Preprints of the 2012 20th Mediterranean Conference on
Control & Automation (MED), Barcelona, Spain, July 3-6,
2012

Risk-Averse Control of Linear Stochastic Systems with Low Sensitivity: A State-Feedback Paradigm

Khanh D. Pham
Air Force Research Laboratory
Space Vehicles Directorate
Kirtland Air Force Base, New Mexico 87117 U.S.A.

Abstract— **The problem of controlling stochastic linear systems with quadratic criterion which includes sensitivity variables is investigated. It is proved that the optimal full state-feedback control law with risk aversion can be realized by the cascade of mathematical statistics of performance uncertainty and a linear feedback. A set of nonlinear matrix equations are obtained, which constitutes the necessary and sufficient conditions that must be satisfied for an optimal solution.**

I. INTRODUCTION

Here, as in the traditional approaches to trajectory and performance index sensitivity reductions, the main objective has been achieving a tradeoff between optimality in the nominal performance and sensitivity to small parameter variations. In fact, discussions on open-loop and feedback control for deterministic linear regulators in [1] and [2] centered largely on the inclusion of a quadratic trajectory sensitivity term in the integrand of the performance index.

Following the order of presentations on parameter sensitivity reduction in linear regulators for deterministic cases [1] and [2], there is an emerging need of explicit recognition of a unified and systematic description of analysis and decision problems within a wider class of stochastic systems, described by higher-order characteristics of performance uncertainty and by controller designs with performance risk aversion. Examples of crucial developments by the author that have addressed the technical challenges aforementioned through the establishment of statistical optimal control include [3] and [4]. But, however, the theory of statistical optimal control can be further extended to the concern of trajectory and performance index sensitivity reduction. With this caveat, the main contribution that is evident in this paper is an effective integration of performance-information analysis into risk-averse feedback control selection for performance robustness and reliability requirements.

The structure of this paper is as follows. Section II contains the control problem and the notion of admissible control. In addition, the development of all the mathematical statistics for performance robustness is carefully discussed. The detailed problem statements and solution method in obtaining the stochastic optimal control are described in Sections III and IV. In Section V, some conclusions and future research extension are provided.

II. MATHEMATICAL STATISTICS FOR PERFORMANCE ROBUSTNESS

In this paper, some classes of problems with complete information shall be investigated. For instance, the stochastic system being controlled on $[t_0, t_f]$ is linear and depends on the constant parameter variable ζ with the known initial condition $x(t_0) = x_0$

$$dx(t) = (A(t,\zeta)x(t) + B(t,\zeta)u(t))dt + G(t)dw(t)\,, \quad (1)$$

where the coefficients $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times m})$ are the functions of the constant parameter variable ζ, except $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times p})$. The process noise $w(t) \in \mathbb{R}^p$ is the p-dimensional stationary Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ adapted over $[t_0, t_f]$ with the correlation of increments for all $\tau_1, \tau_2 \in [t_0, t_f]$ and $W > 0$

$$E\{[w(\tau_1) - w(\tau_2)][w(\tau_1) - w(\tau_2)]^T\} = W|\tau_1 - \tau_2|.$$

Notice that the state $x(t)$ is considered as the function of the constant parameter variable ζ, i.e., $x(t) \equiv \pi(t, x_0, \zeta)$. The traditional approach to sensitivity analysis for the system trajectory has been to define a sensitivity variable by

$$\sigma(t) \triangleq \frac{\partial}{\partial\zeta}\pi(t, x_0, \zeta)\bigg|_{\zeta=\zeta_0}\,. \quad (2)$$

Furthermore, the cost performance for controlling the system (1) is a finite-horizon integral quadratic form (IQF) random cost $J : \mathcal{C}([t_0, t_f]; \mathbb{R}^m) \mapsto \mathbb{R}^+$ such that

$$J(u) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} x^T(\tau)Q(\tau)x(\tau)d\tau + \int_{t_0}^{t_f} \left[u^T(\tau)R(\tau)u(\tau) + \sigma^T(\tau)S(\tau)\sigma(\tau)\right] d\tau\,, \quad (3)$$

where the state and sensitivity weighting symmetric matrices Q and $S \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n})$ are positive semidefinite as is the terminal penalty weighting symmetric matrix $Q_f \in \mathbb{R}^{n\times n}$. The control effort weighting symmetric matrix $R \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times m})$ is positive definite.

