R}^{m \times n}$ such that the closed-loop system with $u(t) = u_k = Lx_k, \forall t \in [t_k, t_{k+1})$, minimizes one of the indexes J_2 or J_∞ that we have discussed in the previous section.

A. \mathcal{H}_2 norm optimization

Applying Lemma 1, we convert the above problem in the one that consists in finding $u_k(t)$ in order to minimize the index J_2 . As usual, it is assumed that $F = 0$. Based on the results of the previous section, this problem can be recast in the equivalent form:

$$\inf_L \|(C_d + D_d L)(zI - (A_d + B_d L))^{-1} E\|_2^2 \quad (31)$$

which is nothing else than a classical LQ problem in discrete-time. Hence, the optimal gain can be determined from the positive definite stabilizing solution of the algebraic Riccati equation (ARE) obtained from

$$(A_d + B_d L)' P (A_d + B_d L) - P + (C_d + D_d L)' (C_d + D_d L) = 0 \quad (32)$$

together with $L = -(B_d' P B_d + D_d' D_d)^{-1} (B_d' P A_d + D_d' C_d)$. This state feedback gain is unique and the best one as far as the control constraint (4) is taken into account. Moreover, whenever it admits a positive definite stabilizing solution, the closed-loop continuous-time system becomes asymptotically stable as a consequence of (12). In this sense, observability of (A, C) and controllability of (A, B) are required.

It is interesting to see that for $T > 0$ arbitrarily small, we have $A_d \approx I + TA$, $B_d \approx TB$, $C_d \approx \sqrt{T}C$ and $D_d \approx \sqrt{T}D$ and the previous Lyapunov equation (32) collapses to

$$(I + T(A + BL))' P (I + T(A + BL)) - P + T(C + DL)' (C + DL) = 0$$

that is

$$(A + BL)' P + P(A + BL) + (C + DL)' (C + DL) + \mathcal{O}(T) = 0$$

where $\mathcal{O}(T) \geq 0$ and as expected, when $T > 0$ goes to zero, we obtain the optimal continuous-time state feedback gain. Of course for $T > 0$ arbitrary the optimal solution does not necessarily coincide with this one.

Example 1: Consider a networked system (29)-(30) defined by matrices

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, F = 0.$$

Figure 1 shows the index J_2 against the sampling period $T > 0$. For $T \rightarrow 0$ it illustrates the fact that the optimal continuous-time solution is generated.

Notice the periodic asymptotes for the discrete-time system clearly indicated in the same figure. These singularities put in evidence the existence of values of the sampling period

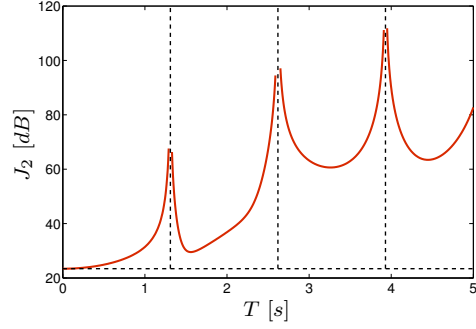


Fig. 1: Values of the performance index J_2 for $T > 0$.

for which the discrete-time ARE does not admit a stabilizing positive definite solution. This figure makes clear that $J_2(\cdot)$ is not a monotonic function of the sampling period T .

B. \mathcal{H}_∞ norm optimization

Following the same steps of the \mathcal{H}_2 case, our concern now is to determine the state feedback gain $L \in \mathbb{R}^{n \times m}$ that minimizes the \mathcal{H}_∞ cost J_∞ . Imposing $u(t) = u_k = Lx_k, \forall t \in [t_k, t_{k+1})$, from Lemma 1 we obtain

$$\inf_L \frac{1}{T} \|(C_d + D_d L)(zI - (A_d + B_d L))^{-1} E_d + F_d\|_\infty^2 \quad (33)$$

Since $T > 0$ is fixed this is a standard \mathcal{H}_∞ problem in discrete-time which can be entirely expressed through LMIs [2]. Indeed, the solution of problem (33) is obtained from

$$\inf_{X > 0, Z, \rho} \rho \quad (34)$$

subject to the LMI constraint

$$\begin{bmatrix} X & \bullet & \bullet & \bullet \\ XA_d' + Z'B_d' & X & \bullet & \bullet \\ E_d' & 0 & I & \bullet \\ C_d X + D_d Z & 0 & F_d & \rho I \end{bmatrix} > 0 \quad (35)$$

which provides the state feedback gain $L = ZX^{-1}$ and the cost $J_\infty = \rho/T$. The advantage of this approach is the linear dependence of all involved variables, including ρ , which avoids the adoption of an interactive method based on the ARE calculation to get the minimum value of the norm. On the other hand, for $T > 0$ arbitrarily small, considering the approximations previously adopted together with $E_d \approx TE$ and $F_d \approx \sqrt{T}F$ the inequality (35) reduces to

$$\begin{bmatrix} AQ + QA' + BR + R'B' & \bullet & \bullet \\ E' & -I & \bullet \\ CQ + DR & F & -(\rho/T)I \end{bmatrix} < 0 \quad (36)$$

where $Q = X/T$ and $R = Z/T$, which is nothing else but the \mathcal{H}_∞ condition in continuous-time. This inequality puts in evidence the scaling $1/T$ appearing in the ρ variable in order to compensate the same scaling in the cost (33).

