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Necessary conditions for solving the nonlinear MIMO output regulation problem

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Abstract— **Solutions of the output regulation problem in a nonlinear setting invariably target systems presented under some "normal form". This paper reviews a recently introduced such form for MIMO nonlinear systems and discusses consequent necessary conditions for solving the MIMO nonlinear output regulation problem. As opposed to the (by now classic) form based on the notion of vector relative degree that is only specific to some square invertible MIMO nonlinear systems, the recent form applies to general (not necessarily invertible, nor square) systems. It turns out that, as far as output regulation goes, the recent form fits better some assumptions that have already been used in the context of square invertible systems and, more important, that can still be used (thanks to this particular normal form) in the context of not square ones.**

I. INTRODUCTION

THE design of feedback controllers such that certain outputs of a given plant track prescribed reference trajectories in the presence of disturbances is a defining problem in control theory. One particular (deterministic) form of this problem is to consider that the dynamics of the system that generates the references and the disturbances the exosystem) are known and consequently design a controller that steers to zero certain outputs of the augmented system plant-plus-exosystem, thus achieving what is called the property of output regulation.

The output regulation problem was extensively (and exhaustively) studied in the case of MIMO linear systems during the 1970s; results culminated with the Internal Model Principle (IMP), which states that a structurally stable solution (i.e. robust to plant parameter variations) necessarily has to use feedback of the regulated variables and incorporate in the feedback path a (possibly redundant) model of the exosystem (see e.g. [1] for a proof).

Although one can trace some attempts to extend the IMP to nonlinear systems also as far back as the 1970s, actual attempts to solve the problem in a (general) nonlinear setting were initiated at the beginning of the 1990s. In particular, the seminal paper [2], while displaying rather weak results (local, nonrobust regulation about an equilibrium), laid the foundations for the subsequent developments by outlining the role of the zero dynamics in the problem and the need to force the existence of an asymptotically attractive invariant set onto which regulation is accomplished. Borrowing from the terminology used in the case of linear systems, the latter idea conveys the internal model property of the controller (along with the stabilizability one).

Two key ideas that enabled the development of results with a higher degree of generality were a general (nonequilibrium) definition of the problem and the fact that the problem of achieving the internal model property can be stated as a nonlinear observer synthesis problem [3]. These ideas were pursued mainly for SISO nonlinear systems, leading to some effective designs (see also [3] for a brief discussion).

The fact that so far there have been limited attempts (e.g. [4]) to solve the problem in the MIMO case is not entirely surprising, and for various reasons. First, MIMO normal forms are not simple extensions of SISO normal forms. For instance, while SISO normal forms lend themselves naturally to the definition of nonlinear equivalents for the linear finite and infinite zero structures and invertibility properties, the extension of these notions to MIMO systems is nowhere near as straightforward. On the other hand, a normal form of some kind represents the only tool to (robustly) handle nonlinear dynamics for control purposes, while normal forms that seem to be adequate for the MIMO nonlinear output regulation problem have been introduced just recently [5]. Second, seeing how the design of observers is instrumental in the design of controllers that solve the output regulation problem, the design of observers in the multiple-output case is known to pose serious technical difficulties, especially in obtaining the right canonical forms that allow a meaningful (constructive) characterization of the observability properties [6]. Last, but not the least, while there is a wealth of stabilization tools for SISO nonlinear systems, there are not so many available for MIMO systems.

This paper presents a first discussion concerning the first of the aforementioned issues—the role of normal forms in MIMO nonlinear output regulation. In particular, based on the recent normal forms of [5], the paper reviews the basic assumptions and necessary conditions for solving the problem. Sufficient conditions will then respond to the remaining two issues (achievement of the internal model property through observer design and stabilization on the invariant set onto which regulation is accomplished).

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The paper is structured as follows: in section 2 the general problem of output regulation for nonlinear MIMO systems is presented, in section 3 the classic and recent results on normal forms used for MIMO nonlinear control problems are introduced and the zeros dynamics of the system is discussed, in section 4 the necessary conditions under which the output regulation problem can be solved are discussed and the assumptions considered in [4] are relaxed and section 5 deals with conclusions and further developments.

