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Approximate feedback linearization of the Acrobot tracking error dynamics with application to its walking-like trajectory tracking

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Abstract— **The purpose of this paper is to provide theoretical framework enabling to design tracking feedback for a general Acrobot trajectory which allows rigorous convergence proof. It is based on the partial exact feedback linearization of the Acrobot model followed by further approximate feedback linearization of the tracking error dynamics for arbitrary target trajectory. The approach presented here enables to prove the convergence in a rigorous way at least for small initial tracking errors. The slight novelty here is that neglecting is made with respect to tracking error along any general trajectory to be tracked, not just in some neighborhood of fixed working point. To demonstrate viability of this approach, simulations of a tracking of a walking-like cyclic trajectory are presented. The walking includes several steps including impacts between them. As a matter of fact, the exponentially stable tracking during the swing phase only is capable to stabilize overall walking, including the effect of the impacts.**

I. INTRODUCTION

This paper aims to provide yet another novel approach to design of the nonlinear feedback to ensure an exponential tracking for a general Acrobot trajectory. Such a design allows to make a rigorous proof of convergence of the Acrobot feedback tracking. The Acrobot is one of the simplest underactuated mechanical systems. It has two links and one actuator placed between them, see Fig. 1. Underactuated mechanical systems [1], [2] are mechanical systems having less actuators than the number of degrees of freedom. Besides the Acrobot, the simplest underactuated mechanical systems are e.g. Inertia wheel pendulum and the Pendubot. The Acrobot motion can be controlled in a way resembling walking. For this purpose it is in the literature usually called as the Biped or Compass gait model. Despite rich literature on this topic, the design of the reference trajectories to be tracked and the design of control laws ensuring its stable tracking is still challenging problem for Compass gait model. For simplicity and shortness, the terminology "Acrobot" and "Acrobot walking" will be used in the sequel.

The extensive study of the Acrobot walking-like movement is a part of even more extensive area of underactuated walking study. Besides the robotics field, this topic is reflected in control research area as well, and goes back perhaps to initial passive walkers study [3], which have limited capabilities and walk down a slope. For the active control setting the corresponding problems are basically related with a) a walking like trajectory planning, and b) a design of the feedback ensuring an exponential tracking of such a trajectory. Recent state of the art is reflected in monographs [4], [5].

The classification of two degrees of freedom mechanical systems may be provided based on feedback transformation and partial exact feedback linearization type. The first results on this classification and certain types of canonical forms was first obtained in [6], [7], [8]. Equivalent classification using differential forms spaces can be found in [9] where systems are classified using terminology *Class 0*, *Class 1* and *Class 2*. For the systems of *Class 0*, e.g. Inertia wheel pendulum, there exists independent function with relative degree four with respect to the input. It means, the systems of *Class 0* are full order exact linearizable. For the systems of *Class 1*, e.g. Acrobot, two independent functions with relative degree 3 exist. These systems have one-dimensional zero dynamics. For the systems of *Class 2*, e.g. Pendubot only one independent function with relative degree 2 exists. These systems have two-dimensional zero dynamics. Partial exact feedback linearization of the general n-link chain system is studied in [10].

Yet, the property of the Acrobot being the *Class 1* is not easy to use for tracking purposes as the one dimensional zero dynamics is unstable along all reasonable walking-like trajectories and depends in a complicated nonlinear way on all variables. After some initial efforts in [6], the first result using these canonical forms for walking was presented in [11], where the efficient design of the walking-like pseudo-passive trajectory was presented including exponentially stable tracking feedback design under quite restrictive assumptions of small velocity of walking. Pseudo-passive trajectory is the one beeing generated by zero virtual input in partial feedback linearized coordinates. Physically this quite interestingly corresponds to keeping constant velocity of the center of mass of the Acrobot. Improvement of the tracking feedback was obtained in [12], [13]. In [12], the appropriate exponentially stable tracking was obtained by solving quadratic stability of a linear system with polytopic uncertainty applying LMI methods. The appropriate exponential tracking presented in [13] was based on the knowledge of the time dependent uncertainty. These treatments were gradually later on extended to the case of many steps walking, including effect of impact map and also including two kinds of observers [14], [15], [16], [17]. For multi-step walking, efficient design of the cyclic trajectory was provided in [14]. As a matter of fact, previously mentioned pseudo-

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passive trajectory was not cyclic and some nonzero control action in linearized coordinates had to be computed to assure that initial conditions at the beginning of the next step after impact are the same as at the beginning of the previous step. All these results presented quite good performance, i.e. in simulations they presented the exponentially stable walking with many steps using limited measurements only. Stability analysis using numerical analysis of Poincare maps along the corresponding cyclic trajectory was provided in [17], [16].

