An iterative LMI scheme for output regulation of continuous-time LPV systems with inexactly measured parameters

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Abstract— **The most part of papers dealing with linear parameter varying systems (LPV) control assumes a perfect parameter knowledge. In this note the scheduling physical parameters are supposed to be inexactly measured. Continuous time polytopic LPV systems are considered and an observer based controller is proposed to guarantee internal stability and performance in terms of exact asymptotic output regulation at a desired set point, despite the presence external disturbances. An iterative LMI procedure is given for computing the maximum parametric uncertainty level that can be tolerated to maintain the exponential stability of the closed loop system. The controller is scheduled by the noisy parameter readings and a quadratic affinely parameter-dependent Lyapunov function is used. This results in a general approach encompassing constant controller gains and/or parameter independent Lyapunov functions.**

I. Introduction

Control of Linear Parameter Varying Systems (LPV) has been receiving a great deal of attention and many approaches have been developed in different settings and frameworks [1]-[9]. A usual assumption of the above methods is that all the parameter values are exactly measured or estimated at each time instant. The case where only some parameters are available for measurement while the others are regarded as uncertainty has been considered in [10],[11]. On the other hand, in the most part of practical situations it is much more realistic to assume a non perfect parameter knowledge of the time varying parameters values because they are very often the result of approximation, estimation or measurements procedures. Hence the stabilizing controller has to be robust with respect to the mismatch between the true and the available parameters. For polytopic discrete-time LPV systems the state reconstruction problem with bounded estimation error has been considered in [12] using the notion of input to state stability. The more general problem of designing an observed based dynamic output controller for discrete-time LPV systems has been recently considered in [13]. LMI's conditions guaranteeing closed-loop stability are derived within some degree of absolute parametric uncertainty. Dynamic output control of polytopic continuous-time LPV have been considered in [14] where a parameter-independent quadratic Lyapunov matrix is used. LMI's solvability conditions are given in the case of relative parametric uncertainty (namely the parametric uncertainty is given as a percentage of the true parameter value). Recent contributions on H_2 and H_∞ gain scheduled control for discrete and continuous-time LPV systems with inexact parameters are given in [15] and [16], using parameter independent quadratic Lyapunov functions.

All the above papers assumed an unknown deterministic mismatch between the true and the measured (or estimated) parameters. The case of parameters affected by a possibly unbounded stochastic error has been considered in [17]. The nature of the problem is such that the observer and feedback gains of the dynamic output controller have to be robustly fixed independently of the time varying plant parameters and only a quadratic stability is obtained.

The purpose of this paper is to give an iterative LMI procedure for the robust output regulation of polytopic continuous-time LPV systems whose parameter values are assumed to be available with a prespecified finite accuracy. Performance measures are given in terms of exact set point regulation and disturbance rejection. As noticed in [18], this kind of performance is not achievable by H_2 controllers. The proposed synthesis procedure consists of two steps. First, separate LMI conditions are stated for the design of the state observer and for the static regulator. This first step requires two independent parameter search lines. If these two parameters are properly chosen, it is possible to maximize the allowed parametric uncertainty determined at the second step, where LMI conditions for the internal exponential stability of the closed-loop system are given. These conditions are relative to the worst-case situation, namely when the uncertainty assumes its "a priori" known maximum theoretical value, which coincides with the size of the interval where each physical parameter takes values. If a solution for the worst case does not exist, an iterative scheme for a single parameter provides the maximum tolerable degree of parametric mismatch.

A remarkable feature of the presented approach is that, within the same general iterative scheme, it encompasses different particular methods for some different control problems, e.g.: constant controller gains and/or parameter independent Lyapunov functions, partly measured parameters, exact parametric knowledge.

The paper is organized in the following way. Section II states some preliminaries and the control problem, section III explains the synthesis procedure, section IV shows how to recover other approaches, a numerical simulation is given in Section V.

II. Preliminaries

Consider the following parametrically affine continuous-time LPV plant Σ_p with p independent time-varying scalar

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parameters $\theta(t) := [\theta_1(t), \cdots, \theta_p(t)]^T$

$$\dot{x}(t) = A_p(\theta(t))x(t) + B_p u(t) + Mw(t),\ x(0) = x_0, \quad (1)$$
$$y(t) = C_p x(t), \quad (2)$$

where $u(t) \in \mathbb{R}^m$ is the control input, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^q$, with $m \geq q$, is the output and $w(t) \in \mathbb{R}^d$ is an unknown constant external disturbance generated as the output response of the following autonomous system Σ_w

$$\dot{x}_w(t) = \mathbf{0}, \qquad x_w(0) = x_{w_0}, \quad (3)$$
$$w(t) = C_w x_w(t). \quad (4)$$

Remark 1: For simplicity of notation, through the paper the dimensions of the null blocks are not specified because they are clear from the context.

It is assumed that: A1): $\theta_i(t) \in [\underline{\theta}_i, \overline{\theta}_i]$, $\forall t \geq 0$, the variation rate $\dot{\theta}_i(t)$ is well defined at all times and satisfies $\dot{\theta}_i(t) \in [\underline{\omega}_i, \overline{\omega}_i]$, $\forall t \geq 0$; A2): each $\theta_i(t)$ is measured on-line according to $\tilde{\theta}_i(t) = \theta_i(t) + \delta_i(t)$ where $|\delta_i(t)| \leq \tilde{\delta}_i$, $\forall t \geq 0$; A3) to account for the known physical bounds on the true parameters, both $\theta(t)$ and $\tilde{\theta}(t)$ are assumed to take values into the same compact set $\Theta \in \mathbb{R}^p$, so that $\tilde{\delta}_i := |\overline{\theta}_i - \underline{\theta}_i|$ indicates the allowable maximum uncertainty level, meanwhile $\dot{\theta}(t) := [\dot{\theta}_1(t), \cdots, \dot{\theta}_p(t)]^T$ and $\delta(t) := [\delta_1(t), \cdots, \delta_p(t)]^T$ lie in compact subsets denoted by $\dot{\Theta}$ and Δ respectively; A4) at each frozen time $\bar{t} \in \mathbb{R}^+$, Σ_p has not transmission zeros in $s = 0$, $\forall \theta(\bar{t}) \in \Theta$. This guarantees that even if $\theta(t)$ remains frozen on a fixed $\bar{\theta}$, $\forall t \geq \bar{t}$, for some $\bar{t} > 0$, the plant does not behave as a differentiator $\forall \bar{\theta} \in \Theta$.

Remark 2 The assumption of constant matrices B and C is not a loss of generality because, as shown in [3], it can be always satisfied if proper LTI filters are applied to the original signals $u(t)$ and $y(t)$.