II. The general output regulation problem for MIMO nonlinear systems

The problem of output regulation considers known the models for the process to be control and the exosystem. The later is supposed to contain the reference and/or the perturbations. A regulator solving the problem in closed loop must assure: the trajectory boundness and error uniform convergence to 0.

We consider a multivariable system given by the following expression:

$$\begin{aligned} \dot{x} &= f(w,x,u) \\ y &= k(w,x) \\ e &= h(w,x) \end{aligned} \tag{1}$$

with: $x \in R^n$ the system state, $u \in R^m$ the control input, $e \in R^p$ regulated output, $y \in R^p$ measured output, $w \in R^r$ exosystem' state.

The exosystem is an autonomous system:

$$\dot{w} = s(w) \tag{2}$$

The functions *f(w,x,u)*, *h(w,x)*, *k(w,x)* and *s(w)* are considered to be of class C^k (sufficiently large) in their arguments. The initial conditions for the system – vary on a fixed closed set $x(0) \in X_0$ and for the exosystem - vary on a invariant compact set $w(0) \in W$.

The exosystem is supposed to be of a certain known class such that *w(t)* – is the solutions family determined by (2) with initial condition in *W*. The regulator is supposed to be of the form:

$$\begin{aligned} \dot{\psi} &= \varphi(\psi, y) \\ u &= \gamma(\psi, y) \end{aligned} \tag{3}$$

with: $\psi \in R^v$ the regulator state and the functions $\varphi(\psi, y)$ and $\gamma(\psi, y)$ are of class C^k. The initial conditions for the regulator can vary on a compact set $\psi(0) \in \Xi$. The system (1), (2) and (3) in closed loop form is:

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x,\gamma(\psi,k(w,x))) \\ \dot{\psi} &= \varphi(\psi,k(w,x)) \\ e &= h(w,x) \end{aligned} \tag{4}$$

Consider that *X* is a compact subset of X_0, the regulator (3) solves the output regulation problem if the positive trajectory on $W \times X \times \Xi$ is bounded and $\lim_{t\to\infty} e(t) = 0$, uniformly on $W \times X \times \Xi$ (when the system is in steady state). The form of the regulator, its initial conditions set and its proprieties are to be determined.

In the context of the output regulation problem as presented in [7] with the notations and lemmas of [7] the trajectories of the system in closed loop are supposed bounded. This leads to the conclusion that if the ω limit set $\omega(W \times X \times \Xi)$ is bounded the steady state error tends to 0 while time tends to infinity if and only if: $\omega(W \times X \times \Xi) \subset \{(w,x,\psi) : h(w,x) = 0\}$.

We further consider that the system is of finite dimension, time invariant, and can be put in a normal form such that it has a well defined relative degrees vector and the zero dynamics is stable (the system is of minimum phase). This approach is followed by Isidori in [4] and also described in [5].

Writing the system in its normal form is vital because this gives, among other information, the zero dynamics. In the following chapter two approaches to represent the system to its normal form are briefly presented.

III. Normal forms and zero dynamics for MIMO nonlinear systems

The normal forms for nonlinear MIMO systems reveal the structural properties of the analyzed system: relative degrees, zero dynamics and invertability properties. The relative degrees refer to the infinite zero structure, while the zero dynamics represents the finite zero structure of the system. These notions are important (as in the case of linear systems) when control problems are of interest. Considering an input affine nonlinear MIMO system it can be represented as in (5):

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{5}$$

where $x \in R^n$ is the system's states, $u \in R^m$ is the input, $y \in R^p$ is the output and *u(x), y(x), h(x)* are vectors.

In the case of square systems, $m=p>1$ the system can be written in the normal form introduced by Isidori in [8]. The system can be put in the normal form with a vector of relative degrees $\{r_1,...,r_m\}$ at x_0 if:

1. $L_{g_j} L_f^k h_i(x) = 0$; $j=1,m$; $i=1,m$; $k<r_i-1$; $\forall x$ in the neighborhood of x_0 and

2. $M = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \dots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \\ L_{g_1} L_f^{r_m-1} h_m(x) & \dots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$ nonsingular at $x_0=x$.