The proof of stability of feedback tracking during the continuous-part, which is done thanks to the sophicticated feedback design, is based on the partial exact feedback linearization of the Acrobot model followed by further approximate feedback linearization of the tracking error dynamics for arbitrary target trajectory. Partial feedback linearization of order 3 has been already used in many ways to track the desired target trajectory. Nevertheless, in previous results tracking error dynamics were treated by robust-like approach and/or by LMI techniques giving either rather quite conservative results, or results without rigorous mathematical tracking convergence proof. The approach presented here enables to prove the convergence in a rigorous way, at least for small initial tracking errors. Approximate feedback linearization technique stems from the ideas presented by Krener in 90's and it basically means to compensate exactly through state transformations and feedback all nonlinearities, where it is possible, while neglecting the others, see [18]. Here, similar ideas are used to further transform the extended system consisting from error dynamics together with target dynamics. The slight novelty here is that neglecting is made with respect to tracking error along any general trajectory to be tracked, not just in some neighborhood of fixed working point.

In addition, to demonstrate this approach, numerical simulations of the first five Acrobot steps based on the new analytical design and a phase-plane of 150 Acrobot steps will be provided.

The rest of the paper is organized as follows. The next section briefly presents the model of the Acrobot together with the main theoretical pre-requisites necessary for the further tracking analysis. The section 3 describes the exact feedback linearization method for the model of the Acrobot. The sections 4 describes the main result of this paper. Simulations of the Acrobot walking are presented in Section 5. Final section draws briefly some conclusions and discusses some open future research outlooks toward an efficient underactuated walking.

II. THE MODEL OF THE ACROBOT

The Acrobot model is a special case of n-link chain with $n-1$ actuators between links supported on the walking surface without slipping as shown in Fig. 1. Actually, the Acrobot has two degree of freedom and one actuator placed between its rigid links. In such a way, the Acrobot in Fig. 1 may serve as the simplest hypothetical walking-like system. For this purpose, the planar model is usually considered and the problem of leg possible hitting the surface is neglected.

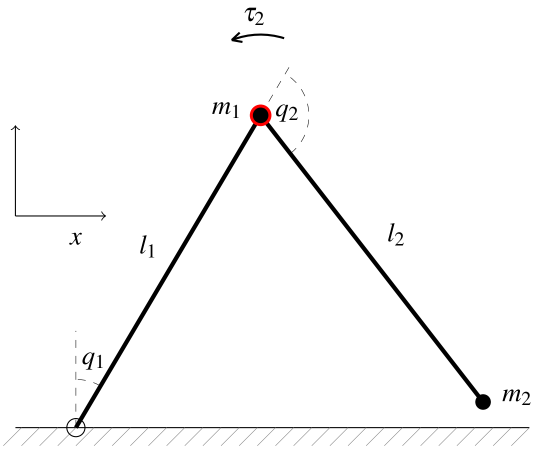

Fig. 1. The Acrobot.

The Acrobot is a typical example of the so-called Lagrangian hybrid systems. As a matter of fact, Acrobot consists of both continuous-time and discrete-time dynamics. The configuration of the Acrobot is described by the generalized coordinates q and is bounded by one-sided constraints $h(q) \leq 0$. Constraints represent the limitation that two solid bodies do not penetrate each other.

The continuous part, when one leg of the Acrobot is in the air, $h(q) > 0$, is modelled by usual Lagrangian approach. The Lagrangian approach yields the well-known resulting Euler-Lagrange equation

$$\begin{bmatrix} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_1} - \frac{\partial \mathscr{L}}{\partial q_1} \\ \vdots \\ \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_n} - \frac{\partial \mathscr{L}}{\partial q_n} \end{bmatrix} = u = \begin{bmatrix} 0 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix}, \tag{1}$$

where u stands for vector of external controlled forces. The system (1) is the so-called **underactuated** mechanical system having the degree of the underactuation equal to one, [1]. Moreover, the underactuated angle is at the pivot point. The zero in the first row of the vector of external controlled forces expresses this fact. The form of this vector is very important for the exact feedback linearization method which will be discussed later. Equation (1) leads to the dynamic equation of motion in the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u \tag{2}$$

where $D(q)$ is the inertia matrix which depends on variable q_2 only, $C(q,\dot{q})$ contains Coriolis and centrifugal terms, $G(q)$ contains gravity terms and u stands for vector of external forces, see [2]. The Acrobot possesses the so-called kinetic symmetry property, i.e. the inertia matrix depends only on the second variable q_2.

The discrete part occures when one leg of the Acrobot touches the ground, $h(q) = 0$, in this case the so-called the impact occurs. The impact is modelled as a contact between two rigid bodies. It is necessary to extend the model of the Acrobot by a Cartesian coordinates of the end of the stance

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leg. The result of this event is instantaneous jump in the angular velocities $\dot{q}$ while angles q remains unchanged,

$$\begin{aligned} D_e(q_e)[\dot{q_e}^+ - \dot{q_e}^-] &= J^\alpha(q_e)^T I_{\lambda_a}, && (3)\\ J^\alpha(q_e)\dot{q_e}^+ &= 0, && (4)\end{aligned}$$

where $D_e(q_e)$ is extend inertia matrix, $\dot{q_e}^+$ (respectively, $\dot{q_e}^-$) is extend angular velocity of the Acrobot just after (respectively, just before) impact. The right hand side of equation (3) corresponds to the contact impulse over the impact duration, see [5], [19] for more details. Equation (3) is accompanied by eguation (4) which corresponds to conditions of no rebound and no slipping of the swing leg.