By A1) $A_p(\theta(t))$ can be written in the polytopic form

$$A_p(\theta(t)) = \sum_{i=1}^{N=2^p} \rho_i(t) A_{p_{v_i}} \quad (5)$$

where $\rho(t) = [\rho_1(t), \cdots, \rho_N(t)]$ belongs to the unit simplex $\Lambda \subset \mathbb{R}^N$, and $A_{p_{v_i}}$ are appropriately defined vertex matrices. The constant external reference $r(t)$ to be tracked is generated as the output response of the following autonomous system Σ_r

$$\dot{x}_r(t) = \mathbf{0}, \quad x_r(0) = x_{r_0}, \quad (6)$$
$$r(t) = C_r x_r(t), \quad r(t) \in \mathbb{R}^q. \quad (7)$$

The series connection of Σ_p with Σ_w is denoted by $\Sigma_{p,w}$, the feedback connections of the controller $\tilde{\Sigma}_c$ to be designed with Σ_p and $\Sigma_{p,w}$ are denoted by Σ_f and $\tilde{\Sigma}_f$ respectively, the series connection of Σ_r with $\tilde{\Sigma}_f$ is denoted by $\tilde{\Sigma}$. Hence the tracking error $e_t(t) = r(t) - y(t)$ is the unforced output response of the autonomous error system Σ_e univocally corresponding to $\tilde{\Sigma}$. It follows that $e_t(t)$ is asymptotically converging to zero for any initial condition of Σ_e, if and only if Σ_e is asymptotically state-output stable.

The problem considered in this paper consists in finding (if it exists) a dynamic output controller $\tilde{\Sigma}_c$ scheduled by the noisy parameter measurements which guarantee the internal exponential stability of the closed-loop system Σ_f, and the asymptotic state-output stability of Σ_e, for each possible $(\theta(t), \dot{\theta}(t)) \in \Theta \times \dot{\Theta}$, and for the given level $\tilde{\delta} := [\tilde{\delta}_1, \cdots, \tilde{\delta}_p]^T$ of mismatch between the true and the available parameters.

III. The controller design procedure.

The series connection $\Sigma_{p,w}$ of Σ_w with Σ_p is given by

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_w(t) \end{bmatrix} = \begin{bmatrix} A_p(\theta(t)) & MC_w \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} + \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix} u(t)$$
$$= \mathscr{A}_{p,w}(\theta(t)) x_{p,w}(t) + \mathscr{B}_{p,w} u(t), \quad (8)$$
$$y(t) = \begin{bmatrix} C_p & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_w(t) \end{bmatrix} = \mathscr{C}_{p,w} x_{p,w}(t). \quad (9)$$

Following [19], the dynamic output feedback controller is given by the connection of a suitable LTI internal model system Σ_m of both $r(t)$ and $w(t)$ with an observer Σ_{ob} of $x_{p,w}(t)$, scheduled by the noisy parameter measurements. Denoting by $x_m(t) \in \mathbb{R}^s$ and $\xi_{p,w}(t) = [\xi_p^T(t), \xi_w^T(t)]^T$ the state of Σ_m and Σ_{ob}, the state of controller $\tilde{\Sigma}_c$ is defined as $z(t) = [x_m^T(t), \xi_{p,w}^T(t)]^T$ with

$$\dot{x}_m(t) = A_m x_m(t) + B_m e_t(t) \quad (10)$$
$$\dot{\xi}_{p,w}(t) = \mathscr{A}_{p,w}(\tilde{\theta}(t))\xi_{p,w}(t) + \mathscr{B}_{p,w} u(t) \quad (11)$$
$$+ \mathscr{L}(\tilde{\theta}(t))(\mathscr{C}_{p,w}\xi_{p,w}(t) - y(t))$$
$$u(t) = \mathscr{K}_m(\tilde{\theta}(t)) x_m(t) + \mathscr{K}_{\xi_{p,w}}(\tilde{\theta}(t))\xi_{p,w}(t) \quad (12)$$
$$= \mathscr{K}_m(\tilde{\theta}(t)) x_m(t) + \begin{bmatrix} \mathscr{K}_{\xi_p}(\tilde{\theta}(t)) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \xi_p(t) \\ \xi_w(t) \end{bmatrix}$$

where $\mathscr{A}_{p,w}(\tilde{\theta}(t))$ is obtained by $\mathscr{A}_{p,w}(\theta(t))$ replacing $\theta(t)$ with $\tilde{\theta}(t)$, $\mathscr{L}(\tilde{\theta}(t)) = [\mathscr{L}_{\xi_p}^T(\tilde{\theta}(t)), \mathscr{L}_{\xi_w}^T(\tilde{\theta}(t))]^T = L_0 + \sum_{i=1}^p \tilde{\theta}_i(t) L_i$, $\mathscr{K}(\tilde{\theta}(t)) = [\mathscr{K}_{\xi_p}(\tilde{\theta}(t)), \mathscr{K}_m(\tilde{\theta}(t))] = K_0 + \sum_{i=1}^p \tilde{\theta}_i(t) K_i$ and A_m and B_m are defined as in [19].

Applying the usual transformation matrix, the feedback connection $\tilde{\Sigma}_f$ of $\tilde{\Sigma}_c$ with $\Sigma_{p,w}$ is described by the triplet $\tilde{\Sigma}_f \equiv (\hat{\mathbf{C}}_f, \hat{\mathbf{A}}_f(\theta(t), \tilde{\theta}(t)), \hat{\mathbf{B}}_f)$, with $\hat{\mathbf{C}}_f := [\mathscr{C}_{p,m}, \mathbf{0}, \mathbf{0}]$, $\hat{\mathbf{B}}_f = [\hat{\mathscr{B}}_{p,m}^T, \mathbf{0}^T, \mathbf{0}^T]^T$,

$$\hat{\mathbf{A}}_f(\theta(t), \tilde{\theta}(t)) = \begin{bmatrix} \hat{A}_f(\theta(t), \tilde{\theta}(t)) & \begin{bmatrix} \mathscr{M} \\ \mathbf{0} \end{bmatrix} \\ \mathbf{0} & A_w \end{bmatrix} \quad (13)$$

to which the state vector $\hat{\mathbf{x}}_f(t) := [[x_{p,m}(t); x_{p,w}(t) - \xi_{p,w}(t)]^T, w^T(t)]^T$ corresponds with $x_{p,m}(t) := [x_p^T(t), x_m^T(t)]^T$, moreover $\hat{A}_f(\theta(t), \tilde{\theta}(t)) :=$