Next, the notion of admissible controls is discussed. In the case of complete information (e.g., state feedback measurement), an admissible control would have the form

$$u(t) = \gamma(t, x(t))\,, \quad t \in [t_0, t_f]\,.$$

As shown in [3], the search for optimal control solutions to the statistical optimal control problem may be consistently and productively restricted to linear time-varying feedback laws generated from the state $x(t)$ by

$$u(t) = K(t)x(t)\,, \quad t \in [t_0, t_f] \quad (4)$$

978-1-4673-2529-5/12/$31.00 ©2012 IEEE

with $K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times n})$ an admissible gain whose further defining properties will be stated shortly.

Then, for an admissible $K(\cdot)$ and the admissible pair (t_0, x_0), it gives a sufficient condition for the existence of $x(t)$ in (1). In view of (4), the controlled system (1) with the initial condition $x(t_0) = x_0$ is rewritten as follows

$$dx(t) = (A(t,\zeta) + B(t,\zeta)K(t))x(t)dt + G(t)dw(t)\,. \quad (5)$$

In addition, the variation of the parameter ζ from the nominal value ζ_0 was assumed to be small, such that the Taylor's expansion of the state and control functions can be approximated by retaining only the first two terms. Differentiating (5) with respect to ζ and evaluating at $\zeta = \zeta_0$ gives

$$\begin{aligned} d\sigma(t) &= (A_\zeta(t,\zeta_0) + B_\zeta(t,\zeta_0)K(t))x(t)dt \\ &\quad + (A(t,\zeta_0) + B(t,\zeta_0)K(t))\sigma(t)dt\,, \quad \sigma(t_0) = 0 \end{aligned} \quad (6)$$

whereby $A_\zeta \triangleq \frac{\partial A}{\partial \zeta}\Big|_{\zeta=\zeta_0}$ and $B_\zeta \triangleq \frac{\partial B}{\partial \zeta}\Big|_{\zeta=\zeta_0}$.

Hence, for the admissible pair (t_0, x_0) and the control policy (4), the aggregation of (5) and (6) is described by the controlled stochastic differential equation

$$dz(t) = F_a(t)z(t)dt + G_a(t)dw(t)\,, \quad z(t_0) = z_0 \quad (7)$$

with the performance measure (3) rewritten as follows

$$J(t_0, z_0) = z^T(t_f)N_f z(t_f) + \int_{t_0}^{t_f} z^T(\tau)N(\tau)z(\tau)d\tau \quad (8)$$

whereby $z = \begin{bmatrix} x^T & \sigma^T \end{bmatrix}^T$ and the continuous-time coefficients are given by

$$F_a(t) = \begin{bmatrix} A(t,\zeta)+B(t,\zeta)K(t) & 0 \\ A_\zeta(t,\zeta_0)+B_\zeta(t,\zeta_0)K(t) & A(t,\zeta_0)+B(t,\zeta_0)K(t) \end{bmatrix}$$

$$G_a(t) = \begin{bmatrix} G(t) \\ 0 \end{bmatrix}, \quad z_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

$$N_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}, N(t) = \begin{bmatrix} Q(t)+K^T(t)R(t)K(t) & 0 \\ 0 & S(t) \end{bmatrix}$$

In the analysis and design of performance-based uncertain systems, it is important to investigate a relation concerning two different subjects of knowledge, e.g,. process information (7) and goal information (8). In the descriptive approach herein, the control designer is concerned with performance variations described by knowledge representation in the form of a complete probabilistic description of the generalized chi-squared random variable (8). The learning process consists here in step by step knowledge validation and updating. At each step all the mathematical statistics associated with (8) in the knowledge updating should be determined before knowledge validation. The results of the successive determination of these performance-measure statistics associated with (8) are used in the formulation of the decisions in a learning decision making with performance risk aversion.

