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Adding and Removing Nodes in Consensus

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Abstract— **The distributed consensus problem has been widely studied in the literature, either with fixed and with time-varying topologies. Typically, the set of agents involved in the consensus does not vary over time.**

In this paper the possibility to dynamically add or remove nodes during consensus is investigated. Specifically, a framework for the achievement of consensus while dynamically adding nodes to the network is provided, together with a stability condition. Moreover, the effects of removing a single node in the network at a given time instant are inspected, characterizing the difference between the asymptotic values with and without the removed node, depending on the removal time instant. A further result provided in this paper is the relation between the node removal at a given time instant and the initial removal of that node (i.e., at the initial time step).

I. Introduction

In the literature, the distributed *consensus* among dynamic agents both in the continuous [3], [4], [5] and discrete-time cases [5], [7] has been widely investigated.

Consensus is intended as the agreement regarding a certain quantity of interest depending on the state of all the agents in the network[6]. More precisely, it can be viewed as a distributed way to calculate a particular function of the initial conditions of the agents (i.e., average, max, min, etc.) by applying inputs that depend only on the state of the agents' neighbors, according to the actual network topology.

The behavior of each subsystem is typically expressed as a single integrator, which represents an agent that maintains its value if isolated [3], [10]. However, the consensus of double integrator systems and high order systems is gaining momentum rapidly [4], [5], [14].

Many extensions have been provided, considering either time varying topologies and time delays (see for instance [3], [9]), or even the consensus over complex networks [11], [12], [13]. Typically, the set of agents involved in consensus has a constant size over time. However, in many realistic cases, the network topology during the consensus is not fixed and changes by adding or removing edges and nodes (i.e., failures or a situation where more agents are dynamically added to refine the global estimation), assuming that the pre-existing edges have a dynamic behavior during the network building.

Indeed, time varying consensus [8], [9] allows the edges to change, and there is the need to guarantee some sort of mutual reachability of the agents in order to achieve an agreement; the case of growing networks, presented in this paper, is different, since new nodes are added dynamically to the existing network. Even more interesting is the case of node removal, and in particular the quantification of the contribution of a given node to the asymptotic value of the network, before it is faulted or gets disconnected.

In this paper we will provide a first step for the quantitative analysis of the effects of adding and removing nodes to the network during the consensus process.

Specifically, a new framework for the achievement of consensus while dynamically adding nodes to the network is provided, together with a stability condition; moreover, the effects of removing a single node in the network at a given time instant are inspected, characterizing the error with respect to the original consensus and the asymptotic values assumed by the agents, depending on the removal time instant. A further result provided in this paper is the relation between the node removal at a given time instant and the complete absence of that node.

The paper is organized as follows: after some preliminary definitions, Section II reviews the distributed consensus problem; a framework for the consensus on growing networks is introduced in Section III, while in Section IV the effects of removing one node are inspected; some simulation results are provided is Section V; finally, some conclusive remarks are collected in Section VI.

II. Consensus Problem

Let us now briefly review the distributed average consensus problem for discrete time single integrators. Let $\Gamma = \{\mathcal{V}, \mathcal{E}, A\}$ be a *directed graph* with n nodes, where the set $\mathcal{V}$ denotes the nodes v_i, $\mathcal{E}$ is the set of edges (v_i, v_j). Matrix $A = \{a_{ij}\}$ is the *adjacency matrix* describing the network topology; elements a_{ij} of A are such that $a_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$, otherwise $a_{ij} = 0$. The graph is said to be *undirected* if, for each edge, $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$. Let $d_i^{out} = \sum_{j=1}^{n} a_{ij}$ be the *out-degree* of node i (i.e., the number of outgoing edges), and let $d_i^{in} = \sum_{j=1}^{n} a_{ji}$ be the *in-degree* of node i (i.e., the number of incoming edges). A directed graph is said to be *balanced* if for each node $v_i \in \mathcal{E}$, $d_i^{out} = d_i^{in}$ (hence, an undirected graph is always balanced). An undirected graph is said to be *connected* if it contains at least a *spanning tree*; a directed graph is said to be *simply connected* if it contains at least a directed spanning tree rooted in a given node v_i, while it is *strongly connected* if for each couple of nodes v_i, v_j there exists a path that connects the nodes respecting the orientation of edges.

