Optimal tracking performance for unstable tall plant models

Patricio E. Valenzuela, Mario E. Salgado and Eduardo I. Silva

***Abstract*— This article focuses on the best achievable tracking performance of unstable tall plant models. The work is presented for discrete time, LTI systems, when an exponentially decaying signal is considered as reference. Closed form expressions for the best tracking performance for one and two degree of freedom control architectures are presented. As an application of those results, they are used in an example to compute the performance gains in the control of originally tall systems which have been squared-up by adding control input channels.**

***Index Terms*— Performance bounds, two degree of freedom control, augmented systems, optimal control, multivariable control.**

I. Introduction

This work focuses on the computation of performance bounds in discrete time MIMO feedback control systems. A performance bound describes the best achievable performance, as measured by a specific cost function, which can be achieved in the control of a given plant [1], [2], [3], [4]. Such an index can be used as a benchmark against which the result of any design method can be compared to.

Performance bounds in control systems have been the subject of much interest in the literature, and significant results have been obtained (see, e.g, [1], [2], [3], [4], [5], [6], and the references therein). The main contribution of these works is the development of closed form expressions for the best achievable performance, when a feedback control system is considered. In [1], the best achievable performance for continuous time feedback control system is studied. The results in [1] show that unstable poles, non-minimum-phase zeros and time delays worsen the optimal tracking performance. Similar results are presented in [2], [3], extending the analysis to discrete time MIMO feedback control systems.

The results in [1], [2], [3] can be only applied to right-invertible plant models. Results for tall plants (a class of non-right invertible plants) have been reported in [5], [7], [8], [9]. In [5], the best achievable tracking performance for SIMO systems is computed. The results in [5] show that not only non-minimum phase zeros and unstable poles affect the optimal performance, but also the total variation of the plant direction with frequency. Similar results are presented in [6], where closed form expressions are derived for discrete time SIMO systems, and an unify view of both continuous and discrete-time results is discussed. The results in [5], [6] assume that the reference vector lies in the subspace spanned by the plant DC-gain. Thus, the results are applicable only to special situations. An alternative is presented in [9], where arbitrary references are considered.

As the main contribution of this paper, we extend the results presented in [8], [9] by considering more general references, in a unique technique to solve the problem of tracking performance in tall plants. In addition, this paper presents a closed form expression for the optimal tracking performance when a two degree of freedom scheme is used to control a tall plant.

The remainder of this paper is organized as follows. Section II introduces notation and preliminaries. Section III presents the best achievable tracking performance for one degree of freedom control schemes. Section IV studies the best achievable tracking performance in two degree of freedom feedback control. Section V presents a case study. Finally, in Section VI, we present conclusions.

II. Notation and preliminaries

$\mathbb{C}$ is the complex field, $\mathbb{C}^{n\times m}$ are $n \times m$ matrices with entries in $\mathbb{C}$, $\mathbb{R}$ is the real field, and $\mathbb{R}^{n\times m}$ are real $n \times m$ matrices. Given a matrix $\mathbf{A} \in \mathbb{C}^{n\times m}$, $\mathbf{A}^T$ and $\mathbf{A}^H$ define its transpose and complex conjugate transpose, respectively. For a complex number x, $\overline{x}$ and $|x|$ are defined as its conjugate and magnitude, respectively; $\mathcal{R}_p^{n\times m}$ is the set of $n \times m$ transfer matrices which are real rational and proper; $\mathcal{R}_{sp}^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational and strictly proper; $\mathcal{RH}_\infty^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational, stable and proper; $\mathcal{RH}_2^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational, stable and strictly proper, and $\mathcal{RH}_2^{\perp\, n\times m}$ the set of $n\times m$ transfer matrices which are constant, improper and/or unstable.

Given any matrix valued function $\mathbf{X}[z] \in \mathbb{C}^{n\times m}$, we define

$$\mathbf{X}[z]^\sim \triangleq \mathbf{X}\left[\overline{z}^{-1}\right]^H , \tag{1}$$

which reduces to $\mathbf{X}[z]^\sim = \mathbf{X}\left[z^{-1}\right]^T$ in the real rational case. A transfer matrix $\mathbf{U}[z]$ is unitary if and only if

$$\mathbf{U}[z]^\sim \, \mathbf{U}[z] = \mathbf{I_m} , \tag{2}$$

where $\mathbf{I}$ is the identity matrix (In this paper we add a subscript to emphasize the size of identity matrices).

