Boundary model predictive control of thin film thickness modelled by Kuramoto-Sivashinsky equation with input and state constraints

Yu Yang and Stevan Dubljevic

Abstract— **In this work, a model predictive control (MPC) synthesis is proposed to regulate, in the presence of naturally present state and input constraints, the thickness of falling film in the vertical tube, modelled by the Kuramoto-Sivashinsky (K-S) equation. The infinite-dimensional state space representation is developed and an exact transformation modifies the boundary control problem into the distributed control problem. The appropriate analysis of K-S spectral operator reveals dissipative structure of the linearized operator which benefits from the applicability of spectral decomposition for the control purpose. The model predictive control synthesis utilizes the finite dimensional representation of the K-S PDE state in the formulation of the optimization functional, while the infinite dimensional K-S PDE state constraints are appropriately defined and cast in a form of constrained quadratic optimization. The simulation study evaluates the performance of proposed methods which achieves both stabilization of the thin film thickness and obeys inputs and states constraints.**

I. INTRODUCTION

The falling film process in a vertical tube has been widely studied for many years [1], [2]. One of the accurate modeling approaches to describe the generation of long waves and its dynamic evolution is given by the well-known Kuramoto-Sivashinsky (K-S) equation [3], [4]. This forth order dissipative partial differential equation (PDE), which have been used to model the plane flame propagation [5] and dendritic fronts in dilute alloys [6], attracted attention of scientific community owing to its complex dynamical features [7], [8]. Although, there is no shortage on the research regarding dynamical complexity and features of K-S equation, very few are centered on control problems associated with the K-S equation. One of the pioneering contributions on control aspects associated with K-S equation is given by [9] and [10], in which actuators are distributed within spatial domain and simple low dimensional state feedback is used to achieve stabilization. However, considering that in the most of cases, the actuators can be only placed at the boundary of the system's domain, it is more appealing to develop the boundary control realization instead of distributed one. Along this line, the contribution regarding stabilization of K-S equation comes from Kristic and co-workers [11], who introduced the nonlinear boundary feedback law to guarantee the global stabilization of K-S equation. Additionally, in [12], the aspects of controllability and boundary stabilization law for the linearized K-S equation are explored.

The control theory of distributed parameter systems, described by parabolic and hyperbolic PDEs has been well established [13]. For the parabolic system, whose dominant dynamics can be approximated only by a few modes, thus, by means of spectrum decomposition, a low dimensional controller based on small number of dominant modes can successfully stabilize the entire infinite dimensional states. Along the same line, in a case of optimal controller synthesis, the infinite dimensional operator's differential Riccati equation is solved to get the optimal feedback control law for the PDE system [14], however this controller synthesis suffers from the excessive computational demands. In recent works, the model predictive control (MPC) strategy is used in [15], [16], to achieve the stabilization and obey naturally present input and state constraints in control of K-S equation.

Build on the aforementioned work, we explore the MPC

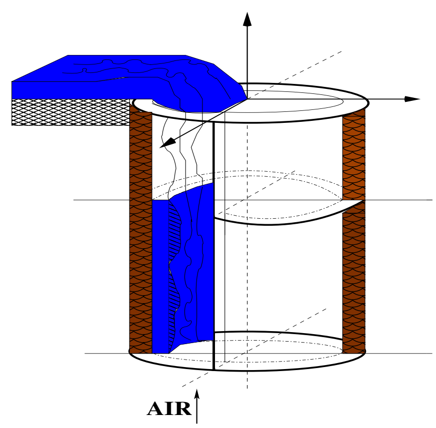

Fig. 1. Two-phase annular flow in vertical tube

synthesis for the K-S equation due to its superior ability to handle states and inputs constrained control problems, commonly present in practice since most of actuators suffer from the physical limitations and key state variables in the system are usually subjected to some constraints arising from performance and/or safety consideration. For example in Fig.1, the airflow in the tube is limited with it's maximum velocity, and the K-S PDE state (the film thickness) can not take negative or large positive values (e.g. flooding of the tubes). The MPC synthesis for finite dimensional systems is already a mature controller synthesis framework, see [17], attributing to its ability to handle optimality, inputs and states constraints and therefore, it provides motivation to cast the K-S PDE control problem into MPC framework. In fact, the MPC framework has already been studied in the case of parabolic PDEs with distributed and boundary placed actuators in [18], [19], [20].

In this paper, compared to previous contributions [15], [16], the exact and more general case accounting for physi-

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cally relevant actuator applied at the boundary is considered. In particular, a new spatial operator which accounts for the entire dynamics of the linearized K-S equation is defined and an appropriate exact transformation for boundary control problem is designed. The infinite-dimensional system representation for K-S PDE state is utilized in MPC controller synthesis by extracting the slow modal state by using K-S PDE spatial operator spectrum features. The MPC synthesis is designed to account for optimization functional given as quadratic optimization of the slow modal states, while input and infinite-dimensional state constraints are cast in the standard constrained optimization framework. The paper is organized as follows: In the preliminaries the model is provided; An infinite-dimensional state space model is developed and the appropriate boundary transformation is given. The MPC formulation is presented in section 3, and the simulation study for the K-S equation with MPC in section 4 shows the performance of the proposed approach. The conclusion is drawn in the section 5.