The set of *neighbors* of a node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. Let $x_i \in \mathbb{R}$ be the state of the i-th

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agent, and $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$. Nodes i and j are said to *agree* if $x_i = x_j$, and consequently the graph Γ agrees if each couple of nodes $v_i, v_j \in \mathcal{V}$ agree. Whenever all the nodes of a network are in agreement, the common value of the nodes is called the *group decision value*.

Let each node in the network be a discrete-time *dynamic agent*, whose dynamics is in the form:

$$x_i(k+1) = q(x_i(k), e_i(k)), \quad \forall v_i \in \mathcal{V} \tag{1}$$

where $e_i(k) \in \mathbb{R}$ represents the input of i-th system.

The stacked dynamics for all the agents is given by:

$$x(k+1) = Q(x(k), e(k)) \tag{2}$$

where $e = \begin{bmatrix} e_1 & \cdots & e_p \end{bmatrix}^T$ and $Q(x(k), e(k))$ is the column-wise concatenation of the elements $Q_i(x(k), e(k)) = q(x_i(k), e_i(k))$.

Let $\chi : \mathbb{R}^n \to \mathbb{R}$; the *$\chi$-consensus problem* in a dynamic graph can be interpreted as a distributed way to calculate $\chi(x(0))$ by using only information depending on the values of the neighbors $\mathcal{N}_i$ as input.

A *protocol* $e_i(k)$ is a feedback law based on the state of the neighbors $j_1, \ldots, j_{\mu_i} \in \mathcal{N}_i$ of the i-th and on the state of the i-th agent itself:

$$e_i(k) = g_i(x_{j_1}(k), \ldots, x_{j\mu_i}(k), x_i(k)) \tag{3}$$

A protocol asymptotically solves the χ-consensus problem if there exists an asymptotically stable equilibrium x^* of (2) such that $x_i^* = \chi(x(0))$ for all $i \in [1, n]$.

In the literature different typologies of consensus have been addressed; in this section we will review the consensus problem for networks of single integrators.

A. Average Consensus for Single Integrators

Consider a network composed of n dynamic agents, each one described by an integrator [3] (for the sake of simplicity we assume, without loss of generality, scalar integrators):

$$\dot{x}_i(t) = e_i(t), \quad \forall v_i \in \mathcal{V} \tag{4}$$

In the discrete-time fashion the system becomes:

$$x_i(k+1) = x_i(k) + \tau e_i(k), \quad \forall v_i \in \mathcal{V} \tag{5}$$

where $\tau > 0$ represents the sampling time. In [3] the following protocol is used to solve the continuous-time average consensus problem:

$$e_i(t) = \textstyle\sum_{j \in \mathcal{N}_i} a_{ij}[x_j(t) - x_i(t)] \tag{6}$$

where a_{ij} are the coefficients of the adjacency matrix of the considered graph. The resulting dynamic system for the n agents is given by

$$\dot{x}(t) = -Lx(t) \tag{7}$$

where L is the $n \times n$ *Laplacian* matrix induced by Γ, whose elements $\{l_{ij}\}$ are in the form:

$$l_{ij} = \begin{cases} \sum_{k=1, k\neq i}^{n} a_{ik} = d_i^{out}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \tag{8}$$

In [3] it is proved that, if the graph is simply connected an agreement is asymptotically reached, while if the graph is strongly connected and balanced the average of the initial condition is achieved.