Any transfer matrix $\mathbf{P}[z] \in \mathcal{RH}_\infty^{n\times m}$ admits an inner-outer factorization

$$\mathbf{P}[z] = \mathbf{P_i}[z] \, \mathbf{P_o}[z] , \tag{3}$$

where $\mathbf{P_i}[z]$ is inner, i.e., belongs to $\mathcal{RH}_\infty^{n\times m}$ and $\mathbf{P_i}[z]^\sim \, \mathbf{P_i}[z] = \mathbf{I_m}$, and $\mathbf{P_o}[z]$ is outer, i.e., belongs to $\mathcal{RH}_\infty^{m\times m}$ and is right invertible in $\mathcal{RH}_\infty$ [10].

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A number $c \in \mathbb{C}$ is said to be a zero of $\mathbf{P}[z] \in \mathcal{R}_{sp}^{n\times m}$ if and only if rank$\{\mathbf{P}[c]\} <$ normalrank$\{\mathbf{P}[z]\}$. If $|c| > 1$, c is called a non-minimum-phase (NMP) zero, otherwise c is called a minimum-phase (MP) zero. For any square plant $\mathbf{H}[z] \in \mathcal{RH}_{\infty}^{n\times n}$, we introduce the factorization

$$\mathbf{H}[z] = \mathbf{E_{I\,,dc}}[z]\ \mathbf{H_{MP}}[z]\ , \tag{4}$$

where $\mathbf{H_{MP}}[z] \in \mathcal{RH}_{\infty}^{n\times n}$ is a minimum-phase (MP) transfer matrix, and $\mathbf{E_{I,\,dc}}[z] \in \mathcal{RH}_{2}^{n\times n}$ is a unitary matrix defined as

$$\mathbf{E_{I,\,dc}}[z]^{-1} \triangleq \mathbf{E_{I,\,c}}[z]^{-1}\ \mathbf{E_{I,\,d}}[z]^{-1}\ , \tag{5}$$

where $\mathbf{E_{I,\,c}}[z]^{-1} \in \mathcal{RH}_2^{\perp\,n\times n}$ and $\mathbf{E_{I,\,d}}[z]^{-1} \in \mathcal{RH}_2^{\perp\,n\times n}$ are given by

$$\mathbf{E_{I,\,c}}[z]^{-1} \triangleq \prod_{k=1}^{n_c}\left\{\frac{1-c_k}{1-\overline{c_k}}\frac{1-z\,\overline{c_k}}{z-c_k}\boldsymbol{\eta}_k\boldsymbol{\eta}_k^H + \mathbf{U}_k\mathbf{U}_k^H\right\}\ , \tag{6}$$

$$\mathbf{E_{I,\,d}}[z]^{-1} \triangleq \prod_{k=1}^{n_z}\left\{z\,\boldsymbol{\eta}_{\infty_k}\boldsymbol{\eta}_{\infty_k}^H + \mathbf{U}_{\infty_k}\mathbf{U}_{\infty_k}^H\right\}\ , \tag{7}$$

for some suitable $\boldsymbol{\eta}_k$, $\boldsymbol{\eta}_{\infty_k} \in \mathbb{C}^{n\times 1}$ and $\mathbf{U}_k$, $\mathbf{U}_{\infty_k} \in \mathbb{C}^{n\times(n-1)}$ satisfying $\boldsymbol{\eta}_k\,\boldsymbol{\eta}_k^H + \mathbf{U}_k\,\mathbf{U}_k^H = \boldsymbol{\eta}_{\infty_k}\,\boldsymbol{\eta}_{\infty_k}^H + \mathbf{U}_{\infty_k}\,\mathbf{U}_{\infty_k}^H = \mathbf{I_n}$, $\forall k$ [3].

The expectation operator is denoted by $\mathcal{E}\{\cdot\}$. The 2 norm of any $\mathbf{B}[z]$ with no poles on $|z| = 1$ is defined as

$$\|\mathbf{B}[z]\|_2 = \sqrt{\text{trace}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{B}\left[e^{j\omega}\right]^H\mathbf{B}\left[e^{j\omega}\right]\,d\omega\right\}}. \tag{8}$$

We note that $\mathcal{RH}_2^{n\times m}$ and $\mathcal{RH}_2^{\perp\,n\times m}$ are orthogonal sets, i.e.,

$$\left\|\{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp} + \{\mathbf{A}[z]\}_{\mathcal{H}_2}\right\|_2^2 = \left\|\{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp}\right\|_2^2 + \left\|\{\mathbf{A}[z]\}_{\mathcal{H}_2}\right\|_2^2. \tag{9}$$

where $\{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp}$ denotes the part of $\mathbf{A}[z] \in \mathcal{RH}_2^{\perp\,n\times m}$, and $\{\mathbf{A}[z]\}_{\mathcal{H}_2}$ denotes the part of $\mathbf{A}[z] \in \mathcal{RH}_2{}^{n\times m}$.

