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# Controllability and Observability of Linear Dynamic Systems Revisited

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*Abstract*— **A modification of well-known controllability and observability tests using Lyapunov equations and Gramians yields a reliable tool for controllability and observability testing for linear continuous and discrete time systems, as well as an alternative procedure to determine controllable, observable and minimal system realizations.**

## I. INTRODUCTION

IN this contribution, it is shown that a modification to well known controllability and observability tests for linear stable continuous time systems using controllability and observability Gramians results in a practical tool for controllability and observability verification. Based thereon, a procedure for determining minimal system realizations for linear systems without restriction to their nature (continuous or discrete) or stability is also given.

Time invariant continuous time and discrete time linear systems in state space representation will be considered. A time invariant linear continuous time system will be described by a continuous state equation and an output equation, as in (1).

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

A time invariant linear discrete time system will be described by a discrete state equation and an output equation, as in (2).

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2}$$

Controllability and observability of (1) and (2) or, alternatively, the controllability of the pair (*A*, *B*) and the observability of the pair (*A*, *C*) are established and widely used concepts (e.g. see [1] or [2]).

## II. REVIEW OF A FEW CONTROLLABILITY AND OBSERVABILITY CONDITIONS

### A. Controllability

The following are well known necessary and sufficient conditions for state controllability [2].

*Condition 1*: The pair (*A*, *B*) is controllable if and only if the controllability matrix

$$C_c(A,B) = [B \quad AB \quad \dots \quad A^{n-1}B]$$

has full row rank.

*Condition 2*: If a continuous-time systems described by (1) is asymptotically stable, then it is controllable if and only if the solution $P_c$ of the Lyapunov equation

$$AP_c + P_c A^T = -BB^T$$

is positive definite, and thus has full rank. $P_c$ is also called the controllability Gramian of (1).

### B. Observability

The following are well known necessary and sufficient conditions for state observability [2].

*Condition 3*: The pair (*A*, *C*) is observable if and only if the observability matrix

$$O_o(A,C) = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$$

has full column rank.

*Condition 4*: The pair (*A*, *C*) is observable if and only if

$$Av_i = \lambda_i v_i, Cv_i \neq 0,$$

for all eigenvector / eigenvalue pairs $(v_i, \lambda_i)$ of *A*.

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### *C. Relationship between controllability and observability conditions*

$C_c(A,B) = O_o(A^T, B^T)^T$ , $O_o(A,C) = C_c(A^T, C^T)^T$ . Thus the following two necessary and sufficient conditions hold.

*Condition 5*: The pair ($A$, $B$) is controllable if and only if

$$A^T p_i = \lambda_i p_i, B^T p_i \neq 0,$$

for all eigenvector / eigenvalue pairs $(p_i, \lambda_i)$ of $A^T$.

*Condition 6*: If a continuous-time systems described by (1) is asymptotically stable, then it is observable if and only if the solution $P_o$ of the Lyapunov equation

$$A^T P_o + P_o A = -C^T C$$

is positive definite, and thus has full rank. $P_o$ is also called the observability Gramian of (1).

## III. MODIFIED CONTROLLABILITY AND OBSERVABILITY CONDITIONS

The modification proposal herein is motivated by the observation that the eigenvalues of $A - \alpha I$ for any real scalar $\alpha$ are those of $A$ subtracted by $\alpha$, while the eigenvectors remain those of $A$. This means that: (a) because of Condition 5, the controllability of ($A$, $B$) and that of ($A - \alpha I$, $B$) are the same; (b) because of Condition 4, the observability of ($A$, $C$) and that of ($A - \alpha I$, C) are the same. Hence, if one chooses $\alpha$ such that all eigenvalues of $A - \alpha I$ lie in the open complex left half plane, Lyapunov equation based controllability and observability conditions can be formulated independently of the stability and nature (discrete or continuous) of the original system. These considerations lead to the formulation of Theorems 1 and 2 below, which allow to test for controllability and observability using Gramian-like solutions to certain Lyapunov equations, regardless of system stability. Up to date such test have been available only for stable systems, via the Conditions 2 and 6 reviewed above.