II. PRELIMINARIES

A. Kuramoto-Sivashinsky equation

The K-S equation is a fourth order partial differential equation, given in the following form:

$$x_t + v x_{\zeta\zeta\zeta\zeta} + x_{\zeta\zeta} + x_\zeta x = 0 \tag{1}$$

$$y(t) = \int_0^l \delta(\zeta - \zeta_c) x(\zeta, t) d\zeta \tag{2}$$

with the boundary and initial conditions:

$$x(0,t) = 0, \quad x(l,t) = \mu(t) \tag{3}$$

$$x_\zeta(0,t) = 0, \quad x_\zeta(l,t) = u(t) \tag{4}$$

$$x(\zeta, 0) = x_0(\zeta) \tag{5}$$

and subject to the constraints:

$$y_{min} \leqslant y(t) \leqslant y_{max} \tag{6}$$

$$u_{min} \leqslant u(t) \leqslant u_{max} \tag{7}$$

$$\mu_{min} \leqslant \mu(t) \leqslant \mu_{max} \tag{8}$$

In the previous research effort [15], only single control $u(t)$ is considered whereas another control input $\mu(t)$ is introduced in this paper to address more realistic considerations of boundary applied regulation, including the Dirichlet boundary condition and Neumann boundary condition together. The $x(\zeta, t)$ denotes the state variable in the separable Hilbert space $\mathcal{H}$; $\zeta \in [0, l]$ is the spatial coordinate and $t \in [0, \infty)$ is the time; $\mu_{min}, \mu_{max}, u_{min}, u_{max}$ represent bounds on $u(t)$ and $\mu(t)$, respectively; the terms $x_{\zeta\zeta\zeta\zeta}$, $x_{\zeta\zeta}$, x_ζ and x_t represent the fourth order, second order, first order spatial derivatives and time derivative, respectively; v is a known parameter of the K-S equation. The PDE state is taken at $\zeta_c \in [0, l]$ by employing an approximated Dirac δ-function.

III. MAIN RESULTS

A. Infinite-dimensional state space model

In this section, the K-S equation given by Eqs.1-2 is linearized around the uniform steady state $x(\zeta, t) = 0$, which is given by:

$$x_t = -v x_{\zeta\zeta\zeta\zeta} - x_{\zeta\zeta} \tag{9}$$

Above equation takes the abstract evolutionary form:

$$\dot{x}(t) = \mathscr{A} x(t) \tag{10}$$

$$\mathscr{B}_1 x(t) = u(t) \tag{11}$$

$$\mathscr{B}_2 x(t) = \mu(t) \tag{12}$$

where $\mathscr{A} := v\frac{d^4}{d\zeta^4} + \frac{d^2}{d\zeta^2}$, with its domain: $\mathscr{D}(\mathscr{A}) = \{\phi(\zeta) \in L_2(0,l) | \phi, \phi_\zeta, \phi_{\zeta\zeta}, \phi_{\zeta\zeta\zeta}$ a.c., $\phi_{\zeta\zeta\zeta\zeta} \in L_2(0,l), \phi(0) = 0, \phi_\zeta(0) = 0\}$; $\mathscr{B}_1$ and $\mathscr{B}_2$ are boundary operators,

$$\mathscr{B}_1 \phi(\zeta) = \frac{d\phi}{d\zeta}(l) \quad \mathscr{B}_2 \phi(\zeta) = \phi(l) \tag{13}$$

with the domain $\mathscr{D}(\mathscr{A}) \subseteq \mathscr{D}(\mathscr{B}_1), \mathscr{D}(\mathscr{A}) \subseteq \mathscr{D}(\mathscr{B}_2)$, where $\mathscr{D}(\mathscr{B}_1) = L_2(0,l)$ and $\mathscr{D}(\mathscr{B}_2) = L_2(0,l)$.

To this end, we define an associate operator $\hat{\mathscr{A}}$ as:

$$\begin{aligned}
&\hat{\mathscr{A}}\phi(\zeta) = \mathscr{A}\phi(\zeta)\\
&\mathscr{D}(\hat{\mathscr{A}}) = \mathscr{D}(\mathscr{A}) \oplus ker(\mathscr{B}_1) \oplus ker(\mathscr{B}_2) = \{\phi(\zeta) \in L_2(0,l)|\\
&\phi, \phi_\zeta, \phi_{\zeta\zeta}, \phi_{\zeta\zeta\zeta} a.c., \hat{\mathscr{A}}\phi(\zeta) \in L_2(0,l),\\
&\phi(0) = 0 = \phi(l), \phi_\zeta(0) = 0 = \phi_\zeta(l)\}
\end{aligned}$$

and explore it's features. A complete characterization of $\hat{\mathscr{A}}$ requires solution to the following eigenvalue problem:

$$\hat{\mathscr{A}}\phi = \lambda\phi \tag{14}$$

Its characteristic equation is $vs^4 + s^2 = \lambda$. We assume that the parameter $v > 0$, however it is not difficult to extend the proposed approach to the case when $v < 0$, so that:

$$\begin{aligned}
&vs^4 + s^2 = \lambda\\
&\rightarrow (s^2 + \frac{1}{2v})^2 - \frac{\lambda}{v} - \frac{1}{4v^2} = 0
\end{aligned} \tag{15}$$

The operator $\hat{\mathscr{A}}$ is self-adjoint operator, and domain of λ is given as $[-\frac{1}{4v}, +\infty)$.

From the equation Eq.15, we consider two different cases. First, for $\lambda^+ > 0$:

$$s_1 = \sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} - \frac{1}{2v}} = \alpha$$

$$s_2 = -\sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} - \frac{1}{2v}} = -\alpha$$

$$s_3 = j\sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = j\beta$$

$$s_4 = -j\sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = -j\beta$$

where $\alpha > 0$, $\beta > 0$ are real numbers.