When the protocol is applied in the discrete fashion considering a sampling time τ, the resulting dynamic is in the form:

$$x(k+1) = P_\tau x(k) \tag{9}$$

where $P_\tau = I_n - \tau L$ is called the *Perron matrix* [3].

It is a well-known result [3] in the literature that protocol (6) solves the consensus problem for a strongly connected topology if $\tau < 1/l^*$, where $l^* = max_i\{l_{ii}\}$.

Note that the condition on the sampling rate $\tau < 1/l^*$ is a sufficient and conservative estimation of the maximum sampling rate that guarantees the convergence of the consensus problem. Note further that the condition is indeed a global condition, since τ depends on the maximum degree of the agents, that is the sum of the weights of the outgoing edges for that agent.

In the following Section a framework for the the first-order discrete-time consensus problem in the case of growing networks will be introduced.

III. Consensus in Growing Networks

Consider a growing network whose nodes are agents and their initial state is $x_i(0) \in \mathbb{R}$.

Starting from a network with $\delta(0)$ nodes, at each step k exactly $\delta(k)$ new nodes are connected to the pre-existing nodes until the maximum number of nodes n is reached. At each time step, during the network growth, the agents perform a consensus step according to the (time varying) Laplacian matrix induced at each step by the present network topology.

Specifically, the above scenario can be represented by the following system:

$$x(k+1) = F(k)x(k), \quad x(0) = x_0 \tag{10}$$

where $x_0 \in \mathbb{R}^n$ is the vector of the initial state of each agent (including those not yet connected to the network at k-th step) and $F(k)$ is a $n \times n$ time-varying matrix given by

$$F(k) = \begin{cases} \begin{bmatrix} I_{\delta^*(k)} - \tau L(\delta^*(k)) & 0 \\ 0 & I_{n-\delta^*(k)} \end{bmatrix} & \text{for } \delta^*(k) < n \\ \\ I_n - \tau L(k) & \text{else} \end{cases} \tag{11}$$

where $L(h)$ is the $h \times h$ laplacian matrix induced by the network topology considering the first h nodes, and $L(k)$ is the topology with all the n agents, while $\delta^*(k) = \sum_{h=0}^{n} \delta(h)$.

Equation (10), therefore, represents a system where, at each step k, the first $\delta^*(k)$ nodes perform a step of consensus according to the current topology, while the state of the other nodes (i.e., not yet added to the network) remains fixed; when the growth phase ends the agents keep performing the consensus according to the topology described by $L(n)$. In order to prove the stability of System (10) the following lemma is required.

Lemma 3.1: Choose $\tau < \frac{1}{l_n^*}$, where l_n^* is the maximum diagonal entry of $L(n)$; if the network topology is undirected and symmetric at each time step k, then

$$||F(k)|| \quad \leq 1 \quad \forall k \in \mathbb{N}_{\geq 0} \tag{12}$$

Proof: Recall that the norm of a square matrix M is equal to the square root of the maximum eigenvalue of $M^T M$. Choose $\tau < \frac{1}{l_n^*}$; then the eigenvalues of $I_{\delta^*(k)} - \tau L(\delta^*(k))$ for $\delta^*(k) \leq n$ are non-negative and the maximum eigenvalue is $\lambda_{max} \leq 1$.

Notice that, since the network topology is undirected, the laplacian matrix $L(\delta^*(k))$ is symmetric; therefore $F(k)$ is hermitian for each k and

$$\begin{aligned}(I_{\delta^*(k)} - \tau L(\delta^*(k)))^T (I_{\delta^*(k)} - \tau L(\delta^*(k))) = \\ = (I_{\delta^*(k)} - \tau L(\delta^*(k)))^2\end{aligned} \tag{13}$$