*Theorem 1*: The pair ($A$, $B$) is controllable if and only if the solution $P_c$ of

$$\tilde{A} P_c + P_c \tilde{A}^T = -BB^T \tag{3}$$

is positive definite for $\tilde{A} = A - \alpha I, \alpha > \max_i \Re e[\lambda_i(A)]$.

*Theorem 2*: The pair ($A$, $C$) is observable if and only if the solution $P_o$ of

$$\tilde{A}^T P_o + P_o \tilde{A} = -C^T C \tag{4}$$

is positive definite for $\tilde{A} = A - \alpha I, \alpha > \max_i \Re e[\lambda_i(A)]$.

In the practical application of Theorems 1 and 2, $\alpha$ can be chosen based on known upper bounds for $\max_i \Re e[\lambda_i(A)]$. Norms and matrix measures of $A$ allow for adequate choices of $\alpha$, because both are upper bounds for $\max_i \Re e[\lambda_i(A)]$ [3]. E.g. selecting the Euclidean norm $\| . \|_2$ yields the possible choice $\alpha = \beta \| A \|_2, \beta > 1$; selecting the associated matrix measure yields the choice $\alpha = \beta \| A + A^T \|_2, \beta > 0.5$ .

## IV. DETERMINING CONTROLLABLE, OBSERVABLE AND MINIMAL REALIZATIONS

Theorems 1 and 2 entail two results that are instrumental in determining controllable and observable systems representations as follows.

*Theorem 3*: Let $U\Sigma U^T$ be the singular value decomposition of the positive semi definite symmetric solution matrix $P_c$ of (3), where $U$ is a unitary matrix, and $\Sigma$ is a diagonal matrix with diagonal elements arranged such that zeros are left at the lower end of the diagonal. Then $U$ is a similarity transformation matrix that transforms ($A$, $B$) into

$$U^T A U = \begin{bmatrix} A_c & A_1 \\ 0 & A_{nc} \end{bmatrix}, U^T B = \begin{bmatrix} B_c \\ 0 \end{bmatrix},$$

where the dimension of the square matrix $A_c$ is equal to rank($\Sigma$) , and ($A_c$, $B_c$) is controllable.

*Proof*: (3) may be rewritten as

$$(A - \alpha I) U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} U^T + U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} U^T (A^T - \alpha I) = -BB^T,$$

where rank($\Sigma_1$) = rank($\Sigma$). Thus

$$U^T (A - \alpha I) U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} U^T (A^T - \alpha I) U = -U^T BB^T U.$$

Using the partitions

$$U^T A U = \begin{bmatrix} A_c & A_1 \\ A_2 & A_{nc} \end{bmatrix}, U^T B = \begin{bmatrix} B_c \\ B_2 \end{bmatrix},$$

where the dimension of the square matrix $A_c$ is equal to rank($\Sigma$), one may write

$$\begin{bmatrix} A_c - \alpha I & A_1 \\ A_2 & A_{nc} - \alpha I \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_c^T - \alpha I & A_2^T \\ A_1^T & A_{nc}^T - \alpha I \end{bmatrix} = -\begin{bmatrix} B_c B_c^T & B_c B_2^T \\ B_2 B_c^T & B_2 B_2^T \end{bmatrix}$$

and

$$\begin{bmatrix} (A_c - \alpha I)\Sigma_1 & 0 \\ A_2 \Sigma_1 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_1 (A_c^T - \alpha I) & \Sigma_1 A_2^T \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} B_c B_c^T & B_c B_2^T \\ B_2 B_c^T & B_2 B_2^T \end{bmatrix}$$