The associate eigenfunction is:

$$\phi^+(\zeta) = C_1 e^{\alpha\zeta} + C_2 e^{-\alpha\zeta} + C_3 cos(\beta\zeta) + C_4 sin(\beta\zeta)$$

The application of boundary conditions renders the following equations:

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ C_1 e^{\alpha l} + C_2 e^{-\alpha l} + C_3 cos(\beta l) + C_4 sin(\beta l) = 0 \\ \alpha C_1 - \alpha C_2 + \beta C_4 = 0 \\ C_1 \alpha e^{\alpha l} - C_2 \alpha e^{-\alpha l} - C_3 \beta sin(\beta l) + C_4 \beta cos(\beta l) = 0 \end{cases}$$

and in order to obtain nonzero solution to above equations, the following is required:

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ e^{\alpha l} & e^{-\alpha l} & cos(\beta l) & sin(\beta l) \\ \alpha & -\alpha & 0 & \beta \\ \alpha e^{\alpha l} & -\alpha e^{-\alpha l} & -\beta sin(\beta l) & \beta cos(\beta l) \end{vmatrix} = 0$$

Finally, we obtain:

$$2\beta - \beta cos(\beta l)(e^{-\alpha l} + e^{\alpha l}) + \frac{sin(\beta l)}{2v\alpha}(e^{-\alpha l} - e^{\alpha l}) = 0$$

for $\beta^2 - \alpha^2 = \frac{1}{v}, \quad \beta > \alpha > 0$, which can be solved numerically.

For the case $-\frac{1}{4v} < \lambda^- < 0$:

$$s_1 = j\sqrt{-\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = j\gamma$$

$$s_2 = -j\sqrt{-\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = -j\gamma$$

$$s_3 = j\sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = j\rho$$

$$s_4 = -j\sqrt{\sqrt{\frac{\lambda}{v} + \frac{1}{4v^2}} + \frac{1}{2v}} = -j\rho$$

where $\gamma > 0$, $\rho > 0$ are real numbers.

The associated eigenfunctions are:

$\phi^-(\zeta) = C_1 cos(\gamma\zeta) + C_2 sin(\gamma\zeta) + C_3 cos(\rho\zeta) + C_4 sin(\rho\zeta)$

and the application of boundary conditions renders the following equations:

$$\begin{cases} C_1 + C_3 = 0 \\ C_1 cos(\gamma l) + C_2 sin(\gamma l) + C_3 cos(\rho l) + C_4 sin(\rho l) = 0 \\ \gamma C_2 + C_4 \rho = 0 \\ -\gamma C_1 sin\gamma l + \gamma C_2 cos(\gamma l) - C_3 \rho sin(\rho l) + C_4 \rho cos(\rho l) = 0 \end{cases}$$

In addition, there is $\gamma^2 + \rho^2 = \frac{1}{v}$. Similarly, to ensure the $C_1 \sim C_4$ has nonzero solution, the determinant of the coefficient matrix of the above equations is set to zero:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ cos\gamma l & sin\gamma l & cos\rho l & sin\rho l \\ 0 & \gamma & 0 & \rho \\ -\gamma sin\gamma l & \gamma cos\gamma l & -\rho sin\rho l & \rho cos\rho l \end{vmatrix} = 0$$

Simplify the above equation, we obtain:

$$2\gamma\rho - \frac{sin(\gamma l)sin(\rho l)}{v} - 2\rho\gamma cos(\rho l)cos(\gamma l) = 0 \quad (16)$$

for $\gamma^2 + \rho^2 = \frac{1}{v}, \quad \rho > \gamma > 0$, which can be solved numerically. Once α, β, γ and ρ are yielded, the coefficients C_i are obtained and used to determine the corresponding eigenfunctions:

$$\phi^+(\zeta) = \kappa^+[\frac{sin(\beta l)/v - 2\alpha\beta e^{-\alpha} + 2\alpha\beta cos(\beta l)}{2\alpha(\alpha e^{-\alpha l} - \beta sin(\beta l) - \alpha cos(\beta l))} e^{\alpha\zeta} - \frac{(\alpha^2 + \beta^2) sin(\beta l)}{2\alpha(\alpha e^{-\alpha l} - \beta sin(\beta l) - \alpha cos(\beta l))} e^{-\alpha\zeta} - \frac{-\beta e^{-\alpha l} + \beta cos(\beta l) - \alpha sin(\beta l)}{\alpha e^{-\alpha l} - \beta sin(\beta l) - \alpha cos(\beta l)} cos(\beta\zeta) + sin(\beta\zeta)]$$

$$\phi^-(\zeta) = \kappa^-[cos(\gamma\zeta) - sin(\gamma\zeta)\frac{\rho}{\gamma}\frac{cos(\gamma l) - cos(\rho l)}{sin(\gamma l)\frac{\rho}{\gamma} - sin(\rho l)} - cos(\rho\zeta) + sin(\rho\zeta)\frac{cos(\gamma l) - cos(\rho l)}{sin(\gamma l)\frac{\rho}{\gamma} - sin(\rho l)}]$$

where κ is the normalizing coefficient to guarantee $\int_0^l \phi^2(\zeta)d\zeta = 1$. Finally, the eigenvalues and eigenfunctions of $\hat{\mathscr{A}}$ are $\lambda(\hat{\mathscr{A}}) = \{\lambda^+, \lambda^-\}$ and $\{\phi^+(\zeta), \phi^-(\zeta)\}$, respectively.

Remark 1: One can show that $\lambda = -\frac{1}{4v}$ is the eigenvalue providing that $sin^2(\sqrt{\frac{1}{2v}}l) = \frac{l^2}{2v}$ and that $\lambda = 0$ is the eigenvalue under the condition $\frac{2cos(\sqrt{1/v}l)-2}{\sqrt{v}} + \frac{lsin(\sqrt{v}l)}{v} = 0$. The procedure is similar with above derivation, thus it is omitted here.

