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# Analogies in the controller design for multi-agent and physically interconnected systems with identical subsystems

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***Abstract*— This paper considers interconnected systems consisting of identical subsystems. The subsystem interconnections are either caused by the physical relations or have to introduced by the controllers. In both cases, decomposition methods are needed to reduce the complexity of the controller design problem.**

**The discrete-time LQR design problem for interconnected systems is considered in this paper. A decomposition approach is introduced which can be uniformly applied to both interconnection structures. It reduces the controller design for the overall system to separate design problems for the modified subsystems, which have the order of a single isolated subsystem.**

***Index Terms*— Interconnected systems, LQR design, decomposition methods**

## I. Introduction

### A. Interconnected systems

This paper deals with the discrete-time LQR design for interconnected systems which consist of a large number of identical subsystems. The subsystem interconnections are either caused by the physical relations between the subsystems (***physically interconnected systems***) or have to be introduced by the controllers to cope with the shared (cooperative) control goals (objectives) (***multi-agent systems***).

***Multi-agent systems*** are composed of physically decoupled subsystems (often referred to as agents), which interact with the other subsystems to accomplish a common (cooperative) goal. With the use of wireless communication networks, information among (even mobile) subsystems can be exchanged, so that tasks beyond the abilities of single subsystems can be completed by groups (teams) of subsystems in an efficient way. Figure 1 shows a multi-agent system with $N$ identical subsystems controlled over a digital network.

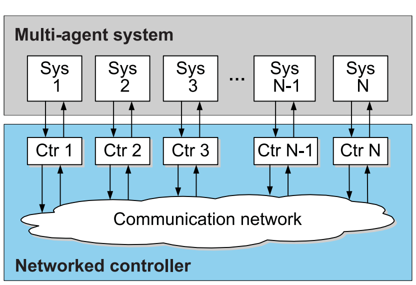


Fig. 1. Networked control of multi-agent systems

***Physically interconnected systems*** consist of the subsystems with usually individual control goals. Examples for physically interconnected systems with similar (ideally identical) subsystems are energy networks, multizone furnaces or adaptive optics. A networked controller structure for a physically interconnected system with $N$ subsystems is shown in Fig. 2.

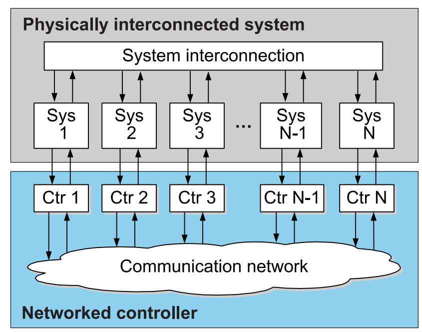


Fig. 2. Networked control of physically interconnected systems

In both cases, the overall system may display a rich and complex behavior. A natural way to deal with the complexity of the controller design problem for interconnected systems is to apply a ***decomposition*** strategy with the intention of decomposing the controller design problem into smaller problems, which can be solved in a more efficient way. This paper introduces a decomposition approach, which can be applied to both interconnection structures. In both cases, an important simplification for the controller design is obtained, since the controller design problem for the overall system can be replaced with the controller design problems for modified subsystems which have the order of a single subsystem.

### B. Literature review

Design of decentralized and distributed controllers for ***physically interconnected systems with complex dynamics*** (often referred to as large scale systems) with different interconnection topologies have been considered in [1]-[7]. In [1], the author obtains simplifications in the controller design using the structural properties of symmetric composite systems. In [6], the same class of physically interconnected systems is studied and it has been shown, that the continuous-LQR design for the overall system can be performed by considering two different modified subsystems independently.

The same design strategy has been applied in [7] to spatially interconnected systems.

The relation between the formation stability and communication topology is firstly considered in [8]. Using a decomposition approach, the authors show that the stability of the vehicle formation is equivalent to the stability of a set of single vehicles, whose outputs are scaled with the eigenvalues of the graph Laplacien matrix. The same idea has been used in [9] to design distributed controllers with $H_2$ and $H_\infty$ performance criterion. Using a decomposition approach, the LMI for the overall system can be decomposed to a set of smaller LMI's which can be solved more easily. In [10], a LQR based design strategy is proposed to design stabilizing distributed controllers. The proposed strategy provides simplification in the computational complexity of the controller design. The focus of the lastly mentioned papers is similar to the focus of this paper, which also introduces a decomposition approach to reduce the complexity of the controller design problem. In contrast to the mentioned references, another design problem is considered in this paper (a discrete LQR problem) and consequently, the output of the design strategy is a optimal (not only stabilizing) but centralized controller.

### C. Contribution and structure of the paper

This paper studies the discrete-time LQR-design for interconnected systems which consist of identical subsystems. The main contribution of this paper is the introduced decomposition strategy, so that the optimal controller for the overall system is obtained as the solution of separate design problems each of which refers to modified subsystems. This approach can be applied uniformly to multi-agent systems and physically interconnected systems. The only assumption is the symmetry in the interconnection matrix, which describes the coupling of the control goals in multi-agent systems and the physical interactions among the subsystems in physically interconnected systems.

The notations which will used throughout the paper together with the system model will be introduced in Section II. The LQR design problem for the multi-agent systems and physically interconnected systems is introduced in Section III. In Section IV, the decomposition approach is applied to the multi-agent systems and in Section V, the same approach is applied to the physically interconnected systems. Concluding remarks are made in Section VI.