IV. SELF-TRIGGERED DESIGN

In this section we introduce what we call a *self-triggered control design* based on the following observation. Contrarily

of what is adopted in the event-triggering strategy [14], where the sampling period is considered as a continuous variable $T \in \mathbb{R}$ to be determined on real time, in order to improve stability and performance, we assume that $T \in \{T_i : i \in \mathbb{K}\}$ where the number N and the values of different time samplings T_i , $i \in \mathbb{K}$, are provided by the designer. They are chosen taking into account two important features of networked control systems: \mathcal{H}_2 performance and bandwidth limitations. The first criterion induces small T while the second one goes in the opposite direction. The main problem to be faced is the determination of a switching rule that orchestrates the use of a particular T_i for some $i \in \mathbb{K}$.

The first step yields the strictly proper state space models $(A_{Li}, E, C_{Li}, 0)$ for all $i \in \mathbb{K}$ as follows. For each T_i , $i \in \mathbb{K}$, the optimal solution of problem (31) provides the closed-loop system matrices $A_{Li} = A_{di} + B_{di}L_i$ and $C_{Li} = C_{di} + D_{di}L_i$ which define the switched linear system

$$x_{k+1} = A_{L\sigma}x_k + Ew_k \quad (37)$$

$$z_k = C_{L\sigma}x_k \quad (38)$$

evolving from $x_0 = 0$, where $\sigma(k) : \mathbb{N} \rightarrow \mathbb{K}$ is the switching function to be designed and $w_k \in \mathbb{R}^r$ is an impulsive external input of the form $w_k = \delta(k)e_\ell$, being e_ℓ the ℓ^{th} column of the identity matrix which is used to define the \mathcal{H}_2 performance

$$J_2(\sigma) = \sum_{\ell=1}^r \|z^\ell\|_2^2 \quad (39)$$

Whenever the switching function is enforced to be equal to $\sigma(k) = i \in \mathbb{K}$ for all $k \in \mathbb{N}$ then $J_2(\sigma)$ equals the optimal performance of the closed-loop system with the sampling T_i . Hence, our purpose is to determine a state dependent switching strategy of the form $\sigma(k) = g(x(k))$ in order to improve the final performance by an appropriate choice of the samplings T_i , $i \in \mathbb{K}$, dynamically, that is, at each instant of time t_k , for all $k \in \mathbb{N}$. Following [7], we consider

$$g(x) = \arg \min_{i \in \mathbb{K}} x' P_i x \quad (40)$$

where $P_i > 0$, $\forall i \in \mathbb{K}$, have to be adequately determined. Indeed, defining the set \mathcal{X}_2 composed by all matrices $\{P_1, \dots, P_N\}$ and a Metzler matrix $\Pi \in \mathcal{M}$ that satisfy the so called Lyapunov-Metzler inequalities

$$A'_{Li} \left(\sum_{j \in \mathbb{K}} \pi_{ji} P_j \right) A_{Li} - P_i + C'_{Li} C_{Li} < 0, \quad i \in \mathbb{K} \quad (41)$$

it is proven in [8] that any feasible matrices $\{P_1, \dots, P_N\}$ whenever plugged in (40) makes the switched linear system asymptotically stable and

$$J_2(\sigma) < \min_{i \in \mathbb{K}} \text{Tr}(E' P_i E) \quad (42)$$

Hence, the minimum upper bound is given by the optimal solution of the problem

$$J_2^{so} = \inf_{\{P_1, \dots, P_N, \Pi\} \in \mathcal{X}_2} \min_{i \in \mathbb{K}} \text{Tr}(E' P_i E) \quad (43)$$

which provides the switching strategy σ_{so} that clearly satisfies $J_2(\sigma_{so}) \leq J_2^{so}$. Unfortunately, this problem is, in general, difficult to solve due to the product of variables $\pi_{ji} P_j$ which makes it nonconvex. To this end, some specific optimization method has to be adopted unless N is small (typically 2 or 3) in which case it is solved by grid search by taking advantage to the fact that for $\Pi \in \mathcal{M}$ fixed, (41) reduces to N LMIs.

In our context, problem (43) exhibits a remarkable property. By construction, all discrete-time subsystems matrices A_{Li} , $i \in \mathbb{K}$, are Schur and consequently the matrix $\Pi = I \in \mathcal{M}$ is feasible. Indeed, plugging this matrix in (41) the LMIs become decoupled, that is

$$A'_{Li} P_i A_{Li} - P_i + C'_{Li} C_{Li} < 0, \quad i \in \mathbb{K} \quad (44)$$

implying that

$$\begin{aligned} J_2(\sigma_{so}) &\leq J_2^{so} \\ &\leq \min_{i \in \mathbb{K}} \inf_{P_i > 0} \{\text{Tr}(E' P_i E) : (44)\} \\ &\leq \min_{i \in \mathbb{K}} \|C_{Li}(zI - A_{Li})^{-1}E\|_2^2 \end{aligned} \quad (45)$$

We can draw the conclusion that the procedure proposed in this paper promotes, in general, a gain of the performance due to the dynamic allocation of time sampling periods.