In the application of the tracking of the reference trajectory it is typical for hybrid systems that the discrete-time part could improve or worsen the tracking error. For the probe of the stability of hybrid systems it is necessary to think over the tracking error after the discrete-time part.

To demonstrate viability of the approach suggested in this paper, the simulations of tracking of walking-like cyclic trajectory are presented. The walking includes several steps and each step is followed by the impact. The exponential stable tracking during the swing phase only is capable to stabilize overall walking, including the effect of impact, thanks to the walking-like trajectory. The design of this trajectory is based on a simple idea of impact mapping of the end point of the tracked trajectory to the desired beginning point of the tracked trajectory which is the same for each step, see [14].

III. PARTIAL EXACT FEEDBACK LINEARIZATION OF THE ACROBOT MODEL

The well known **partial exact feedback linearization** method is based on a system transformation into a new system of coordinates that displays linear dependence between some auxiliary output and new (virtual) input [20]. From the theoretical point of view, the *n*-degrees of freedom mechanical system dynamics is described by 2*n*-dimensional state space. The static state feedback linearization is generated by the suitable output function having the relative degree r yields a linear subsystem of dimension r.

It was shown in [10] that if the generalized momentum conjugated to the cyclic variable is not conserved (as it is the case of Acrobot) then there exists a set of outputs that defines a one-dimensional exponentially stable zero dynamics. That means that it is possible to find a function $\bar{y}(q,\dot{q})$ with relative degree 3 that transforms the original 4 dimensional system (2) by a local coordinate transformation $z = T(q,\dot{q})$ into the new input/output linear system which has 3 dimensional state plus the unobservable nonlinear dynamics of dimension 1.

In the case of the Acrobot there are two independent functions with relative degree 3 which transform the original system into the desired partial linearized form with one dimensional zero dynamics,[1] namely

$$\sigma = \frac{\partial \mathscr{L}}{\partial \dot{q}_1} = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)\dot{q}_1 + (\theta_2 + \theta_3 \cos q_2)\dot{q}_2, \quad (5)$$

$$\begin{aligned} p \;=\; & q_1 + \frac{q_2}{2} + \frac{2\theta_2 - \theta_1 - \theta_2}{\sqrt{(\theta_1+\theta_2)^2 - 4\theta_3^2}} \arctan \\ & \left(\sqrt{\frac{\theta_1+\theta_2-2\theta_3}{\theta_1+\theta_2+2\theta_3}} \tan \frac{q_2}{2} \right). \quad (6)\end{aligned}$$

Actually, functions p and σ were first mentioned in [6], while [21], [8], [7] presented more complete treatment of the canonical forms for 2 DOF underactuated systems.

Based on these results, the following transformation

$$\xi = \mathscr{T}(q,\dot{q}): \quad \xi_1 = p, \quad \xi_2 = \sigma, \quad \xi_3 = \dot{\sigma}, \quad \xi_4 = \ddot{\sigma} \quad (7)$$

$$\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \mathscr{T}^{-1}(\xi): \quad q_1 = \phi_1(\xi_1,\xi_3), \; q_2 = \phi_2(\xi_1,\xi_3), \quad (8)$$

$$\dot{q}_1 = \phi_{1d}(\xi_1,\xi_2,\xi_3,\xi_4), \quad \dot{q}_2 = \phi_{2d}(\xi_1,\xi_2,\xi_3,\xi_4) \quad (9)$$

can be defined. Here, $\phi_{1,2}, \phi_{1d,2d}$ are some functions that can be computed. Notice, that by (5,6) and some straightforward but laborious computations the following relation holds

$$\dot{p} = d_{11}(q_2)^{-1}\sigma, \quad (10)$$

where $d_{11}(q_2) = (\theta_1 + \theta_2 + 2\theta_3 \cos q_2)$ is the corresponding element of the inertia matrix D in (2). Applying (7), (10) to (2) we obtain the Acrobot's dynamics in partial exact linearized form

$$\begin{aligned} \dot{\xi}_1 &= d_{11}(q_2)^{-1}\xi_2, \quad \dot{\xi}_2 = \xi_3, \quad \dot{\xi}_3 = \xi_4, \\ \dot{\xi}_4 &= \alpha(q,\dot{q})\tau_2 + \beta(q,\dot{q}) = w \quad (11)\end{aligned}$$

with the new coordinates ξ and the input w being well defined wherever $\alpha(q,\dot{q})^{-1} \neq 0$. Functions $\alpha(q,\dot{q})$ and $\beta(q,\dot{q})$ can be obtained through straightforward, though laborious computations, see [11].