## II. Preliminaries

### A. Notations

In this paper, $\mathbb{R}$ denotes the field of real numbers and $\mathbb{R}^{mxn}$ the set of real matrices with the dimensions $m \times n$. Throughout this paper, a scalar is denoted by $x \in \mathbb{R}$, a vector $\boldsymbol{x} \in \mathbb{R}^n$ and a matrix $\boldsymbol{X} \in \mathbb{R}^{m\times n}$. A matrix $\boldsymbol{X}$ is orthogonal, if $\boldsymbol{X}' = \boldsymbol{X}^{-1}$. $\boldsymbol{I}_{\mathrm{n}} \in \mathbb{R}^{n\times n}$ denotes the identity matrix. $\mathbf{0}$ denotes a zero matrix or zero vector of appropriate dimensions. $\lambda_i(\boldsymbol{X})$ denotes the $i$-th eigenvalue of the matrix $\boldsymbol{X}$.

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be symmetric. $\boldsymbol{A}$ is positive definite if $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{A}$ is positive semidefinite if $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \geq 0$ for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$. This is denoted by $\boldsymbol{A} \succ 0$ and $\boldsymbol{A} \succeq 0$, respectively.

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{p\times q}$. $\boldsymbol{A} \otimes \boldsymbol{B}$ denotes the Kronecker product (tensor product) of $\boldsymbol{A}$ and $\boldsymbol{B}$.

**Lemma 1.** *[11] Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ , $\boldsymbol{B} \in \mathbb{R}^{r\times s}$ , $\boldsymbol{C} \in \mathbb{R}^{n\times p}$ and $\boldsymbol{D} \in \mathbb{R}^{s\times t}$. Then*

$$\begin{aligned}(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) &= \boldsymbol{A}\boldsymbol{C}\otimes\boldsymbol{B}\boldsymbol{D}\\ (\boldsymbol{A}\otimes\boldsymbol{B})' &= \boldsymbol{A}'\otimes\boldsymbol{B}'.\end{aligned}$$

**Lemma 2.** *[11] If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices, then*

$$(\boldsymbol{A}\otimes\boldsymbol{B})^{-1} = (\boldsymbol{A}^{-1}\otimes\boldsymbol{B}^{-1}).$$

**Lemma 3.** *(Matrix inversion lemma) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ , $\boldsymbol{B} \in \mathbb{R}^{n\times m}$ , $\boldsymbol{C} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{D} \in \mathbb{R}^{m\times m}$. If $\boldsymbol{A}$, $\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}$ and $\boldsymbol{D}$ are invertible matrices, then*

$$(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} = \boldsymbol{A}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1}.$$

### B. System model

*1) Multi-agent systems:* A multi-agent system consisting of $N$ identical subsystems (or agents) is given by

$$\boldsymbol{x}_i(k+1) = \boldsymbol{A}\boldsymbol{x}_i(k) + \boldsymbol{B}\boldsymbol{u}_i(k), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \tag{1}$$

where

- $\boldsymbol{x}_i(k)$ is the $n$-dimensional subsystem state and
- $\boldsymbol{u}_i(k)$ is the control input.

The overall system model can be represented as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_{\mathrm{os}}\boldsymbol{x}(k) + \boldsymbol{B}_{\mathrm{os}}\boldsymbol{u}(k) \tag{2}$$

with

$$\boldsymbol{A}_{\mathrm{os}} = (\boldsymbol{I}_N \otimes \boldsymbol{A}), \quad \boldsymbol{B}_{\mathrm{os}} = (\boldsymbol{I}_N \otimes \boldsymbol{B}). \tag{3}$$

*2) Physically interconnected systems:* A physically interconnected system consisting of $N$ identical subsystems is represented by

$$\boldsymbol{x}_i(k+1) = \boldsymbol{A}\boldsymbol{x}_i(k) + \boldsymbol{B}\boldsymbol{u}_i(k) + \boldsymbol{e}s_i(k), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \tag{4}$$

$$z_i(k) = \boldsymbol{c}_{\mathrm{z}}'\boldsymbol{x}_i(k) \tag{5}$$

$$s_i(k) = \sum_{j=1}^{N} l_{ij} z_j(k) \tag{6}$$

with the same symbols as in (1) and with

- $s_i(k)$ is the scalar physical interconnection input signal and
- $z_i(k)$ is the scalar physical interconnection output signal.

The physical interconnection matrix $\boldsymbol{L}_{\mathrm{p}} \in \mathbb{R}^{N\times N}$ is assumed to be symmetric and describes the relation between the physical interconnection input and output signal of the overall system,

$$\boldsymbol{s}(k) = \boldsymbol{L}_{\mathrm{p}}\boldsymbol{z}(k). \tag{7}$$

The overall system model can be represented as

$$\boldsymbol{x}(k+1) = (\boldsymbol{I}_N \otimes \boldsymbol{A})\,\boldsymbol{x}(k) + (\boldsymbol{I}_N \otimes \boldsymbol{e})\,(\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{c}_z')\,\boldsymbol{x}(k) \tag{8}$$

$$+ (\boldsymbol{I}_N \otimes \boldsymbol{B})\,\boldsymbol{u}(k)$$

$$= \boldsymbol{A}_\mathrm{os}\boldsymbol{x}(k) + \boldsymbol{B}_\mathrm{os}\boldsymbol{u}(k) \tag{9}$$

with

$$\boldsymbol{A}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{A} + \boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{e}\boldsymbol{c}_\mathrm{z}', \quad \boldsymbol{B}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{B}. \tag{10}$$

## III. Problem setup

In this section, the LQR design problem for physically interconnected systems and for multi-agent systems will be introduced.