III. Tracking performance bounds for one DOF control architectures

A. Problem formulation

Consider the 1-dof[1] control scheme depicted in Figure 1. In that figure, $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$ $(n \geq m)$ is the plant model, $\mathbf{C}[z] \in \mathcal{R}_{p}^{m\times n}$ is a feedback controller, and $\mathbf{r}[k] \in \mathbb{R}^n$, $\mathbf{e}[k] \in \mathbb{R}^n$, $\mathbf{u}[k] \in \mathbb{R}^m$ are the reference, tracking error and control signals, respectively. We consider the functional

$$J \triangleq \sum_{k=0}^{\infty}(\mathbf{r}[k] - \mathbf{y}[k])^T(\mathbf{r}[k] - \mathbf{y}[k])\ , \tag{10}$$

when $\mathbf{r}[k] \triangleq \boldsymbol{\nu}\,\lambda^k$, with $\boldsymbol{\nu} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, such that $|\lambda| < 1$, i.e., when the reference is exponentially decaying signal. This reference choice guarantees the convergence of J to a finite value whenever the closed loop is stable.

[1]Degree of freedom.

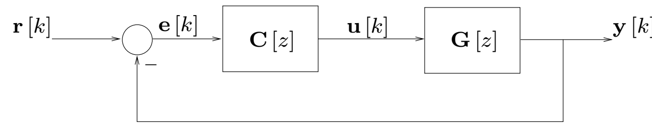

Fig. 1. One Degree of freedom control scheme.

If the closed loop is internally stable, then

$$J = \|\mathbf{E}[z]\|_2^2 \triangleq \|\mathbf{R}[z] - \mathbf{Y}[z]\|_2^2\ , \tag{11}$$

where $\mathbf{E}[z]$ is the $\mathcal{Z}$-Transform of the tracking error $\mathbf{e}[k]$. Using the closed loop description given in Figure 1, J can be written as

$$J = \left\|(\mathbf{I_n} + \mathbf{G}[z]\,\mathbf{C}[z])^{-1}\ \frac{\boldsymbol{\nu}}{z-\lambda}\right\|_2^2. \tag{12}$$

The expression (12) is non-linear in the controller $\mathbf{C}[z]$. To solve this problem, we use a coprime factorization of $\mathbf{G}[z]$ [11], [10], namely

$$\mathbf{G}[z] \triangleq \mathbf{N_D}[z]\,\mathbf{D_D}[z]^{-1} = \mathbf{D_I}[z]^{-1}\,\mathbf{N_I}[z]\ , \tag{13}$$

with $\mathbf{N_I}[z], \mathbf{N_D}[z], \in \mathcal{RH}_{\infty}^{n\times m}$, $\mathbf{D_I}[z] \in \mathcal{RH}_{\infty}^{n\times n}$, $\mathbf{D_D}[z] \in \mathcal{RH}_{\infty}^{m\times m}$, and $\mathbf{D_I}[z]$, $\mathbf{D_D}[z]$ biproper. These factors can be used to describe a stabilizing controller $\mathbf{C}[z]$ as [11], [10]

$$\mathbf{C}[z] \triangleq (\mathbf{Y_D}[z] - \mathbf{D_D}[z]\,\mathbf{Q}[z]) \times (\mathbf{N_D}[z]\,\mathbf{Q}[z] - \mathbf{X_D}[z])^{-1}\ , \tag{14}$$

where $\mathbf{Q}[z] \in \mathcal{RH}_{\infty}^{m\times n}$ is the design parameter, and $\mathbf{Y_I}[z], \mathbf{Y_D}[z] \in \mathcal{RH}_{\infty}^{m\times n}$, $\mathbf{X_I}[z] \in \mathcal{RH}_{\infty}^{m\times m}$, $\mathbf{X_D}[z] \in \mathcal{RH}_{\infty}^{n\times n}$ are such that

$$\begin{bmatrix}\mathbf{X_I}[z] & -\mathbf{Y_I}[z]\\ -\mathbf{N_I}[z] & \mathbf{D_I}[z]\end{bmatrix}\begin{bmatrix}\mathbf{D_D}[z] & \mathbf{Y_D}[z]\\ \mathbf{N_D}[z] & \mathbf{X_D}[z]\end{bmatrix} = \begin{bmatrix}\mathbf{I_m} & \mathbf{0}\\ \mathbf{0} & \mathbf{I_n}\end{bmatrix}. \tag{15}$$