Mathematically stated, the knowledge validation and updating about performance variations of (8) is concerned with modeling and management of cumulants (also known as semi-invariants) associated with (8) as shown below.

Theorem 1: Cumulant-Generating Function.
Let $z(\cdot)$ be a state variable of the stochastic dynamics concerning sensitivity (7) with initial values $z(\tau) \equiv z_\tau$ and $\tau \in [t_0, t_f]$. Further let the moment-generating function be denoted by

$$\varphi(\tau, z_\tau, \theta) = \varrho(\tau,\theta)\exp\left\{z_\tau^T \Upsilon(\tau,\theta) z_\tau\right\} \quad (9)$$

$$\upsilon(\tau,\theta) = \ln\{\varrho(\tau,\theta)\}\,, \qquad \theta \in \mathbb{R}^+\,. \quad (10)$$

Then, the cumulant-generating function has the form of quadratic affine

$$\psi(\tau, z_\tau, \theta) = z_\tau^T \Upsilon(\tau,\theta) z_\tau + \upsilon(\tau,\theta) \quad (11)$$

where the scalar solution $\upsilon(\tau,\theta)$ solves the backward-in-time differential equation with the terminal-value condition $\upsilon(t_f,\theta) = 0$

$$\frac{d}{d\tau}\upsilon(\tau,\theta) = -\mathrm{Tr}\left\{\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\right\} \quad (12)$$

and the matrix solution $\Upsilon(\tau,\theta)$ satisfies the backward-in-time differential equation with the terminal-value condition $\Upsilon(t_f,\theta) = \theta N_f$

$$\begin{aligned} \frac{d}{d\tau}\Upsilon(\tau,\theta) &= -F_a^T(\tau)\Upsilon(\tau,\theta) - \Upsilon(\tau,\theta)F_a(\tau) \\ &\quad - 2\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\Upsilon(\tau,\theta) - \theta N(\tau)\,. \end{aligned} \quad (13)$$

Meanwhile, the scalar solution $\varrho(\tau)$ satisfies the backward-in-time differential equation with the terminal-value condition $\varrho(t_f,\theta) = 1$

$$\frac{d}{d\tau}\varrho(\tau,\theta) = -\varrho(\tau,\theta)\,\mathrm{Tr}\left\{\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\right\}. \quad (14)$$

Proof: For notional simplicity, it is convenient to have $\varpi(\tau, z_\tau, \theta) \triangleq \exp\{\theta J(\tau, z_\tau)\}$ in which the performance measure (8) is rewritten as the cost-to-go function from an arbitrary state z_τ at a running time $\tau \in [t_0, t_f]$, that is,

$$J(\tau, z_\tau) = z^T(t_f)N_f z(t_f) + \int_{\tau}^{t_f} z^T(t)N(t)z(t)dt \quad (15)$$

subject to

$$dz(t) = F_a(t)z(t)dt + G_a(t)dw(t)\,, \qquad z(\tau) = z_\tau\,. \quad (16)$$

By definition, the moment-generating function is $\varphi(\tau, z_\tau, \theta) \triangleq E\{\varpi(\tau, z_\tau, \theta)\}$. Thus, the total time derivative of $\varphi(\tau, z_\tau, \theta)$ is obtained as

$$\frac{d}{d\tau}\varphi(\tau, z_\tau, \theta) = -\varphi(\tau, z_\tau, \theta)\,\theta z_\tau^T N(\tau) z_\tau\,.$$