### A. LQR design problem for multi-agent systems

For multi-agent systems, the performance index depends upon the behavior of the independent subsystems. A reasonable cost function structure is introduced in the following

$$J_\mathrm{M} = \sum_{k=0}^{\infty} \left(\boldsymbol{x}(k)'\boldsymbol{Q}_\mathrm{os}\boldsymbol{x}(k) + \boldsymbol{u}(k)'\boldsymbol{R}_\mathrm{os}\boldsymbol{u}(k)\right) \tag{11}$$

with

$$\boldsymbol{Q}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{Q}_\mathrm{a} + \boldsymbol{L}_\mathrm{q} \otimes \boldsymbol{Q}_\mathrm{k} \text{ and } \boldsymbol{R}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{R}, \tag{12}$$

where $\boldsymbol{R} \succ 0$, $\boldsymbol{Q}_\mathrm{a} \succ 0$ and $\boldsymbol{Q}_\mathrm{k} \succ 0$. Since the subsystems have common objectives, the cost function does not only contain terms which penalize the local system states ($\boldsymbol{Q}_\mathrm{a}$) but also contain terms which penalize the difference state between the subsystems ($\boldsymbol{Q}_\mathrm{k}$). The matrix $\boldsymbol{L}_\mathrm{q}$ is a symmetric matrix and describes the coupling of the control goals.

The design problem is given by,

$$\min_{\boldsymbol{u}(k)} J_\mathrm{M}(\boldsymbol{u}(k), \boldsymbol{x}_0), \tag{13}$$

$$s.t. \quad \boldsymbol{x}(k+1) = \boldsymbol{A}_\mathrm{os}\boldsymbol{x}(k) + \boldsymbol{B}_\mathrm{os}\boldsymbol{u}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{14}$$

with $\boldsymbol{A}_\mathrm{os}$ and $\boldsymbol{B}_\mathrm{os}$ introduced in (3). The control input $\boldsymbol{u}(k)^*$ which solves the optimization problem, can be represented as $\boldsymbol{u}^*(k) = -\boldsymbol{K}^*_\mathrm{os}\boldsymbol{x}(k)$ with the optimal controller matrix

$$\boldsymbol{K}^*_\mathrm{os} = (\boldsymbol{R}_\mathrm{os} + \boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{B}_\mathrm{os})^{-1}\boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{A}_\mathrm{os}. \tag{15}$$

$\boldsymbol{P}^*_\mathrm{os}$ is the positive definite solution of the following discrete algebraic Riccati equation (DARE) for the overall system

$$\boldsymbol{P}^*_\mathrm{os} = \boldsymbol{Q}_\mathrm{os} + \boldsymbol{A}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{A}_\mathrm{os} - \boldsymbol{A}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{B}_\mathrm{os}(\boldsymbol{R}_\mathrm{os} + \boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{B}_\mathrm{os})^{-1}\boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{A}_\mathrm{os} \tag{16}$$

with $\boldsymbol{A}_\mathrm{os}$ and $\boldsymbol{B}_\mathrm{os}$ in (3).

### B. LQR design problem for physically interconnected systems

The physically interconnected systems consist of the subsystems with individual control objectives. A reasonable cost function structure is introduced in the following

$$J_\mathrm{P} = \sum_{k=0}^{\infty} \left(\boldsymbol{x}(k)'\boldsymbol{Q}_\mathrm{os}\boldsymbol{x}(k) + \boldsymbol{u}(k)'\boldsymbol{R}_\mathrm{os}\boldsymbol{u}(k)\right) \tag{17}$$

with

$$\boldsymbol{Q}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{Q}_\mathrm{a} \text{ and } \boldsymbol{R}_\mathrm{os} = \boldsymbol{I}_N \otimes \boldsymbol{R}m \tag{18}$$

where $\boldsymbol{R} \succ 0$ and $\boldsymbol{Q}_\mathrm{a} \succ 0$.

The design problem is given by

$$\min_{\boldsymbol{u}(k)} J_\mathrm{P}(\boldsymbol{u}(k), \boldsymbol{x}_0), \tag{19}$$

$$s.t. \quad \boldsymbol{x}(k+1) = \boldsymbol{A}_\mathrm{os}\boldsymbol{x}(k) + \boldsymbol{B}_\mathrm{os}\boldsymbol{u}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{20}$$

with $\boldsymbol{A}_\mathrm{os}$ and $\boldsymbol{B}_\mathrm{os}$ introduced in (10). The optimal control input can be represented as $\boldsymbol{u}^*(k) = -\boldsymbol{K}^*_\mathrm{os}\boldsymbol{x}(k)$ with $\boldsymbol{K}^*_\mathrm{os}$ introduced in (15) and $\boldsymbol{P}^*_\mathrm{os}$ as the solution of the following DARE

$$\boldsymbol{P}^*_\mathrm{os} = \boldsymbol{Q}_\mathrm{os} + \boldsymbol{A}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{A}_\mathrm{os} - \boldsymbol{A}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{B}_\mathrm{os}(\boldsymbol{R}_\mathrm{os} + \boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{B}_\mathrm{os})^{-1}\boldsymbol{B}'_\mathrm{os}\boldsymbol{P}^*_\mathrm{os}\boldsymbol{A}_\mathrm{os} \tag{21}$$

with $\boldsymbol{A}_\mathrm{os}$ and $\boldsymbol{B}_\mathrm{os}$ in (10). Because of the physical interactions among the subsystems, the optimal controller matrix $\boldsymbol{K}^*_\mathrm{os}$ is a fully occupied matrix, although only local subsystem states are penalized in the cost function.