Using (14) into (12), we get

$$J = \left\|(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\mathbf{D_I}[z]\ \frac{\boldsymbol{\nu}}{z-\lambda}\right\|_2^2. \tag{16}$$

To simplify our analysis, we assume that $\boldsymbol{\nu} \in \mathbb{R}^n$ is a random vector such that $\mathcal{E}\{\boldsymbol{\nu}\} = \mathbf{0}$, $\mathcal{E}\{\boldsymbol{\nu}\boldsymbol{\nu}^T\} = \mathbf{I_n}$, and note that (see (16))

$$\mathcal{E}\{J\} = \left\|(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\mathbf{D_I}[z]\ \frac{1}{z-\lambda}\right\|_2^2. \tag{17}$$

Finally, our study is focused on

Problem 1: Given $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, with no poles or zeros on the unit circle, find

$$J^{\text{opt}} \triangleq \inf_{\mathbf{Q}[z]\in\mathcal{RH}_{\infty}^{m\times n}}\left\|(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\mathbf{D_I}[z]\ \frac{1}{z-\lambda}\right\|_2^2, \tag{18}$$

and, if J^{opt} is achievable, then find $\mathbf{Q^{opt}}[z] \in \mathcal{RH}_{\infty}^{m\times n}$ that achieves J^{opt}. ∎

Problem 1 will be solved in the next section.

B. Optimal tracking performance

Theorem 1: Consider Problem 1, and assume that the plant $\mathbf{G}[z]$ has n_p different unstable poles $p_1, \ldots, p_{n_p}$ such that $\lambda \cdot p_i \neq 1, \forall i \in \{1, \ldots, n_p\}$. Also, consider a coprime factorization given by (13). Also, introduce the inner-outer factorizations $\mathbf{N_D}[z] = \mathbf{N_{Di}}[z]\mathbf{N_{Do}}[z]$ and $\mathbf{D_I}[z]^T = \mathbf{E_{D,c}}[z]^T \mathbf{D_{I,FM}}[z]^T$, with $\mathbf{E_{D,c}}[z]^T$ defined as in (6), and $\mathbf{D_{I,FM}}[z]$ stable, biproper and MP. Then, the solution to the Problem 1 is achieved by choosing $\mathbf{Q}[z] = \mathbf{Q^{opt}}[z]$, where

$$\mathbf{Q^{opt}}[z] = \mathbf{N_{Do}}[z]^{-1} \times (\mathbf{P}_1[\lambda] + \mathbf{P}_2[z])\,\mathbf{D_{I,FM}}[z]^{-1}, \quad (19)$$

and

$$\mathbf{P}_1[z] \triangleq \sum_{i=1}^{n_p} \frac{\mathbf{A}_i}{z - p_i}, \quad (20)$$

$$\mathbf{P}_2[z] \triangleq \mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1} - \mathbf{P}_1[z], \quad (21)$$

$$\mathbf{P}[z] \triangleq \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z], \quad (22)$$

$$\mathbf{A}_i \triangleq \lim_{z \to p_i} (z - p_i)\,\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}. \quad (23)$$

Also, the optimal cost satisfies

$$J^{\mathrm{opt}} = J_s^{\mathrm{opt}} + J_u^{\mathrm{opt}}, \quad (24)$$

where

$$J_s^{\mathrm{opt}} \triangleq \frac{1}{1-\lambda^2}\left(n - \mathrm{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right\}\right), \quad (25)$$

$$J_u^{\mathrm{opt}} \triangleq \frac{1}{1-\lambda^2}\mathrm{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right] - \mathbf{P}_1[\lambda]\right)^T \mathbf{P}_1[\lambda]\right\} + \mathrm{trace}\left\{\sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \overline{p_i} \cdot \frac{\mathbf{A}_i^H \mathbf{A}_j}{(1 - \lambda\overline{p_i})(\overline{p_i} - \lambda)(\overline{p_i}\,p_j - 1)}\right\}. \quad (26)$$

Proof: First, note from (15) that $\mathbf{N_D}[z]\,\mathbf{Y_I}[z] + \mathbf{I_n} = \mathbf{X_D}[z]\,\mathbf{D_I}[z]$. Then, (18) becomes

$$J^{\mathrm{opt}} = \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| (\mathbf{I_n} + \mathbf{N_D}[z]\,\mathbf{Y_I}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z])\,\frac{1}{z-\lambda} \right\|_2^2. \quad (27)$$