B. Boundary transformation

In order to reformulate the original linearized K-S Eq.9 to the abstract boundary problem, given by Eq.10, the following state transformation is used:

$$p(\zeta, t) = x(\zeta, t) - B_1(\zeta)u - B_2(\zeta)\mu \quad (17)$$

To ensure $p \in \mathscr{D}(\bar{\mathscr{A}})$, let $\zeta = 0$, there is $p(0) = x(0) - B_1(0)u - B_2(0)\mu$. Thus, $B_1(0) = 0, B_2(0) = 0$ are imposed to guarantee $p(0) = x(0) = 0$. In addition, with $B_{1\zeta}(0) = 0$ and $B_{2\zeta}(0) = 0$, there is $p_\zeta(l) = x_\zeta(l) - B_{1\zeta}(l)u - B_{2\zeta}(l)\mu = 0$. Given $\mathscr{B}_1 p = u - \mathscr{B}_1 B_1 u - \mathscr{B}_1 B_2 \mu$,

$$\mathscr{B}_1 B_1 = 1, \mathscr{B}_1 B_2 = 0 \rightarrow \mathscr{B}_1 p = 0 \quad (18)$$

Also, consider that $\mathscr{B}_2 p = \mu - \mathscr{B}_2 B_1 u - \mathscr{B}_2 B_2 \mu$,

$$\mathscr{B}_2 B_1 = 0, \mathscr{B}_2 B_2 = 1 \rightarrow \mathscr{B}_2 p = 0 \quad (19)$$

Hence, there should be $B_1(l) = 0, B_2(l) = 1, B_{1\zeta}(l) = 1, B_{2\zeta}(l) = 0$.

We substitute equation Eq.17 into the linearized model of K-S equation Eq.9 to obtain the process dynamic in the form of transformed state:

$$\dot{p}(t) = -\hat{\mathscr{A}}p(t) - \hat{\mathscr{A}}B_1 u(t) - \hat{\mathscr{A}}B_2\mu(t) - B_1\dot{u}(t) - B_2\dot{\mu}(t) \quad (20)$$

In order to benefit from the MPC synthesis, the state space model with decoupled modal states dynamics is sought, which implies that functions $B_1(\zeta)$ and $B_2(\zeta)$ are constructed based on the following two conditions: $\hat{\mathscr{A}}B_1 = 0$ and $\hat{\mathscr{A}}B_2 = 0$ associated with four boundary conditions. The

$B_1(\zeta)$ and $B_2(\zeta)$ can be explicitly solved from the following forth order PDEs:

$$vB_1(\zeta)_{\zeta\zeta\zeta\zeta} + B_1(\zeta)_{\zeta\zeta} = 0 \quad (21)$$
$$B_1(0) = 0, B_{1\zeta}(0) = 0, B_1(l) = 0, B_{1\zeta}(l) = 1 \quad (22)$$
$$vB_2(\zeta)_{\zeta\zeta\zeta\zeta} + B_2(\zeta)_{\zeta\zeta} = 0 \quad (23)$$
$$B_2(0) = 0, B_{2\zeta}(0) = 0, B_2(l) = 1, B_{2\zeta}(l) = 0 \quad (24)$$

Therefore, $B_1(\zeta)$ and $B_2(\zeta)$ are designed as:

$$\begin{aligned} B_1(\zeta) = &- \frac{sin(l\theta) - l\theta}{2\theta cos(l\theta) - 2\theta + l\theta^2 sin(l\theta)} \\ &+ \frac{2\theta sin^2(l\theta/2)}{4\theta sin^2(l\theta/2) - l\theta^2 sin(l\theta)}\zeta \\ &+ \frac{sin(l\theta) - l\theta}{2\theta cos(l\theta) - 2\theta + l\theta^2 sin(l\theta)} cos(\theta\zeta) \\ &- \frac{2sin^2(l\theta/2)}{4\theta sin^2(l\theta/2) - l\theta^2 sin(l\theta)} sin(\theta\zeta) \end{aligned} \quad (25)$$

$$\begin{aligned} B_2(\zeta) = &\frac{\cos(l\theta) - 1}{l\theta \sin(l\theta) - 2 + 2\cos(l\theta)} \\ &+ \frac{\theta \sin(l\theta)}{l\theta \sin(l\theta) - 2 + 2\cos(l\theta)}\zeta \\ &- \frac{\cos(l\theta) - 1}{l\theta \sin(l\theta) - 2 + 2\cos(l\theta)} cos(\theta\zeta) \\ &- \frac{\sin(l\theta)}{l\theta \sin(l\theta) - 2 + 2\cos(l\theta)} sin(\theta\zeta) \end{aligned} \quad (26)$$

where $\theta = \sqrt{1/v}$.
Therefore, the Eq.20 becomes:

$$\dot{p}(t) = -\hat{\mathscr{A}}p(t) - B_1(\zeta)\dot{u}(t) - B_2(\zeta)\dot{\mu}(t) \quad (27)$$

Since $p \in \mathscr{D}(\hat{\mathscr{A}})$, it can be represented by the linear combination of eigenfunctions of the operator $\hat{\mathscr{A}}$, namely, $p(\zeta, t) = \sum_{i=1}^{\infty} a_i(t)\phi_i(\zeta)$. Taking this modal representation into Eq.27,

$$\sum_{i=1}^{\infty} \dot{a}_i(t)\phi_i(\zeta) = -\hat{\mathscr{A}} \sum_{i=1}^{\infty} a_i(t)\phi_i(\zeta) - B_1\dot{u}(t) - B_2\dot{\mu}(t) \quad (28)$$

and projecting Eq.28 on the $\mathcal{H}$ spanned by the eigenfunctions of $\hat{\mathscr{A}}$, one obtains:

$$\dot{a}_i(t) = -\lambda_i a_i(t) - \int_0^l \phi_i(\zeta)B_1 d\zeta\dot{u}(t) - \int_0^l \phi_i(\zeta)B_2 d\zeta\dot{\mu}(t) \quad (29)$$

which leads to the infinite state space model representation:

$$\begin{aligned} \dot{a}(t) = &\begin{bmatrix} -\lambda_1 & 0 & \dots \\ 0 & -\lambda_2 & \dots \\ \dots & \dots & \ddots \end{bmatrix} a(t) - \begin{bmatrix} \int_0^l \phi_1 B_1 d\zeta \\ \int_0^l \phi_2 B_1 d\zeta \\ \vdots \end{bmatrix} \dot{u}(t) - \quad (30) \\ &\begin{bmatrix} \int_0^l \phi_1 B_2 d\zeta \\ \int_0^l \phi_2 B_2 d\zeta \\ \vdots \end{bmatrix} \dot{\mu}(t) \\ = &\Lambda a(t) + \Psi_1\dot{u}(t) + \Psi_2\dot{\mu}(t) \quad (31) \end{aligned}$$

where $a(t) = [a_1(t), a_2(t), \dots]^T$.
Based on the boundary transformation given by $B_1(\zeta)$ and $B_2(\zeta)$, the K-S equation is transferred into the infinite-dimensional state space model which is characterized with decoupled state dynamics. This model representation is then used as model in the MPC synthesis.