Using the standard Ito's formula, it follows

$$\begin{aligned} d\varphi(\tau, z_\tau, \theta) &= E\left\{d\varpi(\tau, z_\tau, \theta)\right\} \\ &= E\Big\{\varpi_\tau(\tau, z_\tau, \theta)\,d\tau + \varpi_{z_\tau}(\tau, z_\tau, \theta)\,dz_\tau \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varpi_{z_\tau z_\tau}(\tau, z_\tau, \theta)G_a(\tau)WG_a^T(\tau)\right\}d\tau\Big\}\,, \\ &= \varphi_\tau(\tau, z_\tau, \theta)d\tau + \varphi_{z_\tau}(\tau, z_\tau, \theta)F_a(\tau)z_\tau d\tau \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varphi_{z_\tau z_\tau}(\tau, z_\tau, \theta)G_a(\tau)WG_a^T(\tau)\right\}d\tau \end{aligned}$$

978-1-4673-2529-5/12/$31.00 ©2012 IEEE

which under the hypothesis of $\varphi(\tau, z_\tau, \theta) = \varrho(\tau,\theta)\exp\left\{z_\tau^T \Upsilon(\tau,\theta) z_\tau\right\}$ and its partial derivatives leads to the result

$$
\begin{aligned}
-\varphi(\tau, z_\tau, \theta)\, \theta z_\tau^T N(\tau) z_\tau &= \frac{\frac{d}{d\tau}\varrho(\tau,\theta)}{\varrho(\tau,\theta)} \varphi(\tau, z_\tau, \theta) \\
&\quad + \varphi(\tau, z_\tau, \theta)\, z_\tau^T \frac{d}{d\tau}\Upsilon(\tau,\theta) z_\tau \\
+ \varphi(\tau, z_\tau, \theta)\, z_\tau^T &\left[F_a^T(\tau)\Upsilon(\tau,\theta) + \Upsilon(\tau,\theta)F_a(\tau)\right] z_\tau \\
+ \varphi(\tau, z_\tau, \theta) &\Big[2 z_\tau^T \Upsilon(\tau,\theta) G_a(\tau) W G_a^T(\tau) \Upsilon(\tau,\theta) z_\tau \\
&\quad + \mathrm{Tr}\left\{\Upsilon(\tau,\theta) G_a(\tau) W G_a^T(\tau)\right\}\Big] .
\end{aligned}
$$

To have constant and quadratic terms being independent of arbitrary z_τ, it requires

$$
\begin{aligned}
\frac{d}{d\tau}\Upsilon(\tau,\theta) &= -F_a^T(\tau)\Upsilon(\tau,\theta) - \Upsilon(\tau,\theta)F_a(\tau) \\
&\quad - 2\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\Upsilon(\tau,\theta) - \theta N(\tau) \\
\frac{d}{d\tau}\varrho(\tau,\theta) &= -\varrho(\tau,\theta)\,\mathrm{Tr}\left\{\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\right\}
\end{aligned}
$$

with the terminal-value conditions $\Upsilon(t_f,\theta) = \theta N_f$ and $\varrho(t_f,\theta) = 1$. Finally, the backward-in-time differential equation satisfied by $\upsilon(\tau,\theta)$ is obtained

$$
\frac{d}{d\tau}\upsilon(\tau,\theta) = -\mathrm{Tr}\left\{\Upsilon(\tau,\theta)G_a(\tau)WG_a^T(\tau)\right\}, \quad \upsilon(t_f,\theta) = 0.
$$

which completes the proof. ■

As it turns out that all the higher-order characteristic distributions associated with performance uncertainty and risk are very well captured in the higher-order performance-measure statistics associated with (8). Subsequently, higher-order statistics that encapsulate the uncertain nature of (8) can now be generated via a Maclaurin series expansion of the cumulant-generating function (11)

$$
\psi(\tau, z_\tau, \theta) = \sum_{r=1}^{\infty} \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi(\tau, z_\tau, \theta)\right|_{\theta=0} \frac{\theta^r}{r!} \tag{17}
$$