Thus clearly $B_2 = 0$ and $A_2 = 0$. According to Theorem 1, $(A_c, B_c)$ is controllable because

$$(A_c - \alpha I)\Sigma_1 + \Sigma_1 (A_c^T - \alpha I) = -B_c B_c^T .$$

*Theorem 4*: Let $U\Sigma U^{\mathrm{T}}$ be the singular value decomposition of the positive semi definite symmetric solution matrix $P_o$ of (4), where $U$ is a unitary matrix, and $\Sigma$ is a diagonal matrix with diagonal elements arranged such that zeros are left at the lower end of the diagonal. Then $U$ is a similarity transformation matrix that transforms $(A, C)$ into

$$U^T A U = \begin{bmatrix} A_o & 0 \\ A_2 & A_{no} \end{bmatrix}, CU = [C_o \quad 0],$$

where the dimension of the square matrix $A_o$ is equal to rank($\Sigma$), and $(A_o, C_o)$ is observable.

A demonstration of Theorem 4 follows along the same lines as that of Theorem 3.

For computing the decomposition of a system in its observable / unobservable or controllable / uncontrollable subsystems, Theorems 3 and 4 provide an alternative to Rosenbrock's method and its variants. (Rosenbrock's method is a modification of the standard algorithm for reducing a matrix to echelon form, and has been used in software such as SLICOT and Matlab [4].)

Observable and controllable, i.e. minimal realizations can be found either by using Theorem 3 to determine the controllable subsystem and then applying Theorem 4 to obtain an observable realization for the controllable subsystem, or alternatively by using Theorem 4 to determine the observable subsystem and then applying Theorem 3 to obtain a controllable realization for the observable subsystem. Such application of these theorems is related to calculating minimal realizations via balancing techniques, using methods as those described in [5]. However, in contrast to the proposal herein, balancing methods do not proceed in two successive steps; they work with the solutions of (3) and (4) both calculated for the original, i.e. the nonreduced system. A further difference is the insight, given here, that the transformation matrices $U$ of Theorems 3 and 4 are suitable for directly decomposing $(A, B, C)$ – as well as $(\tilde{A}, B, C)$, of course – into controllable / uncontrollable or observable / unobservable subsystems, respectively.

## V. APPLICATION

To illustrate the practicality of the concepts detailed in the preceding sections, consider the problem of assessing the observability of systems from a system class described by (1) with

$$A = T\hat{A}T^{-1}, C = \hat{C}T^{-1}$$

and

$$\hat{A} = \left[\begin{array}{c|cccc} R_{r\times 4} & & R_{r\times r} & & \\ & 1 & 1 & 0 & 1 \\ 0_{4\times r} & 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & 0 & 0 & -1 & 0 \end{array}\right], \hat{C} = \begin{bmatrix} 0_{2\times r} & \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix} \end{bmatrix},$$

where $R_{i\times j}$ is a $i\times j$ matrix with normally distributed random entries and $T$ is the orthogonal matrix resulting from the QR (orthogonal-triangular) decomposition of a square matrix with normally distributed random entries. By construction, systems out of this class have four observable modes. All other modes are unobservable. This can be seen from the observability matrix of such system, which is

$$O_o(A,C) = \begin{bmatrix} C \\ CA \\ \ldots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} \hat{C}T^{-1} \\ \hat{C}\hat{A}T^{-1} \\ \ldots \\ \hat{C}\hat{A}^{n-1}T^{-1} \end{bmatrix} = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \ldots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix} T^{-1} .$$

Thus

$$rank(O_o(A,C)) = rank\left(\begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & 1 & 0 \\ 0 & \ldots & 0 & 0 & 1 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 1 & 0 & 2 \\ 0 & \ldots & 0 & 0 & 1 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}\right) = 4 .$$

To compare the practical performance of the classical observability test via observability matrix rank evaluation with the (new) test stemming from Theorem 2, the observability of systems in the aforementioned class with $r = 1,\ldots, 96$ was assessed via:

1) numerical observability matrix rank determination;
2) numerical rank determination of the solution $P_o$ of (4).