C. The Model Predictive Control synthesis for K-S PDE

In order to synthesize the model predictive controller that copes with the infinite-dimensional nature of the K-S PDE, one needs to explore the features of the K-S modal representation and to account for the finite computing capabilities of the optimization algorithms. The feasible way to achieve this goal is to decompose PDE into two subspaces: the slow modal subspace, whose eigenspace is given by diagonal eigenvalue matrix denoted by $\bar{\Lambda}$ and associated with corresponding $\phi_i(\zeta)$, including possibly unstable and stable eigenmodes (λ) distributed close to the imaginary axis, and fast modal subspace associated with all high frequency modes which are inherently stable and bounded. In the modal MPC formulation, the novel feature is that only the slow modal states are considered to contribute to the optimization of the control performance given in terms of the objective function. The fast modal states are incorporated only in the states inequality constraints to guarantee that the full K-S PDE state does not violate any bounds. This approximation scheme cast as the quadratic optimization in the infinite dimensional limit, if feasible, can achieve system stabilization by stabilizing the finite number of unstable modes and ensure satisfaction of input and PDE state constraints.
We define the the finite dimensional approximation of the infinite dimensional state as $\bar{p} = \sum_{i=1}^{k} a_i\phi_i$, where slow modal states are denoted as $\bar{a} = [a_1, a_2, \dots, a_k]^T$, while the fast modal are $\tilde{a} = [a_{k+1}, a_{k+2}, \dots]^T$ and inputs are denoted as $\bar{u}$, $\bar{\mu}$. We consider only $\bar{a}$ and its corresponding output:

$$\dot{\bar{a}} = \bar{\Lambda}_k\bar{a} + \bar{\Psi}_{1k}\bar{u} + \bar{\Psi}_{2k}\bar{\mu} \quad (32)$$
$$\dot{u} = \bar{u} \quad (33)$$
$$\dot{\mu} = \bar{\mu} \quad (34)$$

$$\begin{aligned} \bar{y} = c\bar{x} = c(\bar{p} + B_1 u + B_2\mu) = c\sum_{i=1}^{k} a_i\phi_i + cB_1 u + cB_2\mu \\ = \begin{bmatrix} cB_1 & cB_2 & c\bar{\Phi}^T \end{bmatrix} \begin{bmatrix} u \\ \mu \\ \bar{a} \end{bmatrix} \end{aligned} \quad (35)$$

where $c\bar{\Phi} = [\phi_1(\zeta_c), \phi_2(\zeta_c), \dots, \phi_k(\zeta_c)]$, $cB_1 = B_1(\zeta_c)$ and $cB_2 = B_2(\zeta_c)$.
Since the discrete time system representation is required in the MPC implementation, we rewrite the considered evolutionary system Eqs.32-33-34. Namely, let the function $\Theta(\bar{a}, t, \bar{u}, \bar{\mu})$ be the solution of system Eqs.32-33-34 with inputs $\bar{u}$, $\bar{\mu}$ and initial condition $\bar{a}(0)$. Then, we define the sampled data system with time interval Δ, such that

$$\bar{A}_{q+1} := \Theta(\bar{a}, \Delta, \bar{u}_q, \bar{\mu}_q) \quad (36)$$

where $\bar{A}_{q+1}$ is the sampled modal vector $\bar{a}$ at $(q+1)\Delta$; discrete control input rate $\bar{u}_q$ and $\bar{\mu}_q$ are defined as $\bar{u}_q :=$

$\bar{u}(t)|_{[q\Delta,(q+1)\Delta]}$ and $\bar{\mu}_q := \bar{\mu}(t)|_{[q\Delta,(q+1)\Delta]}$. Similarly, the discrete input u_q and μ_q are also defined as $u_q := u(t)|_{[q\Delta,(q+1)\Delta]}$ and $\mu_q := \mu(t)|_{[q\Delta,(q+1)\Delta]}$. Consequently, the output samples $\bar{y}_q$ at time Δq can be determined according to Eq.35, $\bar{A}_{q+1}$, u_{q+1} and μ_{q+1}. Finally, we construct the quadratic objective function which takes the cost of inputs and the deviation from the set-point in the sampling time instances into account:

$$\begin{aligned}&\min_{\bar{u},\bar{\mu}} \sum_{q=0}^{N} \bar{y}_q^T Q_q \bar{y}_q + \sum_{q=0}^{N-1} [\bar{u}_q^T R_q \bar{u}_q + \bar{\mu}_q^T M_q \bar{\mu}_q] \\ &= \sum_{q=0}^{N} \begin{bmatrix} u_q & \mu_q & \bar{A}_q^T \end{bmatrix} \begin{bmatrix} (cB_1)^T \\ (cB_2)^T \\ (c\bar{\Phi}^T)^T \end{bmatrix} Q_q \begin{bmatrix} cB_1 & cB_2 & c\bar{\Phi}^T \end{bmatrix} \begin{bmatrix} u_q \\ \mu_q \\ \bar{A}_q \end{bmatrix} \\ &+ \sum_{q=0}^{N-1} [\bar{u}_q^T R_q \bar{u}_q + \bar{\mu}_q^T M_q \bar{u}_q] \end{aligned} \quad (37)$$