Let $\sigma(A)$ be the spectral radius of a square matrix A ; it is a standard result that, given two hermitian $n \times n$ matrices R and Q such that $\sigma\{R\} \in [0, r]$ and $\sigma\{Q\} \in [0, q]$, then $\sigma\{RQ\} \in [0, rq]$. Therefore the maximum eigenvalue of

$$(I_{kp} - \tau L(k))^2$$

is $\lambda_{max} \leq 1$ and its square is equal to 1, too; since $F(k)$ is block diagonal with blocks

$$I_{\delta^*(k)} - \tau L(\delta^*(k))$$

and

$$I_{n-\delta^*(k)}$$

it follows that $||F(k)|| \leq 1$. ■

Notice that the condition on τ depends on the total network size and hence is a global condition, while the agents do have only local information. This may lead to very high sampling rates as the number of agents increase; in fact, knowing the final size of the network, $\tau = 1/|\mathcal{V}|$ is the theoretical bound. Indeed this is a general problem of discrete-time consensus, but it becomes even more interesting in the case of growing networks.

A possible solution to this issue is to expand the problem and let the agents reach an agreement also on the most connected node, i.e., performing a leader determination based on the number of links. However such an issue is outside the scope of this paper. Note further that precise bounds for τ can be defined based on the particular strategy chosen for link formation.

We are now in position to prove the stability of System (10), which is a special case of the stability results in [1], [2].

Theorem 3.2: Let the hypotheses of Lemma 3.1 hold and suppose further that the subgraph containing the first k nodes is strongly connected at each time step $k \geq 0$; then System (10) is stable.

Proof: Since, from Lemma 3.1, $||F(k)|| \leq 1$ or, in other terms

$$||F(k)|| = max\Big\{\frac{||F(k)y||}{||y||} : y \in \mathbb{C}, \quad x \neq 0\Big\} \leq 1 \tag{14}$$

it follows that

$$\frac{||F(k)x(k)||}{||x(k)||} \leq 1 \Rightarrow ||x(k+1)|| \leq ||x(k)|| \quad \forall k \in \mathbb{N}_{\geq 0} \tag{15}$$

Therefore the stability of System (10) is proved, since the matrix $F(k)$ is contractive.

■

The following result proves that the asymptotic consensus value for a growing network coincides with that of a static network with all the agents from the initial step.

Theorem 3.3: Let the hypotheses of Theorem 3.2 hold. Then the asymptotic value for consensus in the growing network (10) coincides with that of the standard consensus (9) with the same initial conditions x_0.

Proof: The asymptotic value x_∞ reached by the agents in the growing network (10) is given by:

$$x_\infty = \lim_{k\to\infty} \prod_{h=0}^{k} F(h) x_0 \tag{16}$$

Let us define k^* the time step in which $\delta^*(k^*) \geq n$; since $F(k) = F(k^*)$ is equal to $(I_n - \tau L(k))$ for $k \geq k^*$, it follows that:

$$x_\infty = \lim_{k\to\infty} \prod_{h=k^*}^{k} (I - \tau L(k)) \prod_{h=0}^{k^*} F(h) x_0 \tag{17}$$

Note that, under the hypotheses, $\lim_{k\to\infty} \prod_{h=k^*}^{k} (I - \tau L(k))$ is equal to $\frac{1}{n} 1_n 1_n^T$, and for all $h = 0, \ldots, k^*$, $\frac{1}{n} 1_n 1_n^T F(h) = \frac{1}{n} 1_n 1_n^T$, since the columns of $F(h)$ sum to one and the rows of $\frac{1}{n} 1_n 1_n^T$ contain identical elements. Hence $x_\infty = \frac{1}{n} 1_n 1_n^T x_0$ and each component is the average of the initial conditions. ■

IV. Removing Nodes

Let us now discuss how the removal of nodes affects the consensus and, in particular, what is the effect of the removal of a single node, in terms of the asymptotic consensus value. This represents a complementary issue with respect to previous section.