Using the unitary matrix

$$\mathbf{\Lambda}[z] \triangleq \begin{bmatrix} \mathbf{N_{Di}}[z]^\sim \\ \mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim \end{bmatrix}, \quad (28)$$

(27) can be written as

$$\begin{aligned} J^{\mathrm{opt}} &= \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| \mathbf{\Lambda}[z]\,(\mathbf{I_n} + \mathbf{N_D}[z]\,\mathbf{Y_I}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z])\,\frac{1}{z-\lambda} \right\|_2^2 \\ &= \left\| (\mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim)\,\frac{1}{z-\lambda} \right\|_2^2 \\ &\quad + \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| (\mathbf{N_{Di}}[z]^\sim + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z])\,\frac{1}{z-\lambda} \right\|_2^2. \end{aligned} \quad (29)$$

Also, noting that

$$\left(\mathbf{N_{Di}}[z]^\sim - \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right)\frac{1}{z-\lambda} \in \mathcal{RH}_2^{\perp\, m\times n} \quad (30)$$

$$\left(\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \in \mathcal{RH}_2^{m\times n}, \quad (31)$$

it is clear that

$$\begin{aligned} J^{\mathrm{opt}} &= \left\| (\mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim)\,\frac{1}{z-\lambda} \right\|_2^2 + \\ &+ \left\| \left(\mathbf{N_{Di}}[z]^\sim - \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right)\frac{1}{z-\lambda} \right\|_2^2 \\ &+ \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \right\|_2^2. \end{aligned} \quad (32)$$

It is straightforward to prove that

$$\begin{aligned} &\left\| (\mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim)\,\frac{1}{z-\lambda} \right\|_2^2 + \left\| \left(\mathbf{N_{Di}}[z]^\sim - \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right)\frac{1}{z-\lambda} \right\|_2^2 = \\ &\frac{1}{1-\lambda^2}\left(n - \mathrm{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right\}\right) = J_s^{\mathrm{opt}}, \end{aligned} \quad (33)$$

and thus (32) becomes

$$J^{\mathrm{opt}} = J_s^{\mathrm{opt}} + \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| (\mathbf{P}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z])\,\frac{1}{z-\lambda} \right\|_2^2, \quad (34)$$

with $\mathbf{P}[z]$ defined by (22). On the other hand, if we consider the factorization $\mathbf{D_I}[z] = \mathbf{D_{I,FM}}[z]\,\mathbf{E_{D,c}}[z]$, where $\mathbf{D_{I,FM}}[z]$ is biproper, stable and MP, and $\mathbf{E_{D,c}}[z]$ is unitary, then

$$J^{\mathrm{opt}} = J_s^{\mathrm{opt}} + \inf_{\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1} - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_{I,FM}}[z]\right)\frac{1}{z-\lambda} \right\|_2^2. \quad (35)$$

The expression $\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}$ contains stable and unstable terms. Therefore, it is necessary to make a partial fraction expansion. Given that the unstable terms correspond to the n_p unstable plant poles (with multiplicity one),

$$\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1} \triangleq \mathbf{P}_1[z] + \mathbf{P}_2[z], \quad (36)$$

with $\mathbf{P}_1[z]$ and $\mathbf{P}_2[z]$ as in (20) and (21). It follows that $\mathbf{P}_1[z] \in \mathcal{RH}_2^{\perp\, m\times n}$ and $\mathbf{P}_2[z] \in \mathcal{RH}_\infty^{m\times n}$. Therefore,

$$J^{\rm opt} = J_s^{\rm opt} + \left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 + \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{P}_1[\lambda] + \mathbf{P}_2[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_{I,FM}}[z]\right)\frac{1}{z-\lambda}\right\|_2^2 . \quad (37)$$

We can then note from (37) that $\mathbf{Q}[z]$ can be chosen according to (19) to obtain

$$J^{\rm opt} = J_s^{\rm opt} + \left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 . \quad (38)$$