## IV. Decomposition of the controller design for multi-agent systems

### A. Main Result

**Theorem 1.** *The solution* (16) *of the LQR design problem* (13)-(14) *for the multi-agent system can be represented by*

$$\boldsymbol{P}^*_\mathrm{os} = (\boldsymbol{T} \otimes \boldsymbol{I}_n)\mathrm{diag}\,(\tilde{\boldsymbol{P}}^*_i)(\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1}$$

*in terms of the solutions $\tilde{\boldsymbol{P}}^*_i$ of the following $N$ separate DARE's*

$$\tilde{\boldsymbol{P}}^*_i = \tilde{\boldsymbol{Q}}_i + \boldsymbol{A}'\tilde{\boldsymbol{P}}^*_i\boldsymbol{A} - \boldsymbol{A}'\tilde{\boldsymbol{P}}^*_i\boldsymbol{B}\left(\boldsymbol{R} + \boldsymbol{B}'\tilde{\boldsymbol{P}}^*_i\boldsymbol{B}\right)^{-1}\boldsymbol{B}'\tilde{\boldsymbol{P}}^*_i\boldsymbol{A}, \tag{22}$$

*with*

$$\tilde{\boldsymbol{Q}}_i = \boldsymbol{Q}_\mathrm{a} + \lambda_i(\boldsymbol{L}_\mathrm{q})\boldsymbol{Q}_\mathrm{k} \text{ and } i = 1, \ldots, N.$$

In virtue of Theorem 1, the optimal controller for the multi-agent system can be designed in the transformed domain by solving $N$ independent DARE's, each of which has the order of a single subsystem. The number of the DARE's depend on the spectral properties of the interconnection matrix $\boldsymbol{L}_\mathrm{q}$.

### B. Proof of Theorem 1

Since the matrix $\boldsymbol{L}_\mathrm{q}$ is assumed to be a symmetric, its eigenvalues are real and it exists an orthogonal matrix $\boldsymbol{T}$, which diagonalizes the matrix $\boldsymbol{L}_\mathrm{q}$,

$$\tilde{\boldsymbol{L}}_\mathrm{q} := \boldsymbol{T}^{-1}\boldsymbol{L}_\mathrm{q}\boldsymbol{T} = \begin{pmatrix} \lambda_1(\boldsymbol{L}_\mathrm{q}) & 0 & \ldots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_N(\boldsymbol{L}_\mathrm{q}) \end{pmatrix}. \tag{23}$$

The state transformation

$$\tilde{\boldsymbol{x}}(k) = (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1}\boldsymbol{x}(k) = \boldsymbol{T}^{-1}_\mathrm{os}\boldsymbol{x}(k) \tag{24}$$

is applied to the overall system. In virtue of Lemma 1 and 2, the system matrix $\tilde{\boldsymbol{A}}_{\text{os}}$ has a block diagonal form:

$$\begin{aligned}\tilde{\boldsymbol{A}}_{\text{os}} &= (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{A}_{\text{os}} (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} (\boldsymbol{I}_N \otimes \boldsymbol{A}) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_n)(\boldsymbol{T} \otimes \boldsymbol{A}) \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{A}).\end{aligned} \tag{25}$$

The input matrix $\boldsymbol{B}_{\text{os}}$ in (3) is given after the state transformation as $\tilde{\boldsymbol{B}}_{\text{os}} = (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{B}_{\text{os}}$.

With the transformed states, the cost function (11) can be rewritten as

$$\begin{aligned}J &= \sum_{k=0}^{\infty} \boldsymbol{x}(k)' \boldsymbol{Q}_{\text{os}} \boldsymbol{x}(k) + \boldsymbol{u}(k)' \boldsymbol{R}_{\text{os}} \boldsymbol{u}(k) \\ &= \sum_{k=0}^{\infty} (\boldsymbol{T}_{\text{os}} \tilde{\boldsymbol{x}}(k))' \boldsymbol{Q}_{\text{os}} (\boldsymbol{T}_{\text{os}} \tilde{\boldsymbol{x}}(k)) + \boldsymbol{u}(k)' \boldsymbol{R}_{\text{os}} \boldsymbol{u}(k) \\ &= \sum_{0}^{\infty} \tilde{\boldsymbol{x}}(k)' \tilde{\boldsymbol{Q}}_{\text{os}} \tilde{\boldsymbol{x}}(k) + \boldsymbol{u}(k)' \boldsymbol{R}_{\text{os}} \boldsymbol{u}(k),\end{aligned} \tag{26}$$

where $\tilde{\boldsymbol{Q}}_{\text{os}} = \boldsymbol{T}'_{\text{os}} \boldsymbol{Q}_{\text{os}} \boldsymbol{T}_{\text{os}}$. $\tilde{\boldsymbol{Q}}_{\text{os}}$ is a block diagonal matrix, since