To use the above partially linear form to track any trajectory generated by some open loop control and initial condition, one can proceed as follows. Assume that some open loop control $w^r(t)$, generates a suitable reference trajectory, in partial exact feedback linearized coordinates (11). In other words, the task is to track the following reference system

$$\dot{\xi}_1^r = d_{11}^{-1}(q_2^r)\xi_2^r, \quad \dot{\xi}_2^r = \xi_3^r, \quad \dot{\xi}_3^r = \xi_4^r, \quad \dot{\xi}_4^r = w^r. \quad (12)$$

Denoting $e := \xi - \xi^r$ and subtracting (12) from (11) one obtains the tracking error dynamics as follows

$$\begin{aligned} \dot{e}_1 &= d_{11}^{-1}(\phi_2(\xi_1,\xi_3))\xi_2 - d_{11}^{-1}(\phi_2(\xi_1^r,\xi_3^r))\xi_2^r, \\ \dot{e}_2 &= e_3, \quad \dot{e}_3 = e_4, \quad \dot{e}_4 = w - w^r. \quad (13)\end{aligned}$$

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[1] Actually, by (1) $\dot{\sigma} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_1} = \frac{\partial \mathscr{L}}{\partial q_1}$ and therefore by $\mathscr{L} = K - V$, $K = \dot{q}^T D(q_2)\dot{q}$. $\frac{\partial \mathscr{L}}{\partial q_1} = \frac{\partial K}{\partial q_1} - \frac{\partial V}{\partial q_1} = \frac{\partial \dot{q}^T D(q)\dot{q}}{\partial q_1} - \frac{\partial V}{\partial q_1} = -\frac{\partial V}{\partial q_1}$, e.g. $\dot{\sigma} = -\frac{\partial V(q)}{\partial q_1}$. In other words, $\dot{\sigma}$ has relative degree 2, i.e. σ has the relative degree 3. Moreover, by the straightforward differentiation it holds $\dot{p} = d_{11}(q_2)^{-1}\sigma$, i.e. $\dot{p}$ has relative degree 2, i.e. p should have relative degree 3 as well. As a matter of facc, it is easy to see that product of two functions each of them having relative degree k has again relative degree k.

The most obvious way to handle the complicated first row of this error dynamics would be to use the approximate linearization of the error dynamics along the trajectory to be tracked.

Various techniques are used in [11], [12], [13], [17] provide feedback to stabilize the above error dynamics, among them LMI robust design, high-gain controllers or use of time-varying error dynamics transformations. Their drawbacks are either high degree of conservatism or heuristic character as in case of LMI use. The basic difficulty here is the presence of the term depending linearly on e_3 in the first row with time varying coefficient, which does not make possible for analytic design. As a matter of fact, this linear term can be removed by further exact state and feedback transformation of the extended system (12,13). This constitutes the main result of this paper and will be performed in detail in the next section.

IV. MAIN RESULT

The main result of the paper is the following

Theorem 1. Consider the extended system (12,13) as the dynamical system having 8 dimensional state space, controlled input w and reference input w^r. Then there exists the following change of coordinates and feedback transformation (locally regular in e)

$$\eta = \Phi(\xi_1^r, \xi_2^r, \xi_3^r, \xi_4^r, e_1, e_2, e_3, e_4), \tag{14}$$

$$\mu = \gamma(w, w^r, \xi_1^r, \xi_2^r, \xi_3^r, \xi_4^r, e_1, e_2, e_3, e_4), \tag{15}$$

$$\Phi_1 \equiv \xi_1^r,\ \Phi_2 \equiv \xi_2^r,\ \Phi_3 \equiv \xi_3^r,\ \Phi_4 \equiv \xi_4^r \tag{16}$$

$$\Phi_k(\xi_1^r, \xi_2^r, \xi_3^r, \xi_4^r, 0,0,0,0) \equiv 0, \quad \forall k = 5,6,7,8, \tag{17}$$

which transform the extended system (12,13) into the following form with state η, controlled input μ and reference input w^r:

$$\dot{\eta}_1 = d_{11}^{-1}(q_2^r)\eta_2, \quad \dot{\eta}_2 = \eta_3, \quad \dot{\eta}_3 = \eta_4, \quad \dot{\eta}_4 = w^r. \tag{18}$$

$$\dot{\eta}_5 = \eta_6 + o(\eta), \quad \dot{\eta}_6 = \eta_7, \quad \dot{\eta}_7 = \eta_8, \quad \dot{\eta}_8 = \mu. \tag{19}$$