To complete this proof, we need to prove that the second term on the right hand side of the equality in (38) is $J_u^{\rm opt}$, with $J_u^{\rm opt}$ defined as in (26). For that purpose, consider

$$\left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 = \text{trace}\left\{ \frac{1}{2\pi j}\oint \frac{\left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)^\sim}{1-z\lambda} \times \frac{\left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)}{z-\lambda}\,dz\right\}, \quad (39)$$

where the integral is on $|z| = 1$, counterclockwise oriented. The expression (39) can be computed using the Cauchy's Residue Theorem [12], obtaining

$$\begin{aligned}
\left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 &= \frac{1}{1-\lambda^2}\text{trace}\left\{\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[\lambda]\right\} \\
&\quad + \text{trace}\left\{ \frac{1}{2\pi j}\oint \left[\frac{\mathbf{P}_1[z]^\sim\,\mathbf{P}_1[z] - \mathbf{P}_1[z]^\sim\,\mathbf{P}_1[\lambda]}{(1-z\lambda)(z-\lambda)} \frac{-\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}\right] dz\right\}, \\
&= \text{trace}\left\{ \frac{1}{2\pi j}\oint \frac{\mathbf{P}_1[z]^\sim\,\mathbf{P}_1[z] - \mathbf{P}_1[z]^\sim\,\mathbf{P}_1[\lambda]}{(1-z\lambda)(z-\lambda)}\,dz\right\} \\
&= -\frac{1}{1-\lambda^2}\text{trace}\left\{\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[\lambda]\right\} + \text{trace}\left\{ \frac{1}{2\pi j}\oint \frac{\mathbf{P}_1[z]^\sim\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}\,dz\right\} . \quad (40)
\end{aligned}$$

The last term inside the integral in (40) has, at least, two NMP at infinity. This allows one to use the result reported in [13] to compute (40) as

$$\left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 = \frac{1}{1-\lambda^2}\text{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right] - \mathbf{P}_1[\lambda]\right)^T\mathbf{P}_1[\lambda]\right\} + \text{trace}\left\{\sum_{i=1}^{n_p} \underset{z=\overline{p_i}^{-1}}{\text{Res}}\left\{\frac{\mathbf{P}_1[z]^\sim\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}\right\}\right\}, \quad (41)$$

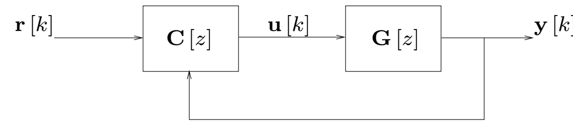

Fig. 2. Two Degree of freedom control scheme.

and, therefore,

$$\left\| \left(\mathbf{P}_1[z] - \mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2 = \frac{1}{1-\lambda^2}\text{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right] - \mathbf{P}_1[\lambda]\right)^T\mathbf{P}_1[\lambda]\right\} + \text{trace}\left\{\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\overline{p_i}\cdot\frac{\mathbf{A}_i^H\mathbf{A}_j}{(1-\lambda\overline{p_i})(\overline{p_i}-\lambda)(\overline{p_i}\,p_j-1)}\right\} . \quad (42)$$

This result requires that the unstable plant poles are simple and that $\lambda\cdot p_i \neq 1$ for $i = 1, 2, ..., n_p$. Using (42) into (38) we obtain (24), concluding this proof. ∎

Theorem 1 characterizes the optimal tracking performance for an unstable plant in terms of the sum of two expressions. The first term $J_s^{\rm opt}$ in (24) is a function of the finite and infinite NMP zeros of $\mathbf{G}[z]$. On the other hand, the second term $J_u^{\rm opt}$ in (24) is a function of the unstable poles of $\mathbf{G}[z]$.

We note that $J^{\rm opt}$ depends explicitly on the reference parameter λ, and on the number of output channels n. Finally, note also Theorem 1 presents a closed form expression for the optimal design parameter $\mathbf{Q^{opt}}[z]$ that achieves $J^{\rm opt}$.

The results presented in this section are obtained by considering a 1-dof control architecture. The next section studies the optimal tracking performance when a 2 parameter controller scheme is used.

IV. TRACKING PERFORMANCE BOUNDS FOR 2-DOF CONTROL ARCHITECTURES

A. Problem formulation

Consider the 2-dof control loop depicted in Figure 2. In this scheme, $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$ $(n \geq m)$ is the plant model, $\mathbf{r}[k] \in \mathbb{R}^n$ is the reference, $\mathbf{y}[k] \in \mathbb{R}^n$ is the system output, and $\mathbf{u}[k] \in \mathbb{R}^m$ is the control signal. We measure performance by

$$J \triangleq \sum_{k=0}^{\infty}\left(\mathbf{r}[k] - \mathbf{y}[k]\right)^T\left(\mathbf{r}[k] - \mathbf{y}[k]\right), \quad (43)$$

when $\mathbf{r}[k] \triangleq \boldsymbol{\nu}\lambda^k$, $\boldsymbol{\nu} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $|\lambda| < 1$. Assuming the closed loop to be stable, and using Parseval's Theorem, (43) can be written as

$$J = \left\|\mathbf{R}[z] - \mathbf{Y}[z]\right\|_2^2 . \quad (44)$$

As before, we consider the double coprime factorization for $\mathbf{G}[z]$ given by (13) and (15). It can be shown that all 2-parameter compensator $\mathbf{C}[z]$ that render the block scheme of Figure 2 internally stable can be written as [11]