$$\begin{bmatrix} \mathscr{A}_{p,m}(\theta(t)) + \mathscr{B}_{p,m}\mathscr{K}(\tilde{\theta}(t)) & -\mathscr{B}_{p,m}\mathscr{K}_{\xi_{p,w}}(\tilde{\theta}(t)) \\ \Delta\mathscr{A}(\theta(t), \tilde{\theta}(t)) & \mathscr{A}_{p,w}(\tilde{\theta}(t)) + \mathscr{L}(\tilde{\theta}(t))\mathscr{C}_{p,w} \end{bmatrix}$$
$$\mathscr{C}_{p,m} := \begin{bmatrix} C_p & \mathbf{0} \end{bmatrix},\ \mathscr{A}_{p,m}(\theta(t)) := \begin{bmatrix} A_p(\theta(t)) & \mathbf{0} \\ -B_m C & A_m \end{bmatrix},$$
$$\Delta\mathscr{A}(\theta(t), \tilde{\theta}(t)) := \begin{bmatrix} A_p(\theta(t)) - A_p(\tilde{\theta}(t)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$
$$\hat{\mathscr{B}}_{p,m} := \begin{bmatrix} \mathbf{0} \\ B_m \end{bmatrix}, \mathscr{M} = \begin{bmatrix} MC_w \\ \mathbf{0} \end{bmatrix}, \mathscr{B}_{p,m} := \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix},$$
$$\mathscr{C}_{p,w} := \begin{bmatrix} C_p & \mathbf{0} \end{bmatrix}.$$

The controller $\tilde{\Sigma}_c$ given by (10)-(12) is required to internally stabilize Σ_f, whose dynamical matrix is the sub-matrix $\hat{A}_f(\theta(t),\tilde{\theta}(t))$ of (13) to which the state vector $\hat{x}_f(t) := [x^T_{p,m}(t), x^T_{p,w}(t) - \xi^T_{p,w}(t)]^T$ corresponds.
Consider the quadratic parameter dependent Lyapunov function $V(\hat{x}_f(t)) = \hat{x}^T_f(t)P_f(\theta(t))\hat{x}_f(t)$, with

$$P_f(\theta(t)) := \begin{bmatrix} \varepsilon P(\theta(t)) & W^T(\theta(t)) \\ W(\theta(t)) & \rho S(\theta(t)) \end{bmatrix} > 0, \quad \forall t \geq 0, \quad (14)$$

where $P(\theta(t))$ and $S(\theta(t))$ are affine parameter dependent positive definite symmetric matrices, ε and ρ are some scalars and $W(\theta(t))$ is an affine parameter dependent matrix. The exponential stability of Σ_f is guaranteed if there exists a matrix $P_f(\theta(t)) > 0$ satisfying the inequality

$$\begin{aligned} & \hat{A}^T_f\left(\theta(t),\tilde{\theta}(t)\right)P_f(\theta(t)) + P_f(\theta(t))\hat{A}_f\left(\theta(t),\tilde{\theta}(t)\right) \\ + \quad & dP_f(\theta(t))/dt < 0, \quad \forall t \geq 0. \end{aligned} \quad (15)$$

On the basis of the above considerations, the goal is to design two gain matrices $\mathscr{K}(\tilde{\theta}(t)) = K_0 + \sum_{i=1}^p \tilde{\theta}_i(t)K_i$ and $\mathscr{L}(\tilde{\theta}(t)) = L_0 + \sum_{i=1}^p \tilde{\theta}_i(t)L_i$ such that condition (15) hold $\forall\ (\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta$.
The synthesis procedure consists of two steps:
1) to find gain matrices $\mathscr{K}(\tilde{\theta}(t))$ and $\mathscr{L}(\tilde{\theta}(t))$, which guarantee the exponential stability of the corresponding time-varying dynamical matrices $\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) := \mathscr{A}_{p,m}(\theta(t)) + \mathscr{B}_{p,m}\mathscr{K}(\tilde{\theta}(t))$, and $\mathscr{A}_l(\tilde{\theta}(t)) := \mathscr{A}_{p,w}(\tilde{\theta}(t)) + \mathscr{L}(\tilde{\theta}(t))\mathscr{C}_{p,w}$, respectively,
2) to prove the existence of LMI's conditions on $\Delta\mathscr{A}(\theta(t),\tilde{\theta}(t)) := \Delta\mathscr{A}(\delta(t))$ guaranteeing the exponential stability of the closed loop system Σ_f.

The solvability of these LMI's for the given $\Delta\mathscr{A}(\delta(t))$ depends on the existence of two scalars ε and ρ and of a matrix $W(\theta(t))$ in (14) which guarantee the fulfillment of (15). In any case it will be shown that if step 1 admits a solution, it is always possible to find, through an iterative scheme, two scalars ε and ρ and a matrix $W(\theta(t))$ such that the internal stabilization of Σ_f can be solved for a smaller parametric mismatch.
The following preliminary Lemma is exploited to implement Step 1.
Lemma: Let Φ be a symmetric matrix and N, M be matrices of appropriate dimensions. The following statements are equivalent:

1) $\Phi < 0$ and $\Phi + NM^T + MN^T < 0$;
2) The LMI problem $\begin{bmatrix} \Phi & M+NF \\ M^T + F^T N^T & -F-F^T \end{bmatrix} < 0$ is feasible with respect to F.

The proof of the Lemma is given in [20].
Step 1A: design of $\mathscr{K}(\tilde{\theta}(t))$
The problem corresponding to step 1A consists in finding a gain $\mathscr{K}(\tilde{\theta}(t))$ such that there exists a positive definite matrix $P(\theta(t))$ satisfying

$$\begin{aligned} & \mathscr{A}^T_k(\theta(t),\tilde{\theta}(t))P(\theta(t)) + P(\theta(t))\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) \quad (16) \\ + \quad & dP(\theta(t))/dt < 0, \forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta. \end{aligned}$$

Assume there exist symmetric matrices $P_i, i = 0,\cdots,p$, such that the affinely parameter dependent matrix $P(\theta(t)) := P_0 + \sum_{i=1}^p \theta_i(t)P_i$ is positive definite $\forall t \geq 0$. Similarly to (5) one has

$$P(\theta(t)) = \sum_{i=1}^{N=2^p} \rho_i(t)P_{v_i}, \quad (17)$$

where the vertex matrices P_{v_i} affinely depend on the matrices $P_0, P_1, \cdots P_p$. Moreover by A2), also $dP(\theta(t)/dt = \sum_{i=1}^{p} \dot{\theta}_i(t)P_i$ can be written as $dP(\theta(t))/dt = \sum_{j=1}^{N=2^p} \eta_j(t)P_{w_j}$ with suitably defined vertex matrices P_{ω_j} affinely dependent on matrices $P_1, \cdots P_p$. Also $\mathscr{K}(\tilde{\theta}(t)) = K_0 + \sum_{i=1}^p \tilde{\theta}_i(t)K_i$ can be written in the polytopic form

$$\mathscr{K}(\tilde{\theta}(t)) = \sum_{\ell=1}^{N=2^p} r_\ell(t)K_{v_\ell} = \sum_{\ell=1}^{N=2^p} r_\ell(t)\tilde{Y}_{v_\ell}G^{-1} = \tilde{Y}(\tilde{\theta}(t))G^{-1}, \quad (18)$$

where G is an invertible square matrix and $\tilde{Y}_{v_\ell}{}'s$ are suitably defined matrices dependent on $p+1$ unknown matrices $Y_0, Y_1, \cdots, Y_p$.