A. Node Removal: Complete Absence of a node

Let us consider a consensus network of n agents and let us assume that the laplacian matrix L is strongly connected. We will also consider the laplacian matrix $\hat{L}_i$ where all the edges that insist on the i-th node have been removed. We will assume that the topology described by $\hat{L}_i$ is still strongly connected for the remaining agents. In the first case the evolution of the system is given by

$$x(k) = (I - \tau L)^k x(0)$$

while in the second case it is

$$\hat{x}(k) = (I - \tau \hat{L}_i)^k x(0)$$

Let us define the error $e(k) = x(k) - \hat{x}(k)$; it follows that:

$$\begin{aligned} e(k) &= [(I-\tau L)^k - (I-\tau \hat{L}_i)^k]x(0) = \\ &= \textstyle\sum_{h=1}^k (-1)^{h+1}\tau^h (L^h - \hat{L}_i^h)x(0) = \Phi_i(k)x(0) \end{aligned} \tag{18}$$

where we define matrix $\Phi_i(k)$ as the *state-error transition matrix*, corresponding to the removal of i-th node.

Note that, given a matrix A, the product IA is commutative and the following property holds:

$$(I - A)^k = I + \sum_{h=1}^{k} (-1)^h \theta_{h+1} A^h \tag{19}$$

where θ_{h+1} is the $h+1$-th entry of the corresponding row of the Binomial Coefficients.

Hence, it follows that:

$$\Phi_i(k) = \sum_{h=1}^{k} (-1)^h \theta_{h+1} \tau^h (L^h - \hat{L}_i^h) \tag{20}$$

We are in particular interested in

$$\bar{\Phi}_i = \lim_{k\to\infty} \Phi_i(k) \tag{21}$$

that we define as the *asymptotic* state-error transition matrix corresponding to the removal of i-th node.

In the general case, there is the need to calculate the above matrix; however under the assumption of strongly connected and balanced laplacians (or undirected connected laplacians), where the average is reached, the matrix has a very simple form.

In fact, in this case, the asymptotic value for the h-th agent for $x_h(k)$ and $\hat{x}_h(k)$ are:

$$\lim_{k\to\infty} x_h(k) = \frac{1}{n}\sum_{j=1}^{n} x_{j0} \tag{22}$$

and

$$\lim_{k\to\infty} \hat{x}_h(k) = \begin{cases} \frac{1}{n-1}\sum_{j=1;j\neq i}^{n} x_{j0} & \text{if } h \neq i \\ x_{i0} & \text{if } h = i \end{cases} \tag{23}$$

Hence it follows that:

$$\lim_{k\to\infty} e_h(k) = \begin{cases} \frac{-1}{n(n-1)}\sum_{j=1;j\neq i}^{n} x_{j0} + \frac{1}{n}x_{i0} & \text{if } h \neq i \\ \\ \frac{1}{n}\sum_{j=1;j\neq i}^{n} x_{j0} - \frac{n-1}{n}x_{i0} & \text{if } h = i \end{cases} \tag{24}$$

And $\bar{\Phi}_i$ is given by:

$$\bar{\Phi}_i = \begin{bmatrix} \frac{-1}{n(n-1)}1_{i-1}1_{i-1}^T & \frac{1}{n}1_{i-1} & \frac{-1}{n(n-1)}1_{n-i}1_{i-1}^T \\ \frac{1}{n}1_{i-1}^T & \frac{1-n}{n} & \frac{1}{n}1_{n-i}^T \\ \frac{-1}{n(n-1)}1_{i-1}1_{n-1}^T & \frac{1}{n}1_{n-1} & \frac{-1}{n(n-1)}1_{i-1}1_{i-1}^T \end{bmatrix} \tag{25}$$

that is matrix $\frac{-1}{n(n-1)}1_n 1_n^T$, after suitably substituting the i-th row and column according to (24).

In the following, the above considerations will be used as a starting point for the analysis of the removal of a node at a given time instant.