$$\begin{aligned}\boldsymbol{T}'_{\text{os}} \boldsymbol{Q}_{\text{os}} \boldsymbol{T}_{\text{os}} &= (\boldsymbol{T}' \otimes \boldsymbol{I}_n) \boldsymbol{Q}_{\text{os}} (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= \left(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_n\right) (\boldsymbol{I}_N \otimes \boldsymbol{Q}_{\text{a}}) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &+ \left(\boldsymbol{T}^{-1} \otimes \boldsymbol{I}_n\right) (\boldsymbol{L}_{\text{q}} \otimes \boldsymbol{Q}_{\text{k}}) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{Q}_{\text{a}}) + \left(\boldsymbol{T}^{-1} \boldsymbol{L}_{\text{q}} \boldsymbol{T} \otimes \boldsymbol{Q}_{\text{k}}\right) \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{Q}_{\text{a}}) + \left(\tilde{\boldsymbol{L}}_{\text{q}} \otimes \boldsymbol{Q}_{\text{k}}\right).\end{aligned}$$

Considering (25) and (26), the DARE for the overall system with the transformed states can be given by

$$\begin{aligned}\tilde{\boldsymbol{P}}^*_{\text{os}} =& \tilde{\boldsymbol{Q}}_{\text{os}} + \boldsymbol{A}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \boldsymbol{A}_{\text{os}} \\ & - \boldsymbol{A}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \tilde{\boldsymbol{B}}_{\text{os}} (\boldsymbol{R}_{\text{os}} + \tilde{\boldsymbol{B}}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \tilde{\boldsymbol{B}}_{\text{os}})^{-1} \tilde{\boldsymbol{B}}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \boldsymbol{A}_{\text{os}}\end{aligned} \tag{27}$$

A relation between $\boldsymbol{P}^*_{\text{os}}$ and $\tilde{\boldsymbol{P}}^*_{\text{os}}$ can be determined, if the value of the cost function (11) with the transformed and original system state is considered:

$$\begin{aligned}J^* = \boldsymbol{x}'_0 \boldsymbol{P}^*_{\text{os}} \boldsymbol{x}_0 &= \tilde{\boldsymbol{x}}'_0 \tilde{\boldsymbol{P}}^*_{\text{os}} \tilde{\boldsymbol{x}}_0 \\ \boldsymbol{x}'_0 \boldsymbol{P}^*_{\text{os}} \boldsymbol{x}_0 &= \left((\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{x}_0\right)' \tilde{\boldsymbol{P}}^*_{\text{os}} (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{x}_0 \\ \boldsymbol{x}'_0 \boldsymbol{P}^*_{\text{os}} \boldsymbol{x}_0 &= \boldsymbol{x}'_0 (\boldsymbol{T} \otimes \boldsymbol{I}_n) \tilde{\boldsymbol{P}}^*_{\text{os}} (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{x}_0\end{aligned}$$

Hence,

$$\tilde{\boldsymbol{P}}^*_{\text{os}} = (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{P}^*_{\text{os}} (\boldsymbol{T} \otimes \boldsymbol{I}_n). \tag{28}$$

$\tilde{\boldsymbol{B}}_{\text{os}} \left(\boldsymbol{R}_{\text{os}} + \tilde{\boldsymbol{B}}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \tilde{\boldsymbol{B}}_{\text{os}}\right)^{-1} \tilde{\boldsymbol{B}}'_{\text{os}}$ in (27) is rewritten as $\boldsymbol{Z}$. Using the matrix inversion lemma, $\boldsymbol{Z}$ can be represented as shown in the following:

$$\begin{aligned}\boldsymbol{Z} =& \tilde{\boldsymbol{B}}_{\text{os}} \left(\boldsymbol{R}_{\text{os}} + \tilde{\boldsymbol{B}}'_{\text{os}} \tilde{\boldsymbol{P}}^*_{\text{os}} \tilde{\boldsymbol{B}}_{\text{os}}\right)^{-1} \tilde{\boldsymbol{B}}'_{\text{os}} \\ =& \tilde{\boldsymbol{B}}_{\text{os}} \Big(\boldsymbol{R}^{-1}_{\text{os}} - \\ & \boldsymbol{R}^{-1}_{\text{os}} \tilde{\boldsymbol{B}}'_{\text{os}} \left((\tilde{\boldsymbol{P}}^*_{\text{os}})^{-1} + \tilde{\boldsymbol{B}}_{\text{os}} \boldsymbol{R}^{-1}_{\text{os}} \tilde{\boldsymbol{B}}'_{\text{os}}\right)^{-1} \tilde{\boldsymbol{B}}_{\text{os}} \boldsymbol{R}^{-1}_{\text{os}}\Big) \tilde{\boldsymbol{B}}'_{\text{os}} \\ =:& \boldsymbol{X} - \boldsymbol{X} \left((\tilde{\boldsymbol{P}}^*_{\text{os}})^{-1} + \boldsymbol{X}\right)^{-1} \boldsymbol{X}\end{aligned} \tag{29}$$

with $\boldsymbol{X} =: \tilde{\boldsymbol{B}}_{\text{os}} \boldsymbol{R}^{-1}_{\text{os}} \tilde{\boldsymbol{B}}'_{\text{os}}$. $\boldsymbol{X}$ is block diagonal since

$$\begin{aligned}\boldsymbol{X} &= \boldsymbol{T}^{-1}_{\text{os}} (\boldsymbol{I}_N \otimes \boldsymbol{B}) (\boldsymbol{I}_N \otimes \boldsymbol{R}^{-1}) (\boldsymbol{T}^{-1}_{\text{os}} (\boldsymbol{I}_N \otimes \boldsymbol{B}))' \\ &= \boldsymbol{T}^{-1}_{\text{os}} (\boldsymbol{I}_N \otimes \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}') \boldsymbol{T}_{\text{os}} \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{B} \boldsymbol{R}^{-1} \boldsymbol{B}').\end{aligned} \tag{30}$$

Hence, all the matrices in (27) are shown to be block diagonal and consequently, the solution $\tilde{\boldsymbol{P}}^*_{\text{os}}$ of the DARE is a block diagonal matrix as well.