in which all $\kappa_r \triangleq \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi(\tau, z_\tau, \theta)\right|_{\theta=0}$ are called performance-measure statistics. Moreover, the series expansion coefficients are computed by using the cumulant-generating function (11)

$$
\begin{aligned}
&\left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\psi(\tau, z_\tau, \theta)\right|_{\theta=0} \\
&\quad = z_\tau^T \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\Upsilon(\tau,\theta)\right|_{\theta=0} z_\tau + \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\upsilon(\tau,\theta)\right|_{\theta=0} .
\end{aligned} \tag{18}
$$

In view of the definition (17), the rth performance-measure statistic therefore follows

$$
\kappa_r = z_\tau^T \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\Upsilon(\tau,\theta)\right|_{\theta=0} z_\tau + \left.\frac{\partial^{(r)}}{\partial\theta^{(r)}}\upsilon(\tau,\theta)\right|_{\theta=0} \tag{19}
$$

for any finite $1 \leq r < \infty$. For notational convenience, the following change of notations

$$
H_r(\tau) \triangleq \left.\frac{\partial^{(r)}\Upsilon(\tau,\theta)}{\partial\theta^{(r)}}\right|_{\theta=0} ; D_r(\tau) \triangleq \left.\frac{\partial^{(r)}\upsilon(\tau,\theta)}{\partial\theta^{(r)}}\right|_{\theta=0} \tag{20}
$$

is introduced so that the next theorem provides an effective and accurate capability for forecasting all the higher-order characteristics associated with performance uncertainty. Therefore, via higher-order performance-measure statistics and adaptive decision making, it is anticipated that future performance variations will lose the element of surprise due to the inherent property of self-enforcing and risk-averse control solutions that are readily capable of reshaping the cumulative probability distribution of closed-loop performance.

Theorem 2: Performance-Measure Statistics.
Let the linear-quadratic stochastic system be described by (7) and (8), in which the pair (A, B) is uniformly stabilizable. For $k \in \mathbb{N}$ fixed, the kth cumulant of performance measure (8) is given by

$$
\kappa_k = z_0^T H_k(t_0) z_0 + D_k(t_0) \tag{21}
$$

where the supporting variables $\{H_r(\tau)\}_{r=1}^k$ and $\{D_r(\tau)\}_{r=1}^k$ evaluated at $\tau = t_0$ satisfy the differential equations (with the dependence of $H_r(\tau)$ and $D_r(\tau)$ upon $K(\tau)$ suppressed)

$$
\frac{d}{d\tau}H_1(\tau) = -F_a^T(\tau)H_1(\tau) - H_1(\tau)F_a(\tau) - N(\tau) \tag{22}
$$

$$
\begin{aligned}
\frac{d}{d\tau}H_r(\tau) &= -F_a^T(\tau)H_r(\tau) - H_r(\tau)F_a(\tau) \\
&\quad - \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} H_s(\tau)G_a(\tau)WG_a^T(\tau)H_{r-s}(\tau)
\end{aligned} \tag{23}
$$

$$
\frac{d}{d\tau}D_r(\tau) = -\mathrm{Tr}\left\{H_r(\tau)G_a(\tau)WG_a^T(\tau)\right\} \tag{24}
$$

whereby the terminal-value conditions $H_1(t_f) = N_f$, $H_r(t_f) = 0$ for $2 \leq r \leq k$ and $D_r(t_f) = 0$ for $1 \leq r \leq k$.

Proof: The expression of performance-measure statistics described in (21) is readily justified by using result (19) and definition (20). What remains is to show that the solutions $H_r(\tau)$ and $D_r(\tau)$ for $1 \leq r \leq k$ indeed satisfy the dynamical equations (22)-(24). Notice that these backward-in-time equations (22)-(24) satisfied by $H_r(\tau)$ and $D_r(\tau)$, are therefore obtained by successively taking derivatives with respect to θ of the supporting equations (12)-(13) under the assumption of (A, B) uniformly stabilizable on $[t_0, t_f]$. ■

To anticipate for a well-posed optimization problem, some sufficient conditions for the existence of solutions to the cumulant-generating equations (22)-(24) are now presented.