B. Delayed Node Removal

Let us now discuss the removal of a node at a given time step, and let us first assume that the network is time varying (either switching links or adding nodes). Specifically, let us suppose to remove the i-th node at a given time instant g, and let us consider the general case of a time varying consensus, either a consensus with switching links or a growing network. We will, moreover, assume that the conditions required for average consensus both on the initial network and on the network without the removed node are verified.

Let $L(k)$ be the time-varying laplacian matrix and $\hat{L}_{i,g}(k)$ be the laplacian matrix of the consensus considering the removal of node i at step g. Note that in this case the error is identically null until the node is removed at g-th step (i.e., $\hat{L}_{i,g}(k) = L(k)$ for $k = 0, \ldots, g-1$).

In this case we have that $e(k) = \Phi_i^g(k)x(0)$, where $\Phi_i^g(k)$ is the state-error transition matrix corresponding to the removal of i-th node at g-th step, and is given by:

$$\Phi_i^g(k) = [\prod_{h=g}^{k}(I-\tau L(k)) - \prod_{h=g}^{k}(I-\tau \hat{L}_{i,g}(k))]\prod_{h=0}^{g-1}(I-\tau L(k)) \tag{26}$$

Note that in the general case the above matrix (and in particular its limit) might be hard to evaluate; however in the case of a consensus with static topology where a node is removed at time step g, matrix $\hat{L}_{i,g}(k)$ is static after step g (and is equal to L before), hence we have that

$$\Phi_i^g(k) = [(I-\tau L)^{k-g} - (I-\tau \hat{L}_{i,g})^{k-g}](I-\tau L)^g \tag{27}$$

or, in other terms

$$\Phi_i^g(k) = \sum_{h=1}^{k-g} (-1)^h \theta_{h+1}\tau^h (L^h - \hat{L}_{i,g}^h)(I-\tau L)^g \tag{28}$$

The above equation can be restated as follows:

$$\Phi_i^g(k) = (\Phi_i(k) - \tilde{\Phi}_i^g(k))(I-\tau L)^g \tag{29}$$

where

$$\tilde{\Phi}_i^g(k) = \sum_{h=k-g+1}^{k} (-1)^h \theta_{h+1}\tau^h (L^h - \hat{L}_{i,g}^h) \tag{30}$$

Note that the error of removing the i-th node at step g is equal to the product of two terms:

- the term $\Phi_i(k) - \tilde{\Phi}_i^g(k)$ that can be regarded as the error transition matrix $\Phi_i^{[0,k-g]}(k)$ for removing the i-th node for the first $k-g$ steps;
- the term $(I-\tau L)^g x_0$ that is the state of the agents in the original consensus at step g (i.e., x(g)).

Hence we have that:

$$\Phi_i^g(k)x_0 = \Phi_i^{[0,k-g]}(k)x(g) \tag{31}$$

Let us now calculate the structure of the asymptotic state-error transition matrix corresponding to the removal of i-th node at step g. Note that, for small values of g and in the limit of $k \to \infty$, the term $\tilde{\Phi}_i^g(k)$ tends to disappear, and the matrix is simply given by:

$$\bar{\Phi}(i, g) = \bar{\Phi}_i(I - \tau L)^g \quad (32)$$

Conversely, in the limit of $k, g \to \infty$, matrix $\Phi_i^{[0,k-g]}(k)$ tends to $\bar{\Phi}_i$, and therefore the limit tends to 0. This is just as expected, since the effect of removing a node is reduced for higher values of g.

V. Simulation Results

Let us now provide some simulation results, both in the case of growing networks and in the case of removing one node.

Figure 1 shows a simulation of a consensus for a growing network where a new node is dynamically added at each time step until a size of $n = 100$ agents is reached. Each new node is randomly connected to a previous node via an undirected link.

Figure 2.(a) shows a simulation of a consensus for a growing network: initially only $\delta(0) = 10$ nodes are in the network, then 10 new nodes are added with random initial condition every 10 time steps until the final size $n = 50$ is reached. Note that, according to Theorem 3.3, the error with respect to the evolution of a consensus with static topology with the same number of agents and the same initial conditions is asymptotically zero (see Figure 2.(b)).