Proof. Consider the general error dynamics (13) and consider its first row in a more detail, namely, one has easily obtain

$$\begin{aligned}\dot{e}_1 \ = \ & \frac{\partial d_{11}^{-1}(\phi_2(\xi_{1,3}^r))}{\partial \xi_1}(\xi_2^r + e_2)e_1 + d_{11}^{-1}(\phi_2(\xi_{1,3}^r))e_2 + \\ & \frac{\partial d_{11}^{-1}(\phi_2(\xi_{1,3}^r))}{\partial \xi_3}(\xi_2^r + e_2)e_3 + d_{11}^{-1}(\phi_2(\xi_{1,3}))\xi_2 - \\ & d_{11}^{-1}(\phi_2(\xi_{1,3}^r))(\xi_2^r + e_2) - \frac{\partial d_{11}^{-1}(\phi_2(\xi_{1,3}^r))}{\partial \xi_1}(\xi_2^r + e_2)e_1 - \\ & \frac{\partial d_{11}^{-1}(\phi_2(\xi_{1,3}^r))}{\partial \xi_3}(\xi_2^r + e_2)e_3\end{aligned}$$

which gives

$$\begin{aligned}\dot{e}_1 \ = \ & \frac{\partial d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r))}{\partial \xi_1}(\xi_2^r + e_2)e_1 + \frac{\partial d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r))}{\partial \xi_3} \\ & (\xi_2^r + e_2)e_3 + d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r))e_2 + (\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|),\end{aligned} \tag{20}$$

where

$$\begin{aligned}(\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|) \ = \ & (\xi_2^r + e_2)\Bigg(d_{11}^{-1}(\phi_2(\xi_1^r + e_1, \xi_3^r + e_2)) - \\ & d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r)) - \frac{\partial d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r))}{\partial \xi_1}e_1 - \frac{\partial d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r))}{\partial \xi_3}e_3\Bigg).\end{aligned} \tag{21}$$

Therefore it holds

$$\begin{aligned}\dot{e}_1 \ = \ & \psi_1(q_1^r, q_2^r)(\xi_2^r + e_2)e_1 + \psi_2(q_2^r)e_2 + \\ & \psi_3(q_1^r, q_2^r)(\xi_2^r + e_2)e_3 + (\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|),\end{aligned} \tag{22}$$

where

$$\begin{aligned}\psi_1(q_1^r, q_2^r) \ := \ & \frac{\partial d_{11}^{-1}(q_2(\xi_1^r, \xi_3^r))}{\partial \xi_1^r} = \frac{\partial d_{11}^{-1}}{\partial q_2}(q_2^r)\frac{\partial q_2}{\partial \xi_1^r}, \\ \psi_2(q_2^r) \ := \ & d_{11}^{-1}(\phi_2(\xi_1^r, \xi_3^r)), \\ \psi_3(q_1^r, q_2^r) \ := \ & \frac{\partial d_{11}^{-1}(q_2(\xi_1^r, \xi_3^r))}{\partial \xi_3^r} = \frac{\partial d_{11}^{-1}}{\partial q_2}(q_2^r)\frac{\partial q_2}{\partial \xi_3^r}.\end{aligned}$$

Summarizing

$$\begin{aligned}\dot{e}_1 \ = \ & \psi_1(q_1^r, q_2^r)(\xi_2^r + e_2)e_1 + \psi_2(q_2^r)e_2 + \\ & \psi_3(q_1^r, q_2^r)(\xi_2^r + e_2)e_3 + (\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|), \\ \dot{e}_2 \ = \ & e_3, \quad \dot{e}_3 = e_4, \quad \dot{e}_4 = w - w^r.\end{aligned} \tag{23}$$

Consider the following transformation

$$\eta_5 := e_1 - \psi_3(q_1^r, q_2^r)\left[\frac{(\xi_2^r + e_2)^2 - (\xi_2^r)^2}{2}\right], \tag{24}$$

the specific form of the transformation enables to make full order linearization of (13) because the term connected with e_3 will be deleted from the first line of (23) and it will appaer in the next line. From (24) we get

$$\begin{aligned}\dot{\eta}_5 \ = \ & \dot{e}_1 - \psi_3(q_1^r, q_2^r)\left[\dot{e}_2(\xi_2^r + e_2) + e_2\dot{\xi}_2^r\right] - \\ & \psi_3^{(\mathrm{I})}(q_1^r, q_2^r)\left[\frac{(\xi_2^r + e_2)^2 - (\xi_2^r)^2}{2}\right].\end{aligned} \tag{25}$$

Substituting from (13) we obtain

$$\begin{aligned}\dot{\eta}_5 \ = \ & \psi_1(q_1^r, q_2^r)(\xi_2^r + e_2)e_1 + \psi_2(q_2^r)e_2 - \\ & \psi_3(q_1^r, q_2^r)e_2\dot{\xi}_2^r - \psi_3^{(\mathrm{I})}(q_1^r, q_2^r)\left[\frac{(\xi_2^r + e_2)^2 - (\xi_2^r)^2}{2}\right] + \\ & (\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|).\end{aligned} \tag{26}$$