The following simple example clarifies the adopted notation. Suppose $p = 1$ so that $\theta(t) \equiv \theta_1(t)$. In this case one has:
$P(\theta(t)) = \sum_{i=1}^{2} \rho_i(t)P_{v_i} = \rho_1(t)(P_0 + \underline{\theta}_1 P_1) + \rho_2(t)(P_0 + \overline{\theta}_1 P_1)$,
$dP(\theta(t))/dt = \sum_{j=1}^{2} \eta_j(t)P_{w_j} = \eta_1(t)(\underline{\omega}_1 P_1) + \eta_2(t)(\overline{\omega}_1 P_1)$,
$\mathscr{K}(\tilde{\theta}(t)) = K_0 + \tilde{\theta}_1(t)K_1 = \sum_{\ell=1}^2 r_\ell(t)\tilde{Y}_{v_\ell}G^{-1} = r_1(t)(Y_0 + \underline{\theta}_1 \cdot Y_1)G^{-1} + r_2(t)(Y_0 + \overline{\theta}_1 \cdot Y_1)G^{-1}$, where $K_0 = Y_0 \cdot G^{-1}$ and $K_1 = Y_1 \cdot G^{-1}$.
Theorem 1: The problem defined by step 1A admits a solution if there exist $p+1$ symmetric matrices Z_i and $p+1$ matrices Y_i, $i = 0,\cdots,p$, an invertible matrix G such that for any real γ satisfying

$$Z_{w_j} - 2\gamma Z_{v_i} < 0, \quad i,j = 1,\cdots,N = 2^p \quad (19)$$

one has

$$\begin{bmatrix} Z_{w_j} - 2\gamma Z_{v_i} & (*)^T \\ Z_{v_i} + (\mathscr{A}_{p,m})_{v_i}G + \mathscr{B}_{p,m}\tilde{Y}_{v_\ell} + \gamma G & -G-G^T \end{bmatrix} < 0, \quad (20)$$

for $i,j,\ell = 1,...,N = 2^p$, where $(\mathscr{A}_{p,m})_{v_i}$ are the vertex matrices of $\mathscr{A}_{p,m}(\theta(t))$ and both Z_{v_i}'s > 0 and Z_{w_j}'s depend affinely on the Z_i's. The matrix $\mathscr{K}(\tilde{\theta}(t)) = K_0 + \sum_{i=1}^p \tilde{\theta}_i(t)K_i$ is characterized by the gains $K_i = Y_i \cdot G^{-1}$, $i = 0,1,\cdots,p$, and the vertex matrices P_{v_i} of $P(\theta(t))$ of (17) are defined as $P_{v_i} = Q^T Z_{v_i} Q$ with $Q := G^{-1}$.
Proof: Inequality (17) can be written as

$$\begin{aligned} & (\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) + \gamma I)^T P(\theta(t)) + P(\theta(t))(\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) + \gamma I) \\ & + dP(\theta(t))/dt - 2\gamma P(\theta(t)) < 0, \end{aligned} \quad (21)$$

$\forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta$. Applying the Lemma, (21) is equivalent to

$$\begin{pmatrix} dP(\theta(t))/dt - 2\gamma P(\theta(t)) & (*)^T \\ P(\theta(t)) + Q^T(\mathscr{A}_{p,m}(\theta(t)) + \mathscr{B}_{p,m}\mathscr{K}(\tilde{\theta}(t)) + \gamma I) & -Q^T - Q \end{pmatrix}$$
$$< 0, \quad \forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta, \quad (22)$$

if there exists a real γ satisfying

$$dP(\theta(t))/dt - 2\gamma P(\theta(t)) < 0 \quad \forall(\theta(t),\dot{\theta}(t)) \in \Theta \times \dot{\Theta}. \quad (23)$$

Left and right-multiplying inequality (22) by $\mathrm{diag}(Q^{-T},Q^{-T})$ and by $\mathrm{diag}(Q^{-1},Q^{-1})$ respectively and putting $Z(\theta(t)) = Q^{-T}P(\theta(t))Q^{-1}$, $dZ(\theta(t))/dt = Q^{-T}dP(\theta(t))/dtQ^{-1}$ and $G = Q^{-1}$ one has

$$\begin{bmatrix} dZ(\theta(t))/dt - 2\gamma Z(\theta(t)) & (*)^T \\ Z(\theta(t)) + (\mathscr{A}_{p,m}(\theta(t)) + \mathscr{B}_{p,m}\mathscr{K}(\tilde{\theta}(t)))G + \gamma G & -G - G^T \end{bmatrix}$$
$$< 0, \quad \forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta. \quad (24)$$

Analogously, left- and right-multiplying inequality (23) by Q^{-T} and by Q^{-1} respectively one has

$$dZ(\theta(t))/dt - 2\gamma Z(\theta(t)) < 0, \quad \forall(\theta(t),\dot{\theta}(t)) \in \Theta \times \dot{\Theta}. \quad (25)$$

Let $(\mathscr{A}_{p,m})_{v_i}$'s , Z_{v_i}'s and Z_{w_j}'s be the vertex matrices of $\mathscr{A}_{p,m}(\theta(t))$, $Z(\theta(t))$ and $Z(\theta(t))/dt$ respectively, by (18), inequalities (24) and (25) hold if and only if conditions (19) and (20) hold.

Step 1B : Design of $\mathscr{L}(\tilde{\theta}(t))$.

Consider the time varying dynamical matrix $\mathscr{A}_l(\tilde{\theta}(t)) := \mathscr{A}_{p,w}(\tilde{\theta}(t)) + \mathscr{L}(\tilde{\theta}(t))\mathscr{C}_{p,w}$. The problem corresponding to step 1B consists in finding a gain $\mathscr{L}(\tilde{\theta}(t))$ such that there exist a positive definite matrix $S(\theta(t))$ satisfying

$$\mathscr{A}_l^T(\tilde{\theta}(t))S(\theta(t)) + S(\theta(t))\mathscr{A}_l(\tilde{\theta}(t)) + dS(\theta(t))/dt < 0$$
$$\forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta. \quad (26)$$

Analogously to step 1A, assume there exist symmetric matrices $S_i, i = 0,\cdots,p$, such that $S(\theta(t)) := S_0 + \sum_{i=1}^{p}\theta_i(t)S_i > 0, \forall t \geq 0$. Both $S(\theta(t))$ and $dS(\theta(t))/dt$ can be written in their respective polytopic forms:

$$S(\theta(t)) = \sum_{i=1}^{N=2^p}\rho_i(t)S_{v_i}, \quad dS(\theta(t))/dt = \sum_{j=1}^{N=2^p}\eta_j(t)S_{w_j},$$

with suitably defined vertex matrices S_{v_i} and S_{w_j} respectively, affinely depending on $S_i, i = 0,\cdots,p$.