Finally Figure 3 shows a simulation of the error induced by a single node removal. In Figure 3.(a) a consensus with static topology for $n = 4$ nodes is plotted, while in Figure 3.(b) the evolution of the state of each agent is plotted when an agent is disconnected from the network at the step $k = 5$. Figure 3.(c) shows the error e between the scenarios depicted in Figures 3.(a) and 3.(b); finally Figure 3.(d) shows the difference between the error of Figure 3.(c) and the asymptotic error computed resorting to the analytical approach.

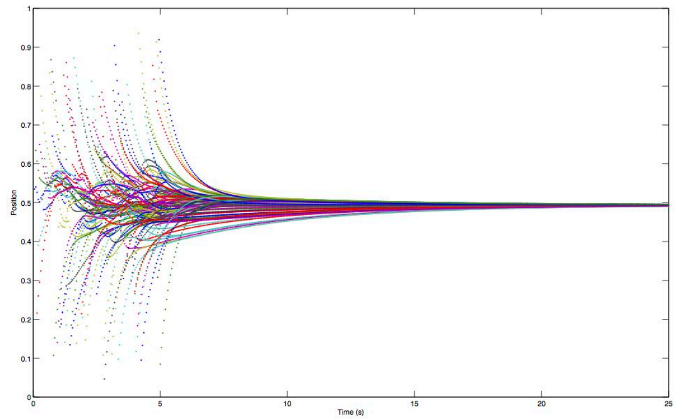

Fig. 1. Simulation of a consensus for a growing network where a new node is dynamically added at each time step until a size of $n = 100$ agents is reached. Each new node is randomly connected to a previous node via an undirected link.

VI. Conclusions

In this paper the possibility to dynamically add or remove nodes during consensus has been investigated. Specifically, a framework for the achievement of consensus while dynamically adding nodes to the network has been provided, together with a stability condition; moreover, the effects of removing a single node in the network at a given time instant have been inspected, characterizing the error with respect to the original consensus and the asymptotic values assumed by the agents, depending on the removal time instant. A further result provided in this paper is the relation between the node removal at a given time instant and the initial removal of that node (i.e., at the initial time step).

Future work will be devoted to inspect the effect of a multiple node removal in distinct or coincident time steps, as well as an integration of the two approaches, thus providing a unified framework for adding and removing nodes to the network.

A challenging issue in this sense will be how to characterize the error of node removal in a unified perspective and how to cope with agent removal and reactivation.

Another very interesting future work direction is to characterize the dynamics of the agents when the addition of new agents never stops, investigating the conditions that may drive the network to a consensus value.

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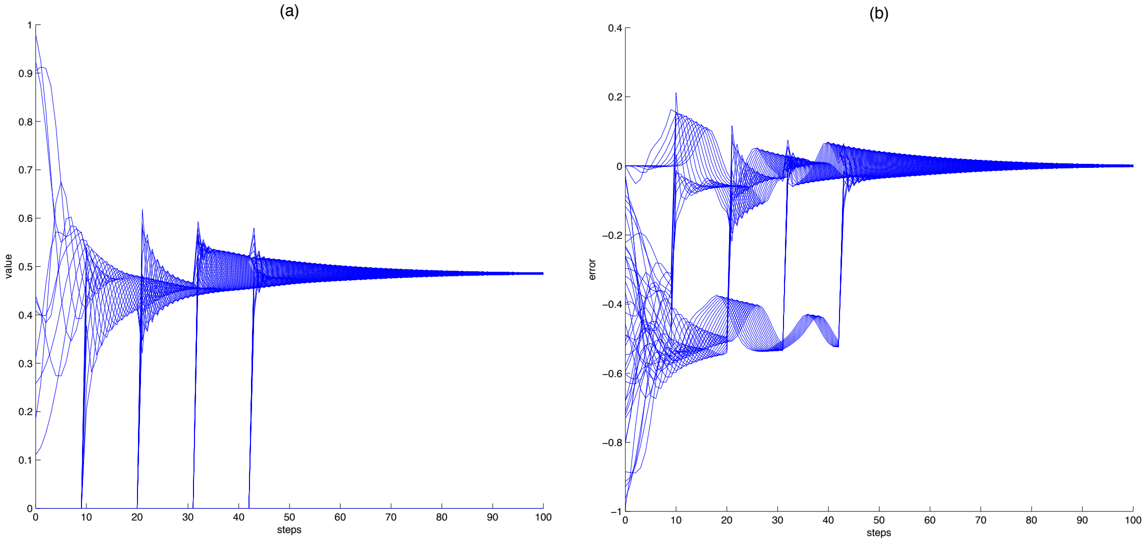