Hence, the DARE for the overall system can be decomposed into following separate equations

$$\begin{aligned}\tilde{\boldsymbol{P}}^*_i =& \tilde{\boldsymbol{Q}}_i + \boldsymbol{A}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{A} \\ & + \boldsymbol{A}' \tilde{\boldsymbol{P}}^*_i \left(\boldsymbol{X}_i - \boldsymbol{X}_i \left((\tilde{\boldsymbol{P}}^*_i)^{-1} + \boldsymbol{X}_i\right)^{-1} \boldsymbol{X}_i\right) \tilde{\boldsymbol{P}}^*_i \boldsymbol{A} \\ =& \tilde{\boldsymbol{Q}}_i + \boldsymbol{A}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{A} - \boldsymbol{A}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \left(\boldsymbol{R} + \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B}\right)^{-1} \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{A},\end{aligned}$$

with

$$\tilde{\boldsymbol{Q}}_i = \boldsymbol{Q}_{\text{a}} + \lambda_i(\boldsymbol{L}_{\text{q}}) \boldsymbol{Q}_{\text{k}} \text{ and } i = (1, \dots, N),$$

which ends the proof.

In the next section, the same decomposition approach will be applied to the physically interconnected systems, where the system interconnection is caused by the physical interaction among the subsystems.

## V. Decomposition of the controller design for physically interconnected systems

### A. Main result

**Theorem 2.** *The solution* (21) *of the LQR design problem* (19)-(20) *for the physically interconnected system can be represented by*

$$\boldsymbol{P}^*_{\text{os}} = (\boldsymbol{T} \otimes \boldsymbol{I}_n) \text{diag}\,(\tilde{\boldsymbol{P}}^*_i) (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1}$$

*in terms of the solutions $\tilde{\boldsymbol{P}}^*_i$ of the following $N$ separate DARE's*

$$\tilde{\boldsymbol{P}}^*_i = \boldsymbol{Q}_{\text{a}} + \tilde{\boldsymbol{A}}'_i \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}_i - \tilde{\boldsymbol{A}}'_i \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \left(\boldsymbol{R} + \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B}\right)^{-1} \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}'_i, \tag{31}$$

*with*

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{A}_i + \lambda_i(\boldsymbol{L}_{\text{p}}) \boldsymbol{e} \boldsymbol{c}_{\text{z}}, \text{ and } i = 1, \dots, N.$$

### B. Analogies in Theorem 1 and Theorem 2

Theorem 1 and Theorem 2 deal with different interconnection structures. However, in both cases, the same decomposition of the controller design problem is obtained, with similar state transformations. For both classes of interconnected systems, the decomposition approach provides an important simplification for the design problem, since the controller design problem (16) for the overall system can be decomposed into design problems for independent subsystems, which are given in (22) for multi-agent systems and in (31) for physically interconnected systems. In (22), the term $\tilde{\boldsymbol{Q}}_i$ contains information on the interconnection of the control goals, while in (31), the term $\tilde{\boldsymbol{A}}_i$ contains information on the physical interconnection of the subsystems. In both cases, the only requirement to apply the design strategy is the symmetry of the interconnection matrices $\boldsymbol{L}_\mathrm{q}$ or $\boldsymbol{L}_\mathrm{p}$.

**Remark 1.** *A more general system class will be considered now, where physically interconnected subsystems share control goals, so that $\boldsymbol{L}_\mathrm{q} \neq \boldsymbol{O}$. The same simplifications for the controller design will be obtained if the matrices $\boldsymbol{L}_\mathrm{q}$ and $\boldsymbol{L}_\mathrm{p}$ are simultaneously diagonalizable. Then, the different diagonal blocks of $\tilde{\boldsymbol{P}}^*_\mathrm{os}$ will be calculated by solving the following DARE's:*

$$\tilde{\boldsymbol{P}}^*_i = \tilde{\boldsymbol{Q}}_i + \tilde{\boldsymbol{A}}_i' \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}_i - \tilde{\boldsymbol{A}}_i' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \left( \boldsymbol{R} + \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \right)^{-1} \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}_i,$$

*for $(i = 1, \ldots, N)$ with*

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{A} + \lambda_i(\boldsymbol{L}_\mathrm{p}) \boldsymbol{e} \boldsymbol{c}_\mathrm{z}', \quad \tilde{\boldsymbol{Q}}_i = \boldsymbol{Q}_\mathrm{a} + \lambda_i(\boldsymbol{L}_\mathrm{q}) \boldsymbol{Q}_\mathrm{k}.$$

### C. Proof of the Theorem 2

It exists an orthogonal transformation matrix $\boldsymbol{T}$, which diagonalizes the matrix $\boldsymbol{L}_\mathrm{p}$ :

$$\boldsymbol{T}^{-1} \boldsymbol{L}_\mathrm{p} \boldsymbol{T} = \tilde{\boldsymbol{L}}_\mathrm{p} = \begin{pmatrix} \lambda_1(\boldsymbol{L}_\mathrm{p}) & 0 & \ldots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_N(\boldsymbol{L}_\mathrm{p}) \end{pmatrix}. \quad (32)$$