The set-point of $\bar{y}$ is specified as $\bar{y} = 0$, which is the spatially uniform state of K-S PDE representing the stable thin film thickness. To simplify the notation, the following variables are specified. The inputs and input rates over the control horizon are $U_N = [u_0, u_1, u_2, \ldots, u_{N-1}]^T$, $\Xi_N = [\mu_0, \mu_1, \ldots, \mu_{N-1}]^T$, $\bar{U}_N = [\bar{u}_0, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{N-1}]^T$ and $\bar{\Xi}_N = [\bar{\mu}_0, \bar{\mu}_1, \ldots, \bar{\mu}_{N-1}]^T$. The weight matrix related to the output is $\mathscr{Q}_q = \begin{bmatrix} cB_1 & cB_2 & c\Phi^T \end{bmatrix}^T Q_q \begin{bmatrix} cB_1 & cB_2 & c\Phi^T \end{bmatrix}$.

It can be deduced that the Q_N represents the penalty associated with the terminal penalty, which is a cost obtained as a bound of the infinite horizon state evolution. Weight matrices $\mathscr{R} = diag\,[R_0, R_1, \ldots, R_{N-1}]$ and $\mathscr{M} = diag\,[M_0, M_1, \ldots, M_{N-1}]$ are used to measure the cost of regulation effort. Note that the decision variable in the optimization is $\bar{u}$ and $\bar{\mu}$, and therefore it is necessary to express $\bar{y}_q$ as the function of $\bar{u}$ and $\bar{\mu}$ explicitly. Although there are many approaches for discretization, we select the finite difference for simplicity, so that without loss of accuracy, we obtain:

$$u_q = u_0 + \Delta \sum_{i=0}^{q-1} \bar{u}_i = u_0 + W_q \bar{U}_N \quad (38)$$

$$\mu_q = \mu_0 + \Delta \sum_{i=0}^{q-1} \bar{\mu}_i = \mu_0 + W_q \bar{\Xi}_N \quad (39)$$

where sampling time interval Δ is selected as $\frac{1}{max|\bar{\Lambda}|}$, and $W_q = [\underbrace{\Delta, \Delta, \ldots, \Delta}_{q}, 0, \ldots, 0]$. For the purpose of discrete system implementation of Eq.36, the $\bar{A}_q$ also can be expressed as the linear function of $\bar{u}$ and $\bar{\mu}$:

$$\begin{aligned}\bar{A}_q &= \bar{\Lambda}_d \bar{A}_{q-1} + \bar{\Psi}_{1d} \bar{u}_{q-1} + \bar{\Psi}_{2d} \bar{\mu}_{q-1} \\ &= \bar{\Lambda}_d^q \bar{A}_0 + \sum_{r=1}^{q} \bar{\Lambda}_d^{q-r} \bar{\Psi}_{1d} \bar{u}_{r-1} + \sum_{r=1}^{q} \bar{\Lambda}_d^{q-r} \bar{\Psi}_{2d} \bar{\mu}_{r-1} \\ &= \bar{\Lambda}_d^q \bar{A}_0 + \Omega_q \bar{U}_N + \Pi_q \bar{\Xi}_N \end{aligned} \quad (40)$$

where $\bar{\Lambda}_d, \bar{\Psi}_{1d}, \bar{\Psi}_{2d}$ are the discrete system matrices. $\Omega_q = [\bar{\Lambda}_d^{q-1} \bar{\Psi}_{1d}, \bar{\Lambda}_d^{q-2} \bar{\Psi}_{1d}, \ldots, \bar{\Psi}_{1d}, 0, \ldots, 0]$ and $\Pi_q = [\bar{\Lambda}_d^{q-1} \bar{\Psi}_{2d}, \Lambda^{q-2} \bar{\Psi}_{2\underline{d}}, \ldots, \bar{\Psi}_{2d}, 0, \ldots, 0]$. Then, we can substitute u_q, μ_q and $\bar{A}_q$ into (37), which yields:

$$\begin{aligned}&\min_{\bar{u},\bar{\mu}} \quad \sum_{q=0}^{N} \bar{y}_q^T Q_q \bar{y}_q + \sum_{q=0}^{N-1} [\bar{u}_q^T R_q \bar{u}_q + \bar{\mu}_q^T R_q \bar{\mu}_q] \\ &= \sum_{q=0}^{N} \{ \begin{bmatrix} u_0^T & \mu_0^T & \bar{A}_0^T (\bar{\Lambda}_d^T)^q \end{bmatrix} + \begin{bmatrix} \bar{U}_N^T & \bar{\Xi}_N^T \end{bmatrix} \begin{bmatrix} W_q^T & 0 & \Omega_q^T \\ 0 & W_q^T & \Pi_q^T \end{bmatrix} \} \\ &\times \mathscr{Q}_q \{ \begin{bmatrix} u_0 \\ \mu_0 \\ \bar{\Lambda}_d^q \bar{A}_0 \end{bmatrix} + \begin{bmatrix} W_q & 0 \\ 0 & W_q \\ \Omega_q & \Pi_q \end{bmatrix} \begin{bmatrix} \bar{U}_N \\ \bar{\Xi}_N \end{bmatrix} \} + \sum_{q=0}^{N-1} \bar{u}_q^T R_q \bar{u}_q \\ &+ \sum_{q=0}^{N-1} \bar{\mu}_q^T M_q \bar{\mu}_q \\ &= \begin{bmatrix} \bar{U}_N^T & \bar{\Xi}_N^T \end{bmatrix} (\sum_{q=0}^{N} \begin{bmatrix} W_q^T & 0 & \Omega_q^T \\ 0 & W_q^T & \Pi_q^T \end{bmatrix} \mathscr{Q}_q \begin{bmatrix} W_q & 0 \\ 0 & W_q \\ \Omega_q & \Pi_q \end{bmatrix} \\ &+ \begin{bmatrix} \mathscr{R} & 0 \\ 0 & \mathscr{M} \end{bmatrix}) \begin{bmatrix} \bar{U}_N \\ \bar{\Xi}_N \end{bmatrix} + 2 \begin{bmatrix} \bar{U}_N^T & \bar{\Xi}_N^T \end{bmatrix} \sum_{q=0}^{N} \begin{bmatrix} W_q^T & 0 & \Omega_q^T \\ 0 & W_q^T & \Pi_q^T \end{bmatrix} \\ &\times \mathscr{Q}_q \begin{bmatrix} u_0 \\ \mu_0 \\ \bar{\Lambda}_d^q \bar{A}_0 \end{bmatrix} + \sum_{q=0}^{N} \begin{bmatrix} u_0^T & \mu_0^T & \bar{A}_0 (\bar{\Lambda}_d^T)^q \end{bmatrix} \mathscr{Q}_q \begin{bmatrix} u_0^T \\ \mu_0^T \\ \bar{A}_0 (\bar{\Lambda}_d^T)^q \end{bmatrix} \end{aligned} \quad (41)$$