Fig. 2. Simulation of a consensus for a growing network: initially only $\delta(0) = 10$ nodes are in the network, then 10 new nodes are added with random initial condition every 10 time steps until the final size $n = 50$ is reached. Figure (a) shows the evolution of the state of the agents, (assuming that the value of not yet added agents is zero until they are added), while Figure (b) shows the error with a consensus with all the agents, which tends asymptotically to zero.

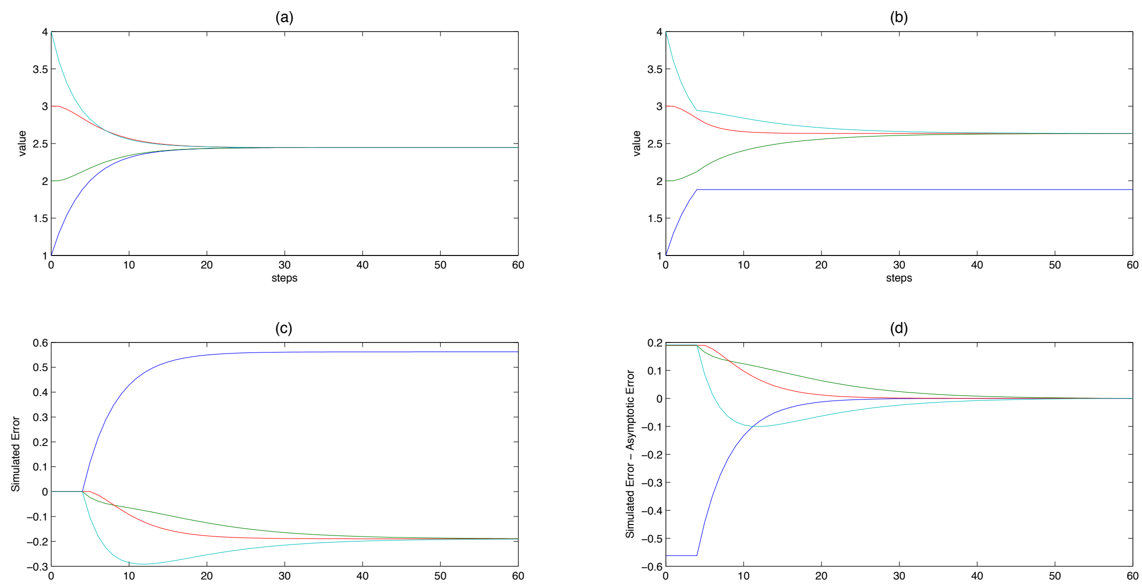

Fig. 3. Simulation of the error induced by a single node removal. In Figure (a) a consensus with static topology for $n = 4$ nodes is plotted, while in Figure (b) the evolution of the state of each agent is plotted when an agent is disconnected from the network at the step $k = 5$. Figure (c) shows the error e between the scenarios depicted in Figures (a) and (b); finally Figure (d) shows the difference between the error of Figure (c) and the asymptotic error computed resorting to the analytical approach.

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