Both $\mathscr{A}_{p,w}(\tilde{\theta}(t))$ and $\mathscr{L}(\tilde{\theta}(t)) = L_0 + \sum_{i=1}^{p}\tilde{\theta}_i(t)L_i$ can be written as

$$\mathscr{A}_{p,w}(\tilde{\theta}(t)) = \sum_{\ell=1}^{N=2^p} r_\ell(t)(\mathscr{A}_{p,w})_{v_\ell}, \quad (27)$$

$$L(\tilde{\theta}(t)) = \sum_{\ell=1}^{N=2^p} r_\ell(t)L_{v_\ell} = \sum_{\ell=1}^{N=2^p} r_\ell(t)V^{-T}\tilde{U}_{v_\ell} = V^{-T}\tilde{U}(\tilde{\theta}(t)),$$

where V is a square invertible matrix, both $(\mathscr{A}_{p,w})_{v_\ell}{}'s$ and $\tilde{U}_{v_\ell}{}'s$ are suitably defined vertex matrices dependent on $p+1$ known matrices $A_0, A_1, \cdots, A_p$ and $p+1$ unknown matrices $U_0, U_1, \cdots, U_p$ respectively.

Theorem 2: The problem defined by step 1B admits a solution if there exist $p+1$ symmetric matrices $S_0, S_1, \cdots, S_p$, $p+1$ matrices $U_0, U_1, \cdots, U_p$, an invertible square matrix V such that for any real β satisfying

$$S_{w_j} - 2\beta S_{v_i} < 0, \quad i,j = 1,\cdots,N = 2^p \quad (28)$$

one has

$$\begin{bmatrix} S_{w_j} - 2\beta S_{v_i} & (*)^T \\ S_{v_i} + V^T(\mathscr{A}_{p,w})_{v_\ell} + \tilde{U}_{v_\ell}\mathscr{C}_{p,w} + \beta V^T & -V - V^T \end{bmatrix} < 0, \quad (29)$$

for $i,j,\ell = 1,...,N = 2^p$, where both S_{v_i}'s > 0 and S_{w_j}'s depend affinely on the S_i's. The matrix $\mathscr{L}(\tilde{\theta}(t)) = L_0 + \sum_{i=1}^{p}\tilde{\theta}_i(t)L_i$ is characterized by the gains $L_i = V^{-T}U_i$, $i = 0,1,\cdots,p$.

Proof: Arguing as in Theorem 1, inequality (26) is equivalent to

$$\begin{pmatrix} dS(\theta(t))/dt - 2\beta S(\theta(t)) & (*)^T \\ S(\theta(t)) + V^T(\mathscr{A}_{p,w}(\tilde{\theta}(t)) + \mathscr{L}(\tilde{\theta}(t))\mathscr{C}_{p,w} + \beta I) & -V - V^T \end{pmatrix}$$
$$< 0, \quad \forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t)) \in \Theta \times \dot{\Theta} \times \Theta, \quad (30)$$

if there exists a real β satisfying

$$dS(\theta(t))/dt - 2\beta S(\theta(t)) < 0, \quad \forall(\theta(t),\dot{\theta}(t)) \in \Theta \times \dot{\Theta}. \quad (31)$$

Denoting by S_{v_i} and S_{w_j} the vertex matrices of $S(\theta(t))$ and $dS(\theta(t))/dt$ respectively, by (27) inequalities (30) and (31) hold if and only if (28) and (29) hold.

Step 2 : Definition of stability conditions for the closed loop system Σ_f.

The results stated at step 1 are now exploited to derive stability conditions for the closed loop system Σ_f, whose dynamical matrix is the sub-matrix $\hat{A}_f(\theta(t),\tilde{\theta}(t))$ of (13). To this purpose let $W(\theta(t))$ of (14) be defined as $W(\theta(t)) = W_0 + \sum_{i=1}^{p}\theta_i(t)W_i$. Both $W(\theta(t))$ and $dW(\theta(t))/dt$ can be written in their respective polytopic form with suitably defined vertex matrices W_{v_i} and W_{w_j} respectively.

Theorem 3: Assume there exists a dynamic output controller $\tilde{\Sigma}_c$ given by (10)-(12) solving step 1. The closed loop system Σ_f is internally exponentially stable for each $(\theta(t)\,\dot{\theta}(t))$ and for the given maximum uncertainty level $\tilde{\delta} := [\tilde{\delta}_1,\cdots,\tilde{\delta}_p]^T$, if there exist $p+1$ matrices $W_0, W_1, \cdots, W_p$ and two scalars ε and ρ such that the LMI's (32) and (33) hold, where $\Delta\mathscr{A}_{v_k}$ are the known vertex matrices of the polytope $\mathscr{P}_{\Delta\mathscr{A}(\delta(t))}$ containing $\Delta\mathscr{A}(\delta(t)) := \begin{bmatrix} -(\sum_{i=1}^{p}\delta_i(t)\cdot A_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, with $|\delta_i(t)| \leq \tilde{\delta}_i$, H_{v_h} and J_{v_h}, $h = 1,\cdots,N_h$ are the vertex matrices of the polytopes $\mathscr{P}_{P(\theta(t))\mathscr{A}_{p,m}(\theta(t))}$ and $\mathscr{P}_{W(\theta(t))\mathscr{A}_{p,m}(\theta(t))}$ containing $P(\theta(t))\mathscr{A}_{p,m}(\theta(t))$ and $W(\theta(t))\mathscr{A}_{p,m}(\theta(t))$ respectively .

Proof: Recalling that $\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) := \mathscr{A}_{p,m}(\theta(t)) + \mathscr{B}_{p,m}\mathscr{K}(\tilde{\theta}(t))$ and $\mathscr{A}_l(\tilde{\theta}(t)) := \mathscr{A}_{p,w}(\tilde{\theta}(t)) + \mathscr{L}(\tilde{\theta}(t))\mathscr{C}_{p,w}$, replacing (14) into (15) gives (34).