Now, we have the almost linearized first equation by setting the new coordinate as follows

$$\begin{aligned}\eta_6 \ = \ & \psi_1(q_1^r, q_2^r)(\xi_2^r + e_2)e_1 + \psi_2(q_2^r)e_2 - \psi_3(q_1^r, q_2^r)e_2\dot{\xi}_2^r - \\ & \psi_3^{(\mathrm{I})}(q_1^r, q_2^r)\left[\frac{(\xi_2^r + e_2)^2 - (\xi_2^r)^2}{2}\right],\end{aligned} \tag{27}$$

which gives the transformed first equation

$$\dot{\eta}_5 \ = \ \eta_6 + (\xi_2^r + e_2)o(\|(e_1, e_3)^{\mathrm{T}}\|). \tag{28}$$

Further, one can easily see that η_6 has the relative degree 3 wrt. the input w. Therefore, one can finish the proof of theorem by performing the usual algorithm of computing further time derivatives of η_6 along system trajectories. This

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has increasing complexity and is left for sake of shortness. Just to give a taste of these lengthy computations, which were, nevertheless, performed in detail and successfully used to stabilize error dynamics later on, consider

$$\begin{aligned}\dot{\eta}_6 = \; & \psi_1^{(\mathrm{I})}(q_1^r,q_2^r)(\xi_2^r{+}e_2)e_1 + \\ & \psi_1(q_1^r,q_2^r)(\dot{\xi}_2^r{+}e_3)e_1 + \psi_1(q_1^r,q_2^r)(\xi_2^r{+}e_2) \\ & \left(d_{11}^{-1}(\phi_2(\xi_1^r{+}e_1,\xi_3^r{+}e_3))(\xi_2^r{+}e_2) - d_{11}^{-1}(\phi_2(\xi_1^r,\xi_3^r))\xi_2^r\right) + \\ & \frac{\partial \psi_2}{\partial q_2^r}(q_2^r)\dot{q}_2^r e_2 + \psi_2(q_2^r)e_3 - \psi_3(q_1^r,q_2^r)e_3\dot{\xi}_2^r - \\ & \psi_3(q_1^r,q_2^r)e_2\ddot{\xi}_2^r - \psi_3^{(\mathrm{II})}(q_1^r,q_2^r)\left[\frac{(\xi_2^r{+}e_2)^2-(\xi_2^r)^2}{2}\right] - \\ & \psi_3^{(\mathrm{I})}(q_1^r,q_2^r)\left[e_3(\xi_2^r{+}e_2)+2e_2\dot{\xi}_2^r\right]. \end{aligned} \tag{29}$$

Denoting the right hand side of (29) as $\eta_7 := \eta_7(\xi_1^r,\ldots,\xi_4^r,e_1,e_2,e_3)$ one has the transformed equation as follows

$$\dot{\eta}_6 = \eta_7(\xi_1^r,\ldots,\xi_4^r,e_1,e_2,e_3). \tag{30}$$

Now, differentiating further η_7 with respective to time along the system trajectories one has that

$$\begin{aligned}\dot{\eta}_7 &= \eta_8(\xi_1^r,\ldots,\xi_4^r,e_1,\ldots,e_4), \\ \dot{\eta}_8 &= \mu(w,w^r,\xi_1^r,\ldots,\xi_4^r,e_1,\ldots,e_4). \end{aligned} \tag{31}$$

Obviously, (31) gives the rest of the transformations mentioned in the theorem formulations, their explicite form is skipped as it quite lenghty. It can be also straightforwardly checked that the overall transformation is locally one-to-one. Therefore, theorem is proved. □

Theorem 1, in fact, presents the so-called aproximate feedback linearization of the error dynamics which goes back to seminal Krener's paper [18]. This technique exactly compensates all nonlinearities where it is possible while neglecting the higher-order nonlinearities that can not be exactly compensated. Aproximate linearization can be clearly used to design efficient exponential tracking of any given reference trajectory.

A cyclic, multi-step, walking-like trajectory for the Acrobot taking into the accout also the impact has been proposed in [14]. In the next section the exponential tracking of this trajectory during swing phases is demonstrated. As a matter of fact, it would be actually stable also including impact effect.

V. APPLICATION TO WALKING TRAJECTORY TRACKING

As already noted, Thoerem 1 provides the approximate feedback linearization of the error dynamics which suggests efficient desing of the locally exponentially stable tracking. As a matter of fact, the error dynamics is transformed in the chain of four integrators perturbed by some higher order nonlinear terms in the first row only. Notice in this respect, that $\eta_{5,6,7,8} = 0$ if and only if $e_{1,2,3,4} = 0$. Therefore, one could easily prove the local in error convergence in a rigorous mathematical way. This approach was realized and tested in numerical simulations.