Considering that $dL_f^j h_i(x)$, $j=0,r_i-1$, $i=1,m$ are linearly independent the system normal form of (5) is:

$$\begin{aligned} \dot{z} &= f_{0i}(z,e) + M(z,e)u \\ \dot{e}_{i,1} &= e_{i,2} \\ &\dots \\ \dot{e}_{i,r_i-1} &= e_{i,r_i} \\ \dot{e}_{i,r_i} &= q(z,e) + \sum_{j=1}^{m} m_{ij}(z,e) u_j \\ y_i &= e_{i,1} \end{aligned} \quad , i=1,m \qquad (6)$$

where $M_{ij}(z,e) = L_{g_j} z_i \circ \Theta^{-1}(z,e)$, and the function $\Theta : x -> (e,z)$ is the diffeomorphism from *x* to *(z, e)*. For the MIMO case, it is possible to find a set of functions such that $L_{g_j} z_i(x) \equiv 0$, $j=1, p$; $i=1,n-\sum_{i=1}^{m} r_i$ only if the distribution spanned by the column vectors $\{g_1, g_2, ... g_m\}$ is involutive in a neighborhood of x_0. Each relative degree r_i is associated to the i^{th} system output. The sum of the relative degrees is at most *n*. Considering the system (6), the zero dynamics is given by the following expression $\dot{z} = f_{0i}(z,0)$.

A generalization of the above algorithm was made in [9]. The assumptions needed for the elaboration of the presented normal form are rather strong. In a recent publication [5], these assumptions are substantially weakened. Moreover, the infinite zero dynamics algorithm [5] allows the representation of MIMO nonlinear systems that are not necessary square and invertible in the following form:

$$\begin{aligned} \dot{z} &= f_e(x) + g_e(x) u_e + \sum_{i=1}^{m_d} \varphi_l(x) v_{d,l} \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_{d,l}, \ j = 1, q_i - 1 \\ \dot{\xi}_{i,q_i} &= v_{d,i} \\ y_\Delta &= h_\Delta(z, z_d) \\ y_{d,i} &= \xi_{i,1}; i = 1, m_d \end{aligned} \qquad (7)$$

with: $q_1 \le q_2 \le ... \le q_{m_d}$, $\xi_i = \{\xi_{i,1}, \xi_{i,1}, ... \xi_{i,q_i}\}$, $i=1,m_d$; $v_{d,i} = a_i(x) + b_i(x)u$; $y_d = col(\xi_{1,1}, \xi_{2,1}, ... \xi_{m_d,1})$ and $col\{b_1(x), b_2(x), ..., b_{m_d}(x)\}$ is nonsingular.

From the infinite zero structure algorithm, the dynamics $\dot{\xi}_{i,j}$ does not depend on v_l, $l>j$ and it follows that :

$$\delta_{i,j,l}(x)=0, \ j<q_l, i=1,m_d \qquad (8)$$

The relations (7), (8) hold under the Assumption B [5]. In (7) m_d represents the largest number for which the system can be transformed in the normal form (this value results by applying the infinite zero structure algorithm). The algorithm also identifies a vector of integer values $\{q_1, q_2, ..., q_m\}$ that represents the infinite zero structure (if the system is linear).

In addition, if the Assumption C [5] holds, there exist a coordinate transform that puts the system in the form :

$$\begin{aligned} \dot{z} &= f_e(x) + g_e(x) u_e \\ \dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x) v_{d,l}, \ j = 1, q_i - 1 \\ \dot{\xi}_{i,q_i} &= v_{d,i} \\ y_e &= h_e(x) \\ y_{d,i} &= \xi_{i,1}; i = 1, m_d \end{aligned} \qquad (9)$$

and $\delta_{i,j,l}(x)=0$, $j<q_l$, $i=1,m_d$.