Recalling that P_{v_i}, S_{v_i}, P_{w_j}, S_{w_j}, K_{v_ℓ}, $(\mathscr{K}_{\xi_{p,w}})^T_{v_\ell}$, L_{v_ℓ}, ΔA_{v_k} are the known vertex matrices of $P(\theta(t))$, $S(\theta(t))$, $dP(\theta(t))/dt$, $dS(\theta(t))/dt$, $\mathscr{K}(\tilde{\theta}(t))$, $\mathscr{K}_{\xi_{p,w}}(\tilde{\theta}(t))$, $\mathscr{L}(\tilde{\theta}(t))$, $\Delta\mathscr{A}(\delta(t))$ respectively and defining H_{v_h} and J_{v_h} the known and unknown vertex matrices of the polytopes $\mathscr{P}_{P(\theta(t))\mathscr{A}_{p,m}(\theta(t))}$ and $\mathscr{P}_{W(\theta(t))\mathscr{A}_{p,m}(\theta(t))}$ respectively, condition (34) is satisfied if the set of LMI's (33) admits a solution. Condition (32) implies that (14) holds and hence (32) and (33) imply that there exists a parameter dependent Lyapunov matrix given by (14) such that condition (15) holds for the given uncertainty degree $\tilde{\delta} := [\tilde{\delta}_1,\cdots,\tilde{\delta}_p]^T$.

Remark 3 The execution of step 1 requires to perform a line search both for γ and β. However, by the separation principle, these two numerical research lines are independent, so that one has not to try all the possible pairs $(\gamma,\beta) \in [\gamma_1,\gamma_2] \times [\beta_1,\beta_2]$, for some fixed γ_1,γ_2, β_1,β_2. The optimal γ and β are determined evaluating the maximum eigenvalues of (16) and (26) respectively at the vertices of the corresponding

$$\begin{pmatrix} \varepsilon P_{v_i} & W_{v_i}^T \\ W_{v_i} & \rho S_{v_i} \end{pmatrix} > 0, \quad i = 1, \cdots, N = 2^p, \tag{32}$$

$$\begin{pmatrix} \varepsilon H_{v_h} + \varepsilon H_{v_h}^T + W_{v_i}^T \Delta\mathscr{A}_{v_k} + \Delta\mathscr{A}_{v_k}^T W_{v_i} + \varepsilon P_{v_i}\mathscr{B}_{p,m}\mathscr{K}_{v_\ell} + \varepsilon\mathscr{K}_{v_\ell}^T\mathscr{B}_{p,m}^T P_{v_i} + \varepsilon P_{w_j} & (*)^T \\ J_{v_h} + W_{v_i}\mathscr{B}_{p,m}\mathscr{K}_{v_\ell} + \rho S_{v_i}\Delta\mathscr{A}_{v_k} - \varepsilon(\mathscr{K}_{\xi_{p,w}})_{v_\ell}^T\mathscr{B}_{p,m}^T P_{v_i} + ((\mathscr{A}_{p,w})_{v_\ell} + L_{v_\ell}C_{p,w})^T W_{v_i} + W_{w_j} & -W_{v_i}\mathscr{B}_{p,m}(\mathscr{K}_{\xi_{p,w}})_{v_\ell} - (\mathscr{K}_{\xi_{p,w}})_{v_\ell}^T\mathscr{B}_{p,m}^T W_{v_i}^T + \rho S_{v_i}((\mathscr{A}_{p,w})_{v_\ell} + L_{v_\ell}C_{p,w}) + \rho((\mathscr{A}_{p,w})_{v_\ell} + L_{v_\ell}C_{p,w})^T S_{v_i} + \rho S_{w_j} \end{pmatrix}$$

$$< 0 \qquad i, j, \ell, k = 1, \cdots, N = 2^p,\ h = 1, \cdots, N_h, \tag{33}$$

$$\begin{pmatrix} \varepsilon[P(\theta(t))\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) + \mathscr{A}_k^T(\theta(t),\tilde{\theta}(t))P(\theta(t)) + dP(\theta(t))/dt] + W^T(\theta(t))\Delta\mathscr{A}(\delta(t)) + \Delta\mathscr{A}^T(\delta(t))W(\theta(t)) & (*)^T \\ W(\theta(t)\mathscr{A}_k(\theta(t),\tilde{\theta}(t)) + \rho S(\theta(t))\Delta\mathscr{A}(\delta(t)) - \varepsilon\mathscr{K}_{\xi_{p,w}}^T(\tilde{\theta}(t))\mathscr{B}_{p,m}^T P(\theta(t)) + \mathscr{A}_l^T(\tilde{\theta}(t))W(\theta(t)) + dW(\theta(t))/dt & -W(\theta(t))\mathscr{B}_{p,m}K(\tilde{\theta}(t)) - \mathscr{K}_{\xi_{p,w}}^T(\tilde{\theta}(t))\mathscr{B}_{p,m}^T W^T(\theta(t)) + \rho[S(\theta(t))\mathscr{A}_l(\tilde{\theta}(t)) + \mathscr{A}_l^T(\tilde{\theta}(t))S(\theta(t)) + dS(\theta(t))/dt] \end{pmatrix}$$

$$< 0 \qquad \forall(\theta(t),\dot{\theta}(t),\tilde{\theta}(t),\delta(t)) \in \Theta\times\dot{\Theta}\times\Theta\times\Delta \tag{34}$$

polytopes and choosing those γ and β yielding the eigenvalues with the most negative real part. The properness of this choice can be easily seen recalling the Schur complement and looking at the structure of (34). △

If condition (33) does not admit a solution for the given $\Delta\mathscr{A}(\delta(t))$, in any case, there surely exists a smaller polytope $\mathscr{P}_{\alpha\Delta A(\delta(t))} \subset \mathscr{P}_{\Delta A(\delta(t))}$, with vertex matrices $\Delta\mathscr{A}'_{v_k}$ affinely depending on some $\tilde{\delta}'_i = \alpha\cdot\tilde{\delta}_i$, with $0 \le \alpha < 1$, such that (33) is satisfied. Hence one can define a single-line iterative LMI procedure for different decreasing values of $\alpha \in [0,1)$. This iterative procedure gives an alternative solution to the original problem in terms of the maximum polytope $\mathscr{P}_{\alpha\Delta\mathscr{A}(\delta(t))}$ of the above kind satisfying (33).