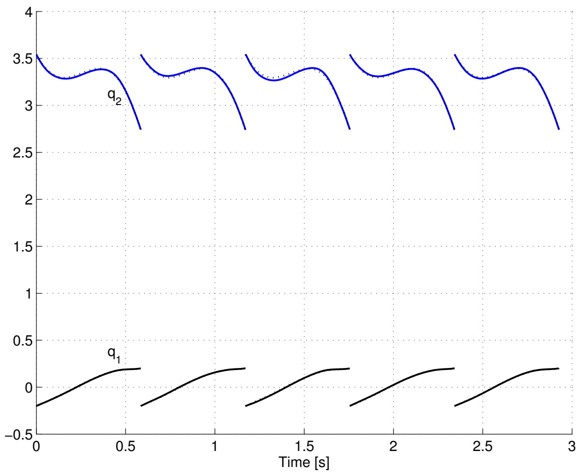


Fig. 2. Angular positions q_1, q_2 and references (dotted line) for 5 steps walking using the new derived tracking feedback.

All these transformations and feedback were recomputed in order to provide tracking feedback in original coordinates $q_{1,2}, \dot{q}_{1,2}$. For details about the tracked walking-like trajectory see [14]. By reason of comparison with our earlier results, the model parameters of the Acrobot, initial conditions and the used impact matrix for simulations are same as in [14]. The computed feedback does not respect any physical limit to computed torque τ_2. Of course, the real physical torque is limited by used engines. Just to test behaviour of feedback tracking numerically in a more realistic circumstances, the corresponding torque τ_2 was saturated with the realistic saturation limit in the range $\pm 15\,\mathrm{Nm}$. From our experiences, we can note that strong gain combined with the saturation appears to be useful for the walking because the time convergence is important to take place within a short time period, as the step has finite time duration. Moreover, this saturation limit practically does not affect the quality of tracking, except the initial stage of the step, see [13], [14].

The corresponding simulations of angular positions and velocities are shown in Figs. 2, 3. Fig. 4 shows phase-plane plots of variables q_1 and q_2. From simulation of approximately 150 steps one can easily see the walking convergences towards a periodic motion. Thin curves correspond to motion of the real Acrobot in the first few steps which are plotted in detail in Fig. 2, 3. Thin curves are caused by addition an initial error. This error is minimized during few steps in the beginning of walking. Thin curves convergence towards a periodic motion, the thick curve. The straight line corresponds to the impact phase, where the angular velocities of the Acrobot changes instantaneously.

Animation of the corresponding Acrobot walking is shown in Fig. 5. In the upper part is the animation of the first step and the animation of the second step is at the bottom part.

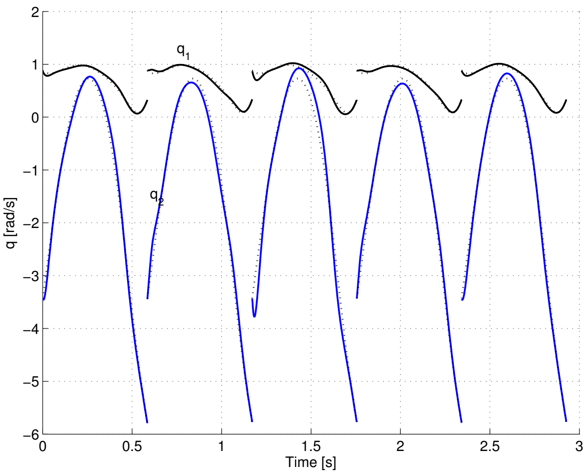
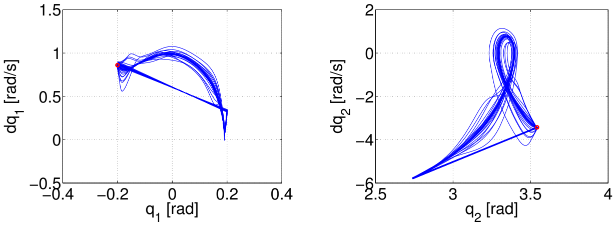


Fig. 3. Angular velocities $\dot{q}_1$, $\dot{q}_2$ and references (dotted line) for 5 steps walking using the new derived tracking feedback.



(a) (b)

Fig. 4. Phase-plane plots for (a) q_1, (b) q_2. The initial state is represented by a red circle.

VI. CONCLUSIONS

This paper presented, after transforming Acrobot into the partially linear form, another error dynamics transformation giving its approximate feedback linearization. This enables to track any bounded target trajectory of Acrobot in a locally exponentially stable way and prove its convergence in a rigorous mathematical way. This approach was also successfully demonstrated in simulations of exponetial tracking of walking-like trajectory with possibly unlimited number of steps, including impact effect.

The numerical simulations and animations of 5 steps show nicely truly hybrid convergence of the tracking error. As it was indicated, 5 steps are presented for the sake of limited space in this paper, otherwise unlimited number is possible. In simulations, up to 150 steps were reached.