Moreover, form (9) gives an insight on the system invertability properties. If the term u_e is absent, the system is left invertible; if the term y_e is absent, the system is right invertible; if both terms u_e and y_e are absent, the system is invertible; if both terms u_e and y_e exist, the system is degenerate [5].

In the case of square invertible systems with $m=p=m_d$, the terms u_e si y_e do not exist and the system takes the following normal form [5]:

$$\dot{z} = f_e(z,\xi)$$
$$\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1}\delta_{i,j,l}(z,\xi)v_l, j = 1, q_i - 1 \quad (10)$$
$$\dot{\xi}_{i,q_i} = v_i$$
$$y_i = \xi_{i,1}; i = 1, m_d$$

with $q_1 \le q_2 \le ... \le q_m$; $\delta_{i,j,l}(x)=0,\ j<q_l,\ i=1,m.$ (11)

It can be observed that the system in (10) form has a triangular structure [5] between $\dot{\xi}_{i,j}$ and the inputs. Relation (11) reveals the fact that there is also a triangular dependency of $\delta_{i,j,l}$ on the state variables [5].

The system's (10) zero dynamics is given by the expression: $\dot{z} = f_e(z,0)$.

This type of normal form with a structure in which the commands are entering the system in a triangular fashion and the $\delta_{i,j,l}$ functions have triangular dependencies on the systems state presents valuable properties. Using them to solve the nonlinear MIMO output regulation problem hasn't been considered.

In what follows we are going to present the output regulation problem in the case of MIMO systems and to compare the conditions for the problem solvability in case of using the two normal forms presented above in terms of assumptions restrictions.

IV. Necessary conditions for solving the output regulation problem MIMO nonlinear systems

A. The system in the normal form (6)

In this case the system with the exosystem (2) has the form:

$$\begin{aligned}\dot{z} &= f_0(z,w) + f_1(z,e,w)e \\ \dot{e} &= q(z,e,w) + M(z,w)u\end{aligned}\ , i=1,m \quad (12)$$

with: initial conditions in the set $Z \times E \times W$ where Z is fixed and compact and E is bounded and the functions f_0, f_1, g, s, M are smooth enough.

The coupling matrix $M(z,w)$ is considered, this means that system (12) has a vector of relative degrees: $\{1,1,...1\}$ between the control input u and the regulated output e. A system with the relative degrees vector $\{r_1,...r_m\}$ can always be transformed into the form (12).

Considering that a controller of the form (3) solves the problem of output regulation and applying lemma 2[7] the steady state locus of the system in closed loop must be a subset of the set for which the error is zero ($e=0$).

If the system's (7) zero dynamics is in steady state the following conditions are respected: the steady state locus of the system in closed loop $\omega(W \times Z \times E \times \Xi)$ is a subset of: $R^s \times R^{n-m} \times \{0\} \times R^v$ and the restriction of the system in closed loop to the steady state locus $\omega(W \times Z \times E \times \Xi)$ is (with the zero dynamics given by the first two relations):

$$\dot{w} = s(w)$$
$$\dot{z} = f_0(w,z)\text{; with } e=0 \text{ and}$$
$$\dot{\psi} = \varphi(\psi,0)$$
$$\forall\ (w,z,0,\dots,0,\psi) \in \omega(W \times Z \times E \times \Xi)$$

It was considered that the positive trajectory of the exosystem (2) on W *is* bounded if the trajectories asymptotically approach $\omega(W)$. This assumption does not diminish the generality of the problem because it can be considered that $W=\omega(W)$ which means the exosystem is in steady state.

Assumption MIMO: The set $W \subset R^n$ is compact and invariant under (2). If the positive orbit of the set $W \times Z \times E \times \Xi$ under (1), (2) and (3) system dynamics in closed loop is the graph of a function defined on the whole of W parts. Noting

$$A_{ss} = \{(w,z) \mid (w,z,0,\dots,0,\psi) \in \omega(W \times Z \times E \times \Xi), \psi \in R^v\} \quad (13)$$

the function

$$u_{ss} : A_{ss} \to R^m$$
$$(w,z) \to -M^{-1}(w,z)q(w,z,0)$$

is considered. By construction the set described by (13) is the codomain of a function defined on the whole of W, which is invariant under the zero dynamics of the system in the normal form (6) and the exosystem:

$$\begin{aligned}\dot{w} &= s(w) \\ \dot{z} &= f_0(w,z)\end{aligned} \quad (14)$$

The function u_{ss} is the control law that forces the system to evolve on A_{ss}. In conclusion, if the controller (3) solves the output regulation problem for the system in normal form with the exosystem, then there is a function defined on the whole of W which has the codomaine A_{ss} and A_{ss} is invariant under (14).