Theorem 3: Existence of Performance-Measure Statistics.
Let (A, B) be uniformly stabilizable. Then, any given $k \in \mathbb{N}$, the time-backward matrix differential equations (22)-(24) admit unique and bounded solutions $\{H_r(\tau)\}_{r=1}^k$ and $\{D_r(\tau)\}_{r=1}^k$ on $[t_0, t_f]$.

Proof: Under the assumption of stabilizability, there always exists a feedback control gain such that the continuous-time composite state matrix $F_a(t)$ is uniformly exponentially stable on $[t_0, t_f]$. In other words, there exist positive constants η_1 and η_2 such that the pointwise matrix norm of the closed-loop state transition matrix satisfies the inequality

$$
||\Phi_a(t,\tau)|| \leq \eta_1 e^{-\eta_2(t-\tau)}, \qquad \forall\ t \geq \tau \geq t_0 .
$$

According to the results in [6], the state transition matrix, $\Phi_a(t,t_0)$ associated with the continuous-time composite state matrix $F_a(t)$ has the sequel properties

$$\frac{d}{dt}\Phi_a(t,t_0) = F_a(t)\Phi_a(t,t_0)\,; \qquad \Phi_a(t_0,t_0) = I$$

$$\lim_{t_f\to\infty} ||\Phi_a(t_f,t)|| = 0\,; \quad \lim_{t_f\to\infty} \int_{t_0}^{t_f} ||\Phi_a(t_f,t)||^2 dt < \infty\,.$$

By the matrix variation of constant formula, the unique solutions to the time-backward matrix differential equations (22)-(24) together with the terminal-value conditions are then written as follows

$$H_1(\tau) = \Phi_a^T(t_f,\tau)N_f\Phi(t_f,\tau) + \int_\tau^{t_f} \Phi_a^T(t,\tau)N(t)\Phi_a(t,\tau)dt$$

$$H_r(\tau) = \int_\tau^{t_f} \Phi_a^T(t,\tau)\cdot \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} H_s(t)G_a(t)WG_a^T(t)H_{r-s}(t)\Phi_a(t,\tau)dt$$

$$D_r(\tau) = \int_\tau^{t_f} \mathrm{Tr}\left\{H_r(t)G_a(t)WG_a^T(t)\right\}dt\,, \quad 1 \le r \le k\,.$$

As long as the growth rate of the integrals is not faster than the exponentially decreasing rates of two factors of $\Phi_a(\cdot,\tau)$, it is therefore concluded that there exist upper bounds on the solutions $H_r(\tau)$ and $D_r(\tau)$ during the interval $[t_0,t_f]$. ■

III. PROBLEM STATEMENTS

The purpose of this section is to provide appropriate statements of statistical optimal control with the addition of the necessary and sufficient conditions for optimality for risk-averse feedback strategies with low sensitivity that are considered in this investigation. The statistical optimal control of linear stochastic systems with low sensitivity here is distinguished by the fact that the evolution in time of all mathematical statistics (21) associated with the random performance measure (8) of the generalized chi-squared type are described by means of matrix and scalar-valued differential equations (22)-(24).