After the state transformation

$$\tilde{\boldsymbol{x}}(k) = (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{x}(k) = \boldsymbol{T}_\mathrm{os}^{-1} \boldsymbol{x}(k), \quad (33)$$

the system matrix gets the following block diagonal form:

$$\begin{aligned} \tilde{\boldsymbol{A}}_\mathrm{os} &= (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{A}_\mathrm{os} (\boldsymbol{T} \otimes \boldsymbol{I}_n) \quad (34) \\ &= (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} (\boldsymbol{I}_N \otimes \boldsymbol{A} + (\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{e} \boldsymbol{c}_\mathrm{z}')) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} (\boldsymbol{I}_N \otimes \boldsymbol{A}) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &\quad + (\boldsymbol{T} \otimes \boldsymbol{I}_n)^{-1} (\boldsymbol{L}_\mathrm{p} \otimes \boldsymbol{e} \boldsymbol{c}_\mathrm{z}') (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{A}) + (\tilde{\boldsymbol{L}}_\mathrm{p} \otimes \boldsymbol{e} \boldsymbol{c}_\mathrm{z}'). \quad (35) \end{aligned}$$

According to (26), the cost function (11) with the transformed states can be given by

$$J = \sum_0^\infty (\tilde{\boldsymbol{x}}(k)' \tilde{\boldsymbol{Q}}_\mathrm{os} \tilde{\boldsymbol{x}}(k) + \boldsymbol{u}(k)' \boldsymbol{R}_\mathrm{os} \boldsymbol{u}(k)),$$

where

$$\begin{aligned} \tilde{\boldsymbol{Q}}_\mathrm{os} &= \boldsymbol{T}_\mathrm{os}' \boldsymbol{Q}_\mathrm{os} \boldsymbol{T}_\mathrm{os} \\ &= \left( \boldsymbol{T}^{-1} \otimes \boldsymbol{I}_n \right) (\boldsymbol{I}_N \otimes \boldsymbol{Q}_\mathrm{a}) (\boldsymbol{T} \otimes \boldsymbol{I}_n) \\ &= (\boldsymbol{I}_N \otimes \boldsymbol{Q}_\mathrm{a}) = \boldsymbol{Q}_\mathrm{os}. \end{aligned}$$

Consequently, the DARE for the overall system with the transformed states can be given by

$$\begin{aligned} \tilde{\boldsymbol{P}}^*_\mathrm{os} = &\boldsymbol{Q}_\mathrm{os} + \tilde{\boldsymbol{A}}_\mathrm{os}' \tilde{\boldsymbol{P}}^*_\mathrm{os} \tilde{\boldsymbol{A}}_\mathrm{os} \\ &- \tilde{\boldsymbol{A}}_\mathrm{os}' \tilde{\boldsymbol{P}}^*_\mathrm{os} \tilde{\boldsymbol{B}}_\mathrm{os} (\boldsymbol{R}_\mathrm{os} + \tilde{\boldsymbol{B}}_\mathrm{os}' \tilde{\boldsymbol{P}}^*_\mathrm{os} \tilde{\boldsymbol{B}}_\mathrm{os})^{-1} \tilde{\boldsymbol{B}}_\mathrm{os}' \tilde{\boldsymbol{P}}^*_\mathrm{os} \tilde{\boldsymbol{A}}_\mathrm{os}. \quad (36) \end{aligned}$$

According to (29), the matrix

$$\boldsymbol{Z} = \tilde{\boldsymbol{B}}_\mathrm{os} \left( \boldsymbol{R}_\mathrm{os} + \tilde{\boldsymbol{B}}_\mathrm{os}' \tilde{\boldsymbol{P}}^*_\mathrm{os} \tilde{\boldsymbol{B}}_\mathrm{os} \right)^{-1} \tilde{\boldsymbol{B}}_\mathrm{os}'$$

can be represented as

$$\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{X} \left( (\tilde{\boldsymbol{P}}^*_\mathrm{os})^{-1} + \boldsymbol{X} \right)^{-1} \boldsymbol{X}.$$

Since the matrix $\boldsymbol{X}$ is block diagonal, the DARE (36) for the overall system can be decomposed in independent DARE's for the modified subsystems. The independent DARE's are given by

$$\tilde{\boldsymbol{P}}^*_i = \boldsymbol{Q}_\mathrm{a} + \tilde{\boldsymbol{A}}_i' \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}_i - \tilde{\boldsymbol{A}}_i' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \left( \boldsymbol{R} + \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \boldsymbol{B} \right)^{-1} \boldsymbol{B}' \tilde{\boldsymbol{P}}^*_i \tilde{\boldsymbol{A}}_i'$$

which completes the proof.

## VI. Application example: Controller design for a platoon with identical vehicles

In this section, a discrete LQR controller will be designed for a platoon. Firstly, the platoon will be considered as a multi-agent system. Secondly, a platoon with an internal distance controller will be considered and the controller design will be performed for the distance controlled platoon. The interactions among the subsystems introduced by the distance controller can be also interpreted as a physical interconnection.