A. The tracking problem

Let $H(s,\theta(t),\tilde{\theta}(t))$ be the system function of the LPV system Σ_f, as defined in [21], namely: $H(s,\theta(t),\tilde{\theta}(t)) = \int_{-\infty}^{\infty} W(\theta(t),\tilde{\theta}(t),\tau)e^{-s\tau}d\tau$, $W(\theta(t),\tilde{\theta}(t),\tau)$ being the impulsive response of Σ_f, and let $C_0(\theta(t),\tilde{\theta}(t)) := \lim_{s\to 0} H(s,\theta(t),\tilde{\theta}(t))$. The steady-state output response of Σ_f to $r(t) = r_0\cdot\delta_{-1}(t)$ is given by $\lim_{t\to\infty} r_0\cdot C_0(\theta(t),\tilde{\theta}(t))$ [22], and, by A4), $\lim_{t\to\infty} C_0(\theta(t),\tilde{\theta}(t)) \neq 0$. Assuming that $\theta(t)$ and $\tilde{\theta}(t)$ are such that $C_0(\theta(t),\tilde{\theta}(t))$ satisfies A5): $\int_{t_0}^{\infty}(r_0\cdot C_0(\theta(t),\tilde{\theta}(t)) - r_0)dt \neq 0$, for some $t_0 \geq 0$, the following corollary can be proved.

Corollary: The controller $\tilde{\Sigma}_c$ internally exponentially stabilizing Σ_f for some level of uncertainty $\alpha\Delta\mathscr{A}(\delta(t))$, is such that the error system Σ_e results to be state-output stable within the same level of uncertainty.

Proof: The internal exponential stability of Σ_f implies $C_0(\theta(t),\tilde{\theta}(t)) < \infty$, so that, by A5), one necessarily has $\lim_{t\to\infty} r_0\cdot C_0(\theta(t),\tilde{\theta}(t)) = \lim_{t\to\infty} y(t) = r_0\cdot\delta_{-1}(t)$ and $\lim_{t\to\infty} e_t(t) = 0$. In fact the internal model Σ_m of $r(t)$ integrates $e_t(t)$ and if the tracking error $e_t(t) = y(t) - r(t) = r_0\cdot C_0(\theta(t),\tilde{\theta}(t)) - r_0$, would not converge to zero, A4) and A5) would imply $\lim_{t\to\infty} r_0\cdot C_0(\theta(t),\tilde{\theta}(t)) = \infty$, against the hypothesis of internal stability of Σ_f. It follows that Σ_e is asymptotically state output stable. △

Remark 4 In essence, A5) is a rather weak condition on the possible trajectories of $\theta(t)$ and $\tilde{\theta}(t)$: the admissible trajectories can not be characterized by periodic oscillations inducing an asymptotic zero-mean periodic oscillating behavior of $y(t) = r_0\cdot C_0(\theta(t),\tilde{\theta}(t))$ around the chosen r_0. △

IV. Encompassing other cases

The purpose of this section is to briefly outline how the general approach followed to derive the required LMI based stability conditions may be used to recover other approaches and control situations. To this purpose it is enough to modify $\mathscr{K}(\tilde{\theta}(t))$, $\mathscr{L}(\tilde{\theta}(t))$, $P(\theta(t))$, $S(\theta(t))$, $W(\theta(t))$ as $\mathscr{K}(\tilde{\theta}(t)) = K_0 + \sum_{i=1}^p a_i\tilde{\theta}_i(t)K_i$, $\mathscr{L}(\tilde{\theta}(t)) = L_0 + \sum_{i=1}^p b_i\tilde{\theta}_i(t)K_i$, $P(\theta(t)) = P_0 + \sum_{i=1}^p c_i\theta_i(t)P_i$, $S(\theta(t)) = S_0 + \sum_{i=1}^p d_i\theta_i(t)S_i$, $W(\theta(t)) = W_0 + \sum_{i=1}^p e_i\theta_i(t)W_i$. where the a_i, b_i, c_i, d_i, e_i are dummy binary variables that may only assume the values 0 and 1. If some variables are set equal to 0, then the corresponding parameters θ_i and $\tilde{\theta}_i$ are excluded from the definition of the relative gain or Lyapunov matrix. Afterwards, the corresponding stability conditions can be derived by following the same procedure of Section

II, which clearly corresponds to all variables equal to 1. This easily allows the approach to be applied to the case where only some parameters are on line measurable [11]. Analogously, the quadratic stability approach (see e.g. [14], [15], [16]), is recovered putting $c_i = d_i = e_i = 0$, $i = 1, \cdots, p$. The case of exact parameter measures is obtained putting $A_p(\theta(t)) = A_p(\tilde{\theta}(t))$, whence $\Delta\mathscr{A}(\theta(t), \tilde{\theta}(t)) = 0$, and matrix $\hat{A}_f(\theta(t)\tilde{\theta}(t))$ assumes an upper triangular form. The closed-loop stability can be studied through the diagonal matrix $P_f(\theta(t)) = \text{diag}[\varepsilon P(\theta(t)), S(\theta(t))]$, obtained from (14), choosing $W(\theta(t)) = 0$, $\rho = 1$. The corresponding stability conditions (32) and (33) result to be greatly simplified. The first one is trivially satisfied $\forall \varepsilon > 0$, while, using the Schur complement, it is easily seen that the second one is satisfied for a large enough ε. Hence, in the case of exact parameter knowledge, it is enough to only solve the first step of the whole procedure.

V. Numerical example

The following mass spring damper system with time varying spring stiffness has been considered in [23]:

$$A_p(\theta(t)) = \begin{bmatrix} 0 & 1 \\ -0.5 - 0.5\theta(t) & -0.2 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_p = \begin{bmatrix} -1 & 0 \end{bmatrix}.$$

It is here assumed that: $|\theta(t)| \leq 1$, $|\dot{\theta}(t)| \leq 1$, $\forall t \geq 0$, $\theta(t)$ is measurable on line according to $\tilde{\theta}(t) = \theta(t) + \delta(t)$ with $|\delta(t)| \leq \tilde{\delta} = 2$, $\forall t \geq 0$. The state is also assumed to be affected by a scalar unitary disturbance $w(t)$ premultiplied by $M = [1, 1]^T$, namely $A_w = 0$, $C_w = 1$ and $x_w(0) = 1$. The external reference $r(t)$ to be tracked consists of a scalar unitary step signal so that $A_r = 0$, $C_r = 1$ and $x_r(0) = 1$. By [19] the internal model Σ_m is given by $A_m = 0$ and $B_m = 1$. By varying γ with a logarithmic scale inside $[10^{-5}, 10^5]$, on account of Remark 3 it is found that the best choice is $\gamma = 10^4$, for which Theorem 1 provides

$$\begin{aligned} \mathscr{K}(\tilde{\theta}(t)) &= [\mathscr{K}_{\xi_p}(\tilde{\theta}(t)) | \mathscr{K}_m(\tilde{\theta}(t))] = K_0 + \tilde{\theta}(t) K_1 \\ &= 10^6 \left[\begin{array}{cc|c} -3.1632 & -0.0103 & -1.1386 \end{array}\right] \\ &+ \tilde{\theta}(t) \cdot 10^{-5} \left[\begin{array}{cc|c} 0.2241 & 0.0017 & 0.0906 \end{array}\right] \end{aligned}$$

with

$$P(\theta(t)) = \begin{bmatrix} 4.3106 \cdot 10^{11} & 4.3162 \cdot 10^7 & 1.5687 \cdot 10^{11} \\ 4.3162 \cdot 10^7 & 4.4991 \cdot 10^3 & 1.5697 \cdot 10^7 \\ 1.5687 \cdot 10^{11} & 1.5697 \cdot 10^7 & 1.0798 \cdot 10^{12} \end{bmatrix}$$
$$+\theta(t) \begin{bmatrix} -1.8482 \cdot 10^5 & -26.3192 & -1.1026 \cdot 10^5 \\ -26.3192 & -0.0272 & -13.8482 \\ -1.1026 \cdot 10^5 & -13.8482 & -0.8813 \cdot 10^5 \end{bmatrix}.$$