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$$\left[\mathbf{C}_1[z] \;\vdots\; \mathbf{C}_2[z]\right] \triangleq \left(\mathbf{X_I}[z] - \mathbf{R_0}[z]\,\mathbf{N_I}[z]\right)^{-1} \times \left[\mathbf{Q}[z] \;\vdots\; \mathbf{Y_I}[z] - \mathbf{R_0}[z]\,\mathbf{D_I}[z]\right], \quad (45)$$

for some $\mathbf{Q}[z], \mathbf{R_0}[z] \in \mathcal{RH}_\infty^{m\times n}$. This factorization can be used to write J as

$$J = \left\| \left(\mathbf{I_n} - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right) \frac{\boldsymbol{\nu}}{z-\lambda} \right\|_2^2, \quad (46)$$

and, hence,

$$\mathcal{E}\{J\} = \left\| \left(\mathbf{I_n}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right) \frac{1}{z-\lambda} \right\|_2^2, \quad (47)$$

where we assumed $\boldsymbol{\nu}$ to be as in Section III-A.

Finally, the problem to be solved in this section is

Problem 2: Given $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, with no zeros on the unit circle, find

$$J^{\text{opt}} \triangleq \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{I_n}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right) \frac{1}{z-\lambda} \right\|_2^2, \quad (48)$$

and, if J^{opt} is achievable, then find $\mathbf{Q^{opt}}[z] \in \mathcal{RH}_\infty^{m\times n}$ that achieves J^{opt}. ∎

B. Optimal tracking performance

Corollary 1: Consider Problem 2, and assume that the plant $\mathbf{G}[z]$ has the coprime factorization in (13). Consider the inner-outer factorization $\mathbf{N_D}[z] \triangleq \mathbf{N_{Di}}[z]\,\mathbf{N_{Do}}[z]$. Then, a solution to Problem 2 is achieved by choosing $\mathbf{Q}[z] = \mathbf{Q^{opt}}[z]$, where

$$\mathbf{Q^{opt}}[z] = \mathbf{N_{Do}}[z]^{-1}\,\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T, \quad (49)$$

and the optimal performance is given by

$$J^{\text{opt}} = \frac{1}{1-\lambda^2}\left(n - \text{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right\}\right). \quad (50)$$

Proof: Direct from the proof of Theorem 1, replacing $\mathbf{Y_I}[z] = \mathbf{0}$ and $\mathbf{D_I}[z] = \mathbf{I_n}$. ∎

The result given in Corollary 1 shows that, when 2-dof architectures are used, the optimal tracking performance depends only on plant NMP zeros. If we compare this result with those obtained in Section III, we can observe that, for stable plants, the optimal tracking cost in 1 and 2-dof configurations are equal. These facts are of course consistent with previous literature [2], [14], [4].

V. Case Study

The previous results are valid for arbitrary tall systems and, in particular, for square ones. Here, we will exploit them to study the benefits, in terms of performance improvement, of adding new control inputs to a tall system squaring it up. Of course, our results can also be used to study the effect of deleting control inputs from a square system.

Example 1: Consider a tall plant $\mathbf{G_A}[z] \in \mathcal{R}_{sp}^{2\times 1}$ defined as

$$\mathbf{G_A}[z] \triangleq \begin{bmatrix} \dfrac{z-0.5}{z(z-p)} & \dfrac{z-0.5}{z(z-0.8)} \end{bmatrix}^T, \quad (51)$$

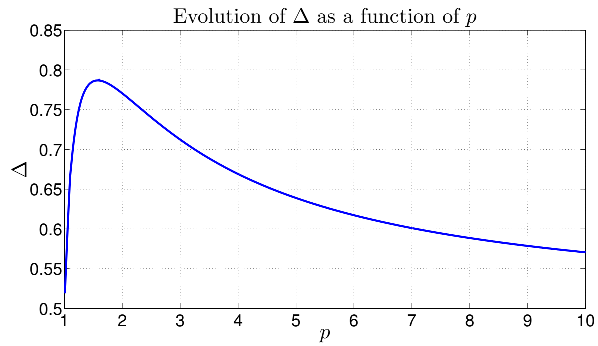

Fig. 3. Evolution of Δ as a function of p.