Example 2: Consider the following marginally stable dynamic system of the form (29)-(30), given by

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad F = 0. \end{aligned}$$

To build the discrete-time switched linear system we suppose that transmission is allowed only with sampling periods equal to $T_1 = 0.80$ [s] and $T_2 = 1.1$ [s]. Using the procedure developed in the previous section, we obtain the associated costs $J_2(\sigma = 1) = 8.8501$ and $J_2(\sigma = 2) = 9.9731$. In addition, solving problem (43) by performing a grid search in the box $[0, 1] \times [0, 1]$ we have determined the minimum guaranteed \mathcal{H}_2 cost $J_2^{so} = 8.6059$.

We also solved this problem using the event-triggering strategy proposed by [14]. Through simulation, the event-triggered regulator achieves an \mathcal{H}_2 gain of 7.80 and our approach provides a cost of 8.31. However, to accomplish this performance, in various intervals of time, the event-triggered compensator works with very small sampling periods, which may not be possible on limited bandwidth communication channels. This behavior is shown in Figure 2, which displays the sampling period for both strategies. As it can be seen in this figure, the upper curve describes the sampling period with respect to time for our switching approach and the second one represents the sampling period for the event-triggered controller. With this in mind, we impose a threshold value $\tau = 0.80$ [s] for the sampling time corresponding to a maximum bandwidth. In this situation the event-triggering technique provides a cost of 11.39, which shows that our technique provides a gain of 27% in

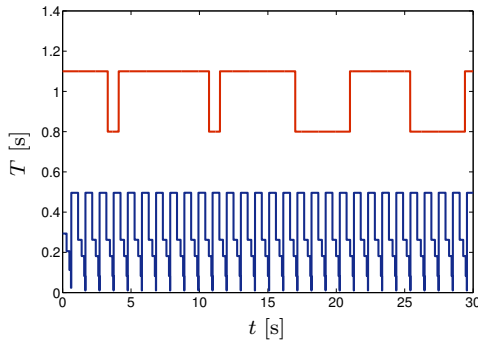


Fig. 2: Sampling period for both techniques.

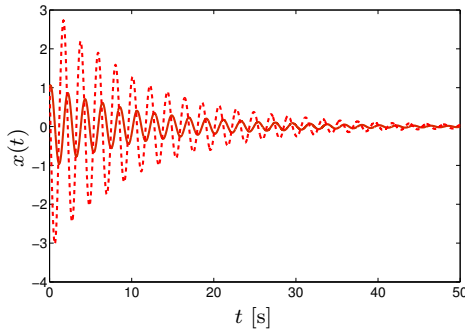


Fig. 3: Self-triggered closed-loop system.

performance, approximately. It is also important to state that, if we use our approach and allow sampling with a period less than 0.55 [s], we obtain better \mathcal{H}_2 costs than event-triggering and may demand less network resources. Simulation results are shown in Figures 3 and 4.

This final example makes clear that our approach is well-adapted to limited bandwidth constraints. When applied to unstable systems, the event-triggering technique does not guarantee stability with the minimum sampling period constraint (threshold).

V. CONCLUSION

In this paper we have presented results in Networked Control Systems design, subject to limited bandwidth communication constraints. To this end, our first approach consisted in obtaining an optimal state-feedback gain that stabilizes

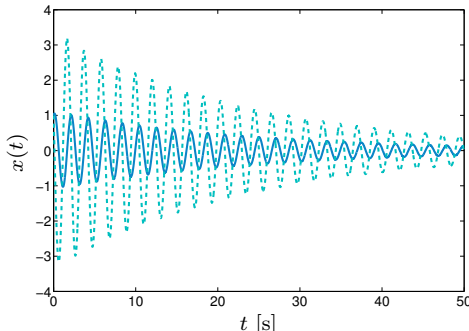


Fig. 4: Event-triggered closed-loop system ($\tau = 0.8$ [s]).

the sampled time system through the solution of a linear quadratic problem of a specific discrete-time system. This important result has been used afterwards in order to develop \mathcal{H}_2 and \mathcal{H}_∞ performance indexes for the limited bandwidth problem. With these indexes, we state and solve optimal \mathcal{H}_2 and \mathcal{H}_∞ control problems for the NCS.

Finally, the self-triggering control system has been designed using a discrete-time switched control approach. It has been shown that this strategy is well-adapted to a networked environment and it may improve the overall performance of the closed-loop system. It is important to remember that most of the conditions are given by LMIs or AREs, which can be efficiently solved numerically.

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