Applying a similar procedure to Theorem 2, it is found that the best β is $\beta = 10^{-1}$, for which one has

$$\begin{aligned} \mathscr{L}(\tilde{\theta}(t)) &= [\mathscr{L}_{\xi_p}^T(\tilde{\theta}(t)) | \mathscr{L}_w^T(\tilde{\theta}(t))]^T = L_0 + \tilde{\theta}(t) L_1 \\ &= \left[\begin{array}{c} 8.6179 \\ 2.6410 \\ \hline 7.4014 \end{array}\right] + \tilde{\theta}(t) \left[\begin{array}{c} -0.0024 \\ -0.4984 \\ \hline -0.0023 \end{array}\right], \end{aligned}$$

with

$$\begin{aligned} S(\theta(t)) &= \begin{bmatrix} 4.2009 & -0.1024 & -1.6059 \\ -0.1024 & 2.3189 & -0.7731 \\ -1.6059 & -0.7731 & 1.8362 \end{bmatrix} \\ &+ \theta(t) \cdot 10^{-6} \begin{bmatrix} 0.3551 & -0.2016 & -0.1249 \\ -0.2016 & -0.8745 & -0.3629 \\ -0.1249 & -0.3629 & 0.2424 \end{bmatrix}. \end{aligned}$$

Theorem 3 does not admit a solution for the maximum theoretical uncertainty level $\tilde{\delta} = 2$. At any way, by varying α inside $[0 : 0.05 : 1]$ the iterative LMI procedure admits a solution for the smaller polytope $\mathscr{P}_{\alpha\Delta A(\delta(t))} \subset \mathscr{P}_{\Delta A(\delta(t))}$, with vertex matrices $\Delta\mathscr{A}'_{v_k}$ affinely depend on $\tilde{\delta}' = \alpha \cdot \tilde{\delta}$ with $\alpha = 0.99$. The Lyapunov matrix $P_f(\theta(t) > 0$ which guarantees the satisfaction of (15) is caracterized by $\rho = 745.621$, $\varepsilon = 8.6551 \cdot 10^{-10}$ and

$$\begin{aligned} W(\theta(t)) &= \begin{bmatrix} -568.9834 & -0.057 & -221.7947 \\ 17.0108 & 0.0017 & -74.0166 \\ 31.9933 & 0.0032 & -83.7587 \end{bmatrix} \\ &+ \theta(t) \begin{bmatrix} -0.009 & -4.0041 \cdot 10^{-6} & -0.0324 \\ -0.0022 & 3.9543 \cdot 10^{-7} & 0.0049 \\ -0.0044 & 5.6444 \cdot 10^{-7} & -0.0016 \end{bmatrix}. \end{aligned}$$

A tracking simulation of $r(t) = \delta_{-1}(t)$, has been performed starting from the initial condition $x_f(0) = [x_p^T(0), x_m^T(0), \xi_{x_p}^T(0), \xi_w^T(0), w^T(0)]^T = [[0,0]^T, 0, [0.1, 0.1]^T, 0.1, 1]^T$,and supposing that $\theta(t)$ and $\tilde{\theta}(t)$ are varying inside $\Theta := [-1, 1]$ as shown in Fig. 1. The trajectory of the controlled output $y(t)$ is shown in Fig.2.

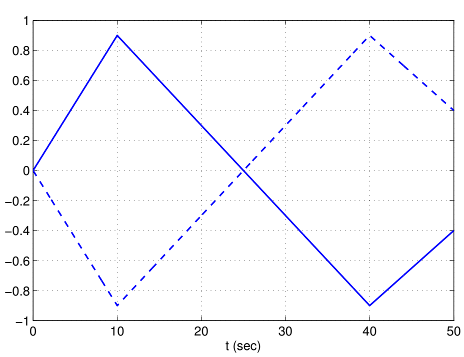

Fig. 1. Trajectories of $\theta(t)$ (continuous line) and $\tilde{\theta}(t)$ (dashed line)

VI. Conclusions

Up to now the LPV control problem has been mostly faced under the assumption of an exact parameter knowledge. Only few papers have recently considered the more realistic situation of a parametric mismatch. This paper dealt with the robust output regulation problem for continuous-time LPV system whose parameters are affected by a bounded uncertainty. A dynamic observer based controller has been designed exploiting the separation principle and LMI conditions have been stated to guarantee the internal exponential stability of the feedback system and the state-output stability of the error system. This approach results on a general

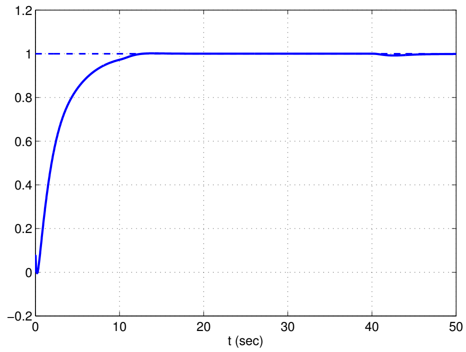

Fig. 2. Output response $y(t)$ of the plant

framework including other methods as particular cases. E.g. both parameter dependent ([12], [13],[24]) and parameter independent ([14], [15], [16],) Lyapunov functions can be considered properly choosing the binary variables c_i, d_i, e_i relative to the modified $P(\theta(t))$, $S(\theta(t))$ and $W(\theta(t))$ respectively, and deriving the corresponding LMI conditions following the same approach of Section III. As in general, there are not valid reasons to "a priori" state the least conservative synthesis method, this feature allows the designer to decide on the basis of an "a posteriori" comparison between different techniques, all derived within the same general framework. The cost of this procedure is in terms of an (off-line) increased computational effort.

For brevity, only the case of constant exogenous signals has been considered, but it is possible to show that, under assumption A5, the present approach can be extended to deal with a more general class of polynomial inputs.

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