Apart from the objective function, another essential element of MPC realization is the explicit account for input and state constraints, given by Eqs.6−8. The input constraints are expressed as the constraints on $\bar{u}$ and $\bar{\mu}$:

$$\begin{aligned}&y_{min} - \underline{Y} \leqslant c\Phi^T (\bar{\Lambda}_d^q \bar{A}_0 + \Omega_q \bar{U}_N + \Pi_q \bar{\Xi}_N) \\ &+ cB_1 (u_0 + W_q \bar{U}_N) + cB_2 (\mu_0 + W_q \bar{\Xi}_N) \leqslant y_{max} - \overline{Y} \end{aligned} \quad (42)$$

$$U_{min} \leqslant U^0 + W\bar{U}_N \leqslant U_{max} \quad (43)$$

$$\Xi_{min} \leqslant \Xi^0 + W\bar{\Xi}_N \leqslant \Xi_{max} \quad (44)$$

where $U^0 = [\underbrace{u_0, u_0, \ldots, u_0}_{N}]^T$ and $\Xi^0 = [\underbrace{\mu_0, \mu_0, \ldots, \mu_0}_{N}]^T$. The $\overline{Y}$ and $\underline{Y}$ are the upper and lower bound of the outputs corresponding to the fast modal states evolution. Namely, the infinite dimensional fast modal states evolution plays the role of the slack variable in the constrained quadratic programming optimization. The fast modal states are inherently stable and coupled to the slow modes only through the input, so that due to the constrained input the fast modal state evolution can be expressed in some appropriate norm $\|\tilde{a}\|_p = \overline{Y}$, where $p = 1, 2, \infty$. Since the slow modal subspace has included all possibly unstable states, the stability of the process can be ensured by the terminal equality constraints, cast as $u_N = 0$, $\mu_N = 0$, $\lambda_{unstable} := \sigma(\bar{A}_N) = 0$, which requires that unstable modes are zero at the end of the horizon, and the resulting constrained optimization problem can be made feasible by enlarging the control horizon N appropriately.

$$u_0 + W_N \bar{u}_N = 0 \quad (45)$$

$$\mu_0 + W_N \bar{\mu}_N = 0 \quad (46)$$

$$\bar{\Lambda}_d^N \bar{A}_0 + \Omega_N \bar{U}_N + \Pi_N \bar{\Xi}_N = 0 \quad (47)$$

To summarize, the MPC formulation is:

$$\min_{\bar{U}_N,\bar{\Xi}_N} \begin{bmatrix}\bar{U}_N^T & \bar{\Xi}_N^T\end{bmatrix} (\sum_{q=0}^{N} \begin{bmatrix}\mathscr{R} & 0\\ 0 & \mathscr{M}\end{bmatrix} + \begin{bmatrix}W_q^T & 0 & \Omega_q^T\\ 0 & W_q^T & \Pi_q^T\end{bmatrix}$$
$$\times \mathscr{Q}_q \begin{bmatrix}W_q & 0\\ 0 & W_q\\ \Omega_q & \Pi_q\end{bmatrix}) \begin{bmatrix}\bar{U}_N\\ \bar{\Xi}_N\end{bmatrix} + 2\begin{bmatrix}\bar{U}_N^T & \bar{\Xi}_N^T\end{bmatrix}$$
$$\times \sum_{q=0}^{N} \begin{bmatrix}W_q^T & 0 & \Omega_q^T\\ 0 & W_q^T & \Pi_q^T\end{bmatrix} \mathscr{Q}_q \begin{bmatrix}u_0\\ \mu_0\\ \bar{\Lambda}_d^q \bar{A}_0\end{bmatrix} \quad (48)$$

$$\text{s.t.}\quad u_0 + W_N \bar{U}_N = 0 \quad (49)$$
$$\mu_0 + W_N \bar{\mu}_N = 0 \quad (50)$$
$$\bar{\Lambda}_d^N \bar{A}_0 + \Omega_N \bar{U}_N + \Pi_N \bar{\Xi}_N = 0 \quad (51)$$
$$y_{min} - \underline{Y} \leqslant c\Phi^T(\bar{\Lambda}_d^q \bar{A}_0 + \Omega_q \bar{U}_N + \Pi_q \bar{\Xi}_N)$$
$$+ cB_1(u_0 + W_q \bar{U}_N) + cB_2(\mu_0 + W_q \bar{\Xi}_N) \leqslant y_{max} - \overline{Y} \quad (52)$$
$$U_{min} \leqslant U^0 + W\bar{U}_N \leqslant U_{max} \quad (53)$$
$$\Xi_{min} \leqslant \Xi^0 + W\bar{\Xi}_N \leqslant \Xi_{max} \quad (54)$$

Remark 2: Note that the third term of Eq.41 is ignored in the MPC because it represents the state initial condition contribution to the cost and therefore doesn't affect the calculation of $\bar{u}$ or $\bar{\mu}$.