REFERENCES

[1] M. Spong, *Underactuated mechanical systems, control problems in robotics and automation*. London: Springer Verlag, 1998.

[2] I. Fantoni and R. Lozano, *Non-linear control of underactuated mechanical systems*. Heidelberg: Springer Verlag, 2002.

[3] T. McGeer, “Passive dynamic walking,” *International Journal of Robotics Research*, vol. 2, no. 2, pp. 62–82, 1990.

[4] E. Westervelt, J. Grizzle, C. Chevallereau, J. Choi, and B. Morris, *Feedback control of dynamic bipedal robot locomotion*. CRC Press, 2007.

[5] C. Chevallereau, G. Bessonnet, G. Abba, and Y. Aoustin, *Bipedal Robots: Modeling, Design and Walking Synthesis*. Wiley-ISTE, 2009.

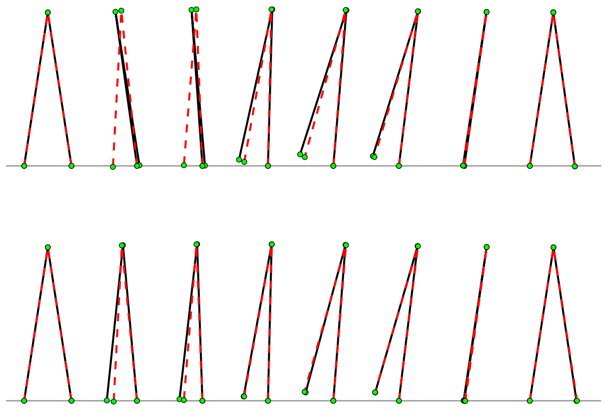


Fig. 5. The animation of the two single steps shown in time moments with gaps $\Delta t = 0.08\,\mathrm{s}$ between them. Dashed line is reference, the full one represents the actual Acrobot.

[6] L. Cambrini, C. Chevallereau, C. Moog, and R. Stojic, “Stable trajectory tracking for biped robots,” in *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, 2000.

[7] R. Olfati-Saber, “Normal forms for underactuated mechanical systems with symmetry,” *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 305–308, feb 2002.

[8] ——, “Control of underactuated mechanical systems with two degrees of freedom and symmetry,” in *Preprints of the American Control Conference, 2000*, vol. 6, 2000, pp. 4092 –4096 vol.6.

[9] J. Zikmund and C. Moog, “The structure of 2-body mechanical systems,” in *Proceedings of the IEEE Conference on Decision and Control*, San Diego, USA, 2006, pp. 6464–6469.

[10] J. Grizzle, C. Moog, and C. Chevallereau, “Nonlinear control of mechanical systems with an unactuated cyclic variable,” *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 559–576, may 2005.

[11] S. Čelikovský, J. Zikmund, and C. Moog, “Partial exact linearization design for the acrobot walking,” in *Preprints of the American Control Conference, 2008*, Seattle, USA, 2008, pp. 874–879.

[12] M. Anderle, S. Celikovsky, D. Henrion, and J. Zikmund, “Advanced lmi based analysis and design for acrobot walking,” *International Journal of Control*, vol. 83, no. 8, pp. 1641–1652, 2010.

[13] M. Anderle and S. Čelikovský, “Analytical design of the acrobot exponential tracking with application to its walking,” in *Proccedings of the IEEE International Conference on Control and Automation, ICCA 2009*, 2009, pp. 163–168.

[14] ——, “Sustainable acrobot walking based on the swing phase exponentially stable tracking,” in *Proceedings of the ASME 2010 Dynamic Systems and Control Conference*. Cambridge, Massachusetts, USA: ASME, 2010, pp. 1/8–8/8.

[15] ——, “Position feedback tracking of the acrobot walking-like trajectory based on the reduced velocity observer,” in *Preprints of the 8th IFAC Symposium on Nonlinear Control Systems*. Bologna, Italy: IFAC, 2010, pp. 1011–1016.

[16] ——, “Stability analysis of the acrobot walking with observed geometry,” in *Preprints of the 18th World Congress IFAC*. Milan, Italy: IFAC, 2011.

[17] ——, “Feedback design for the acrobot walking-like trajectory tracking and computational test of its exponential stability,” in *Proceedings of the IEEE Multi-Conference on Systems and Control*, Denver, Colorado, 2011.

[18] A. Krener, “Approximate linearization by state feedback and coordinate change,” *Systems and Control Letters*, vol. 5, no. 3, pp. 181–185, 1984.

[19] B. Brogliato, *Nonsmooth Mechanics: Models, Dynamics and Control*. London: Springer Verlag, 1996.

[20] A. Isidori, *Nonlinear Control Systems*. New York: Springer Verlag, 1996.

[21] R. Olfati-Saber and A. Megretski, “Controller design for a class of underactuated nonlinear systems,” in *Proceedings of the 37th IEEE Conference on Decision and Control*, vol. 4, 1998, pp. 4182 –4187 vol.4.