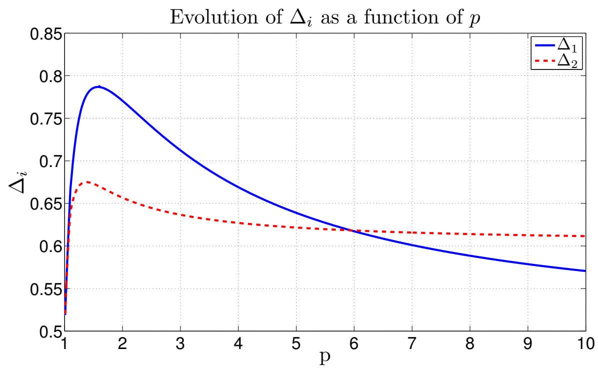

Fig. 4. Evolution of Δ_i as a function of p.

with $p \in \mathbb{R}$, such that $|p| > 1$. The system presented in (51) has one unstable pole at $z = p$, and one NMP zero at infinity. To improve the tracking performance of the system, we propose to add one control channel to get

$$\mathbf{G_S}[z] \triangleq \begin{bmatrix} \mathbf{G_A}[z] & \begin{matrix} \dfrac{z-0.2}{z(z-0.6)} \\[2ex] \dfrac{z-0.4}{z(z-0.6)} \end{matrix} \end{bmatrix}. \quad (52)$$

The augmented system (52) has the same unstable pole at $z = p$, two NMP zeros at infinity and one finite zero at $z = c \triangleq (0.2\,p - 0.32)\,/\,(p-1)$. Note that $\partial c/\partial p > 0$, and $|c| > 1$ for $1 < p < 1.1$.

To study the benefits of adding this new control input in a 1-dof control scheme, we consider the reference $\mathbf{r}[k] \triangleq (0.9)^k\,\boldsymbol{\nu}$, with $k \in \mathbb{N}_0$ and $\boldsymbol{\nu} \in \mathbb{R}^2$. Defining J_A^{opt} as the optimal tracking performance of $\mathbf{G_A}[z]$, and J_S^{opt} as the optimal tracking performance of $\mathbf{G_S}[z]$, we study

$$\Delta \triangleq \frac{J_A^{\text{opt}} - J_S^{\text{opt}}}{J_A^{\text{opt}}}, \quad (53)$$

as a function of p. Figure 3 shows Δ as a function of the unstable pole p. We can observe that the benefits of adding a new control input are always over 50% in this case.

Example 2: Consider the setup of Example 1.

We define $J_{A,1}^{\text{opt}}$ as the optimal tracking performance of the tall system in a 1-dof control scheme, $J_{A,2}^{\text{opt}}$ as the optimal tracking performance of the tall system in a 2-dof control scheme, $J_{S,1}^{\rm opt}$ as the optimal tracking performance of the augmented system in a 1-dof control scheme, and $J_{S,2}^{\rm opt}$ as the optimal tracking performance of the augmented system in a 2-dof control architecture. To quantify the benefits of using the augmented system, we define

$$\Delta_i \triangleq \frac{J_{A,i}^{\rm opt} - J_{S,i}^{\rm opt}}{J_{A,i}^{\rm opt}}, \tag{54}$$

where $i \in \{1, 2\}$. This index allows to compare the benefits of adding a new control input in a closed loop system. Also, we can study the effect of a new design parameter on the achievable performance.

Figure 4 shows Δ_i as a function of unstable pole p, for $i \in \{1, 2\}$. The results show that, when the pole p is less than 6 approx., the improvement of tracking performance in 1-dof control architectures is better than the improvement achieved in 2-dof schemes. However, the improvement achieved by 2-dof architectures is better than the improvement in 1-dof schemes when the pole p is greater than 6. Therefore, the value of p affects the benefits obtained in both control schemes, and the number of parameters to be designed depends on the location of the unstable pole p, in order to obtain a better improvement when a new control input is available and the number of design parameters can be chosen.

VI. Conclusions

This paper has studied the best achievable performance for tall systems, when a decaying reference is considered. Closed form expressions for the optimal performance have been computed both one and 2-dof architectures. The results show that NMP zeros, and unstable poles in one-dof scheme, have a deleterious effect on the tracking performance.

As a potential application of the results we have studied the benefits of adding additional control channels to a tall plant. The results suggest that the performance gains can be significant for a wide-range of cases.

Future work should focus on more realistic scenario including, for instance, energy constraints, or communication constraints in the additional channels.

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