Moreover, for each $(w_0, z_0) \in A_{ss}, \exists\, \psi_0 \in R^v$ such that the integral curve of (14) is exactly the integral curve of

$\dot{\psi} = \varphi(\psi, 0)$ starting in ψ_0 and satisfying $u_{ss}(w(t), z(t)) = \gamma(\psi(t), 0), \forall t \in R.$ In other words, one can build a controller that reproduces the input for the steady state such that the regulated error is 0 (internal model for nonlinear system).

Considering this approach it can be observed that Assumption MIMO can be considered only if the coupling matrix *M* is invertible.

The invertability implies two aspects: the system is square and the inverse of *M* must be formally computed – which represents a major drawback for real implementation cases. This is probably why no publication follows the article giving this solution in [4]. In what follows we propose an alternative to this approach.

B. The system in the normal form (10)

Following the above reasoning, we consider that the system is written in the normal form (10) with the exosystem (2):

$$\begin{aligned}
&\dot{w} = s(w)\\
&\dot{z} = f_e(z, w)\\
&\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, w, \xi) v_l, j = 1, q_i - 1\\
&\dot{\xi}_{i,q_i} = v_i\\
&y_i = \xi_{i,1}; i = 1, m_d
\end{aligned}$$

New assumption MIMO: The set $W \subset R^n$ is compact and invariant under/for system's (10) and (2) zero dynamics. If the positive orbit of the set $W \times Z \times E \times \Xi$ under (10), (2) and (3) system dynamics in closed loop is the graph of a function defined on the whole of *W* parts.

If we note:

$$\begin{aligned}
&A'_{ss} = \{(w, z) \mid (w, z, 0, \ldots, 0, \psi)\\
&\in \omega(W \times Z \times E \times \Xi), \psi \in R^v\}
\end{aligned} \tag{15}$$

and if in relation (8) $q_l \le 1$ we can consider the function:

$$\begin{aligned}
&u'_{ss} : A'_{ss} \to R^m\\
&(w, z) \to \begin{bmatrix} -a_1(w, z, 0) / b_1(w, z, 0) \\ f_p(u'_1, .., u'_{p-1}, a_1(w, z, 0), ..., \\ a_{p-1}(w, z, 0), b_1(w, z, 0), ..., b_{p-1}(w, z, 0)) \end{bmatrix}.
\end{aligned}$$

If the condition: $q_l \le 1$ in (8) does not hold, there is still a better premises for inverting the system due to a block diagonal form comparing to [4].

By construction the set described by (15) is invariant under the zero dynamics of the system in the normal form (10) and the exosystem (2):

$$\begin{aligned}
&\dot{w} = s(w)\\
&\dot{z} = f_e(z, 0)
\end{aligned} \tag{16}$$

The function u'_{ss} is the control law that forces the system to evolve on A'_{ss}. The "triangular" properties of the new normal form (10) lead to a feasible solution for writing the necessary conditions for the output regulation problem in a real case, without formally inverting the coupling matrix.

V. CONCLUSIONS

In the light of recent advancements on normal forms we consider that a more suitable approach for solving the nonlinear output regulation problem in the case of square MIMO systems is to use the normal form proposed in [5]. In this case we propose new conditions under which the problem of output regulation is solvable. These more relaxed from the following points of view: the relative degrees vector is not required; there is no assumption that implies the fact that the coupling *M* matrix is to be inverted explicitly, nonsquare MIMO systems could be considered for control too; the assumptions needed for the [5] normal form are less restrictive than the ones for the classical normal form.

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