These numerical experiments were performed using Matlab R2011a running under Ubuntu 10.04 (Linux) on a Intel Core i7 based PC. The following standard Matlab routines were employed: *obsv* for observability matrix calculation, *rank* for rank determination (with default tolerance), and *lyap* for Laypunov equation solution.

Numerical results are summarized in Figs. 1 and 2. In Fig. 1 the correct number of observable states is 4, for all systems orders (as shown above). Thus Fig. 1 reports the failure of the observability matrix based test for system orders 8, 9 and all orders higher than 10. In contrast, the test based on Theorem 2 performed well for all orders. This illustrates: (a) the numerical robustness and reliability of the concepts proposed herein; (b) the failure-prone performance of the observability matrix based test for systems with medium to high order, due to numerical reasons.

Fig. 2 shows that a higher computational load is the price paid for the better performance of the observability test based on Theorem 2. The computational effort for this test is roughly ten times larger than the effort demanded in the observability matrix based test, which, however, does not yield reliable results in this example, as shown in Fig. 1.

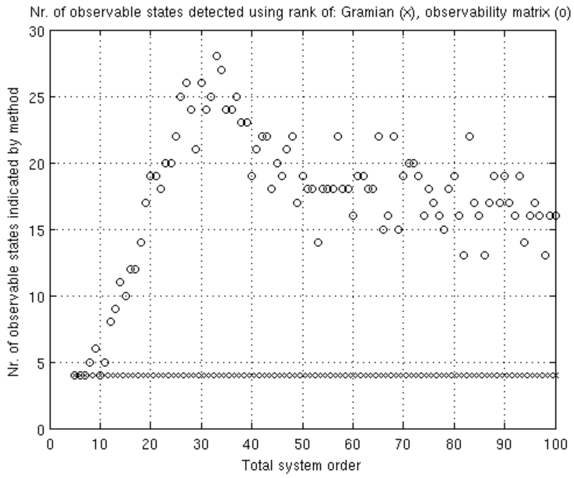

Fig. 1. Results of observability assessments via rank determination of the observability matrix and of the Gramian matrix from Theorem 2.

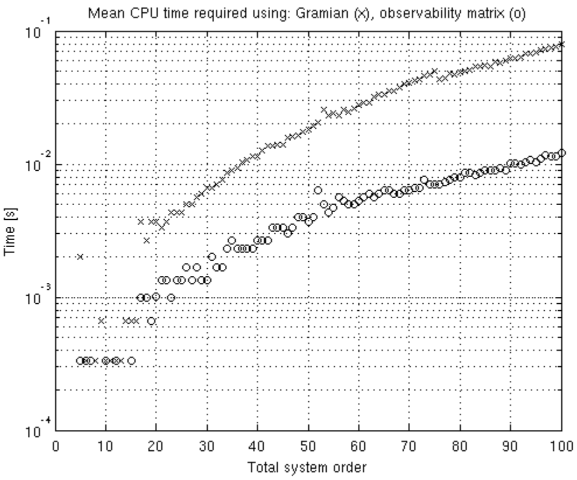

Fig. 2. Computational effort for the observability assessments via rank determination of the observability matrix and of the Gramian matrix from Theorem 2summarized in Fig. 1.

## VI. CONCLUSION

This contribution points out that continuous time Lyapunov equations can be used in testing observability and controllability of continuous as well as discrete time linear systems. The proposed observability / controllability tests are alternatives to existing tests: the solution of continuous time Lyapunov equation and non-singularity tests of such solution must be computed. Controllable, observable and minimal realizations can be determined by properly using the products of the singular value decompositions of the solutions to Lyapunov equations (3) and (4), and applying similarity transformations indicated in Theorems 3 and 4.

An application example allows an assessment of the robustness of the proposed approach as well as the computational effort required.

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