IV. Simulation Study

The K-S equation (1) and (2) with parameter $v = 0.7$, $l = 9$ is considered in the simulation study. By solving the eigenvalue problems, the slow modal states with $\lambda_1 = -0.1294$, $\lambda_2 = -0.1841$ and $\lambda_3 = 0.3487$, are selected for the state evolution in the MPC formulation. As it can be seen in the slow subspace there are one unstable and two stable modes, which in the extended formulation for the MPC design are augmented by finite subspace arising from $u(t)$ and $\mu(t)$ Eqs.33-34. The fast modal incorporates 5 states, whose corresponding eigenvalues are all stable. The sampling time is specified as 0.0217s, selected as mentioned before. For the MPC, let $R_t = 0.01$, $M_t = 1$ and $Q_t = 1$ and locate the sensor in the position $\zeta_c = 7$. The control horizon is set $N = 180$, which guarantees the feasibility and performance. The parameters in output and input constraints are $y \in [y_{min}, y_{max}] = [-5, 5]$, $u \in [u_{min}, u_{max}] = [-2, 10]$, $\mu \in [\mu_{min}, \mu_{max}] = [-10, 10]$. The initial conditions for the modal states are $a_1 = 0.1$, $a_2 = 0.2$ and $a_3 = 0.15$, and all other modes are initialized at zero. Along the spatial coordinate, 800 points are selects to show the evolution of the K-S equation. The Fig.2 demonstrates successful stabilization of the K-S equation with MPC given by Eqs.48-54. The Fig.3 shows that the output $y(t)$, which reflects that the K-S PDE state at $\zeta = 7$, does not violate imposed performance constraints. The Figs.4-5 demonstrate the control input $u(t)$ and $\mu(t)$ evolutions. Due to particular constraints in the MPC formulation, one can see that both of inputs are active in the early stage, which implies that the MPC is likely to use extreme regulation to drive the process towards the set-point.

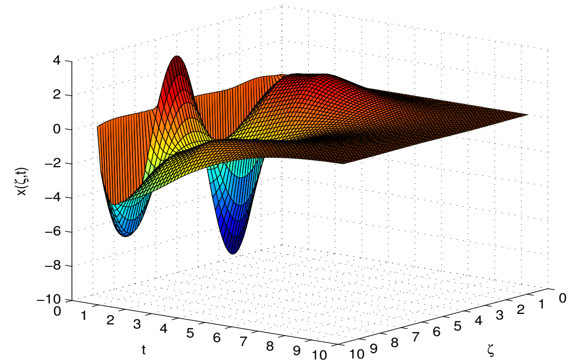

Fig. 2. State evolution of the K-S PDE under the MPC control law given by Eqs.48-54.

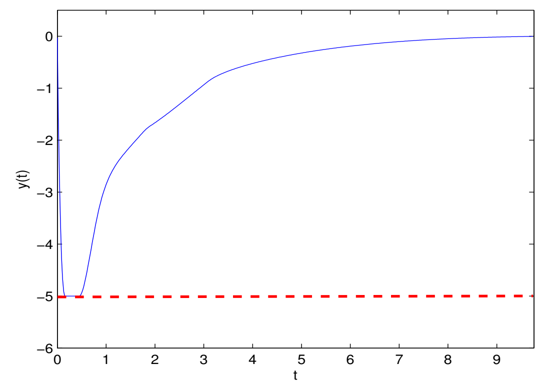

Fig. 3. The output evolution of the state $y(\zeta_c)$ at $\zeta_c = 7$ under the MPC control law given by Eqs.48-54.

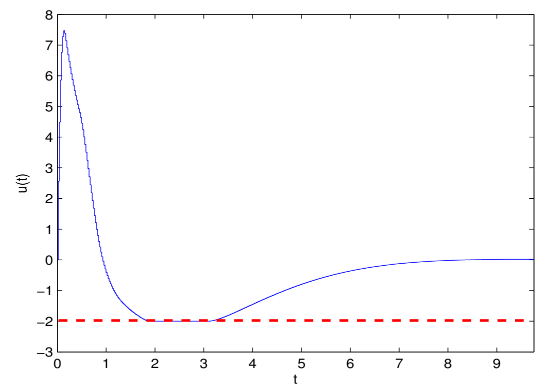

Fig. 4. The input evolution $u(t)$ under the MPC control law given by Eqs.48-54.

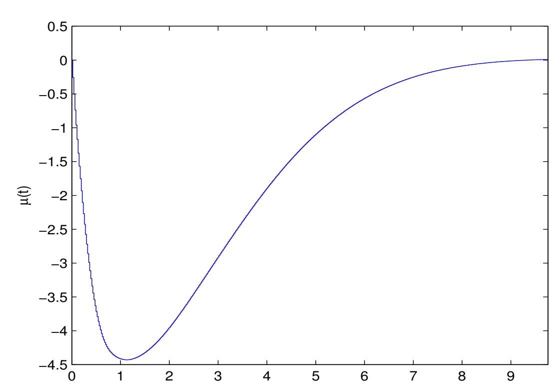

Fig. 5. The input $\mu(t)$ under the MPC control law given by Eqs.48-54.

V. Conclusion

In this work, a boundary model predictive control scheme is proposed to regulate the thin film thickness in the falling film process, modelled by the K-S equation. The original PDE system is transferred to the infinite dimensional state space representation and a reduction technique is employed to obtain a finite dimensional model, which in the limit satisfies the infinite dimensional PDE state constraints. The MPC formulation is developed, explicitly incorporating output, input and terminal constraints to guarantee the stability and performance satisfaction. The simulation study demonstrates the effectiveness of the algorithm.

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