### A. Vehicle platoon as a multi-agent system

The model of a single vehicle with double integrator dynamics in continuous time is given in the following in discrete-time with a step time $T = 0.005$

$$\dot{\boldsymbol{x}}_i(k+1) = \boldsymbol{A} \begin{pmatrix} p_i(k) \\ v_i(k) \end{pmatrix} + \boldsymbol{b} u_i(k)$$

and with the system parameters

$$\boldsymbol{A} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ T \end{pmatrix}. \quad (37)$$

where $p_i(k)$ represents the position of a vehicle and $v_i(k)$ the velocity of a vehicle.

The cost function structure (11) with the weighting matrices

$$\boldsymbol{Q}_\mathrm{a} = \begin{pmatrix} 10 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad \boldsymbol{Q}_\mathrm{k} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

and $r = 0.1$ and with the interconnection matrix

$$\boldsymbol{L}_\mathrm{q} = \begin{pmatrix} N-1 & -1 & \ldots & \ldots & -1 \\ -1 & N-1 & -1 & \ldots & \vdots \\ -1 & \ddots & \ddots & \ddots & -1 \\ \vdots & \ddots & -1 & N-1 & -1 \\ -1 & \ldots & \ldots & -1 & N-1 \end{pmatrix}.$$

is considered. The spectrum of the interconnection matrix consists of two distinct eigenvalues. Consequently, only two different DARE's must be solved, to determine the optimal controller. Since the relative states of all subsystems are equally penalized in the cost function, the optimal controller evaluates information from all subsystems identically:

$$\boldsymbol{K}^*_\mathrm{os} = (\boldsymbol{I}_N \otimes \boldsymbol{K}_\mathrm{a}) + (\boldsymbol{L}_\mathrm{q} \otimes \boldsymbol{K}_\mathrm{k}). \quad (39)$$

## B. Vehicle platoon with an internal distance controller

Now, it is assumed that the vehicles can measure the positions of their nearest neighbors and an internal distance controller is implemented in the vehicles. The model (4) of a single vehicle has the parameters $\boldsymbol{A}$ and $\boldsymbol{b}$ introduced in (37) and

$$\boldsymbol{c}'_{\mathrm{z}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \boldsymbol{e} = \boldsymbol{b}.$$

The output of the internal distance controller is denoted with $s_i(k)$. Using the interconnection matrix $\boldsymbol{L}_{\mathrm{p}}$, $\boldsymbol{s}(k)$ is represented as

$$\boldsymbol{s}(k) = (\boldsymbol{L}_{\mathrm{p}} \otimes \boldsymbol{e}\boldsymbol{c}'_{\mathrm{z}})\, \boldsymbol{x}(k),$$

with

$$\boldsymbol{L}_{\mathrm{p}} = -10 \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

The cost function structure (11) with the weighting matrix

$$\boldsymbol{Q}_{\mathrm{a}} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

is considered. Although the vehicles have only local control objectives, the optimal controller is fully occupied, since the vehicles are interconnected through the distance controller.

## C. Simulation results

It is assumed that only the first vehicle deviates from the operation point. Emphasize that the initial deviation of the first vehicle can be seen as a disturbance. The deviation of the control error for different vehicles is shown in Fig. 3 for multi-agent systems and in Fig. 4 for physically interconnected systems.

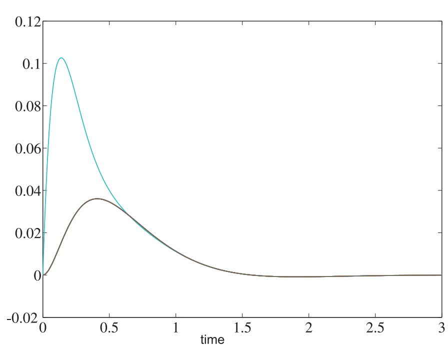


Fig. 3. Control error of vehicles with optimal controller

In the multi-agent case, since the difference states of all subsystem pairs are identically penalized in the cost function, the response of the vehicles on the disturbance acting on the first vehicle is identical (Fig. 3).

In physically interconnected systems, the system interconnection (distance controller) causes a deviation from the operation point in order to reduce the distance with the

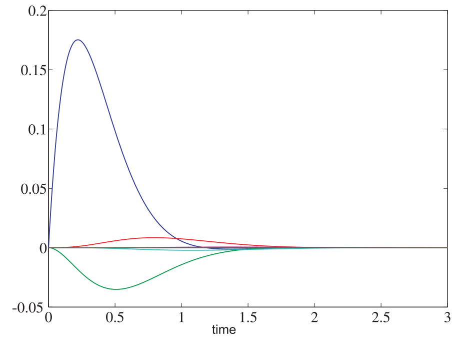


Fig. 4. Control error of distance controlled vehicles with optimal controller

disturbed vehicle. However, as the vehicles have only local control goals, the optimal controller tries to hold the vehicles at the operation point. Therefore, the system interconnection has an effect like a disturbance and hence, the vehicles, which are stronger interconnected with the first vehicle deviates more from the operation point.

# VII. Conclusions

This paper studies the discrete-time LQR design problem for interconnected systems with identical subsystems. Two different interconnection structures have been considered. The main contribution of this paper is the introduced decomposition approach, which provides for both interconnection structures important simplifications for the optimal controller design. The necessary condition to apply the decomposition approach is the symmetry in the interconnection matrix.

The communication load related with the implementation of the designed controller is not considered in this paper. The future research will study discrete-time control strategies with event-based or situation-dependent communication to reduce the communication load.

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