For such problems it is important to have a compact statement arised from statistical optimal control so as to aid mathematical manipulations. To make this more precise, one may think of the k-tuple state variables $\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot),\ldots,\mathcal{H}_k(\cdot))$ and $\mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot),\ldots,\mathcal{D}_k(\cdot))$ whose continuously differentiable states $\mathcal{H}_r \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{2n\times 2n})$ and $\mathcal{D}_r \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ having the representations $\mathcal{H}_r(\cdot) \triangleq H_r(\cdot)$ and $\mathcal{D}_r(\cdot) \triangleq D_r(\cdot)$ with the right members satisfying the dynamics (22)-(24) are defined on $[t_0,t_f]$. In the remainder of the development, the bounded Lipschitz continuous mappings are introduced as

$$\mathcal{F}_r : [t_0,t_f] \times (\mathbb{R}^{2n\times 2n})^k \mapsto \mathbb{R}^{2n\times 2n}$$
$$\mathcal{G}_r : [t_0,t_f] \times (\mathbb{R}^{2n\times 2n})^k \mapsto \mathbb{R}$$

where the rules of action are given by

$$\mathcal{F}_1(\tau,\mathcal{H}) \triangleq -F_a^T(\tau)\mathcal{H}_1(\tau) - \mathcal{H}_1(\tau)F_a(\tau) - N(\tau)$$

$$\mathcal{F}_r(\tau,\mathcal{H}) \triangleq -F_a^T(\tau)\mathcal{H}_r(\tau) - \mathcal{H}_r(\tau)F_a(\tau) - \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s(\tau)G_a(\tau)WG_a^T(\tau)\mathcal{H}_{r-s}(\tau)$$

$$\mathcal{G}_r(\tau,\mathcal{H}) \triangleq -\mathrm{Tr}\left\{\mathcal{H}_r(\tau)G_a(\tau)WG_a^T(\tau)\right\}\,, \quad \le 1 \le r \le k\,.$$

The product mappings that follow are necessary for a compact formulation

$$\mathcal{F}_1 \times \cdots \times \mathcal{F}_k : [t_0,t_f] \times (\mathbb{R}^{2n\times 2n})^k \mapsto (\mathbb{R}^{2n\times 2n})^k$$
$$\mathcal{G}_1 \times \cdots \times \mathcal{G}_k : [t_0,t_f] \times (\mathbb{R}^{2n\times 2n})^k \mapsto \mathbb{R}^k$$

whereby the corresponding notations $\mathcal{F} \triangleq \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ and $\mathcal{G} \triangleq \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$ are used. Thus, the dynamic equations of motion (22)-(24) can be rewritten as

$$\frac{d}{d\tau}\mathcal{H}(\tau) = \mathcal{F}(\tau,\mathcal{H}(\tau)), \qquad \mathcal{H}(t_f) \equiv \mathcal{H}_f \tag{25}$$

$$\frac{d}{d\tau}\mathcal{D}(\tau) = \mathcal{G}\left(\tau,\mathcal{H}(\tau)\right), \qquad \mathcal{D}(t_f) \equiv \mathcal{D}_f \tag{26}$$

whereby the k-tuple values $\mathcal{H}_f \triangleq (N_f,0,\ldots,0)$ and $\mathcal{D}_f = (0,\ldots,0)$.

Notice that the product system uniquely determines the state matrices $\mathcal{H}$ and $\mathcal{D}$ once the admissible feedback gain K being specified. Henceforth, these state variables will be considered as $\mathcal{H} \equiv \mathcal{H}(\cdot,K)$ and $\mathcal{D} \equiv \mathcal{D}(\cdot,K)$. The performance index in optimal statistical control problems can now be formulated in K. For the given terminal data $(t_f,\mathcal{H}_f,\mathcal{D}_f)$, the classes of admissible feedback gains are next defined.

Definition 1: Admissible Feedback Gains.
Let compact subset $\overline{K} \subset \mathbb{R}^{m\times n}$ be the set of allowable feedback gain values. For the given $k \in \mathbb{N}$ and sequence $\mu = \{\mu_r \ge 0\}_{r=1}^k$ with $\mu_1 > 0$, the set of feedback gains $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ is assumed to be the class of $\mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n})$ with values $K(\cdot) \in \overline{K}$ for which solutions to the dynamic equations (25)-(26) with the terminal-value conditions $\mathcal{H}(t_f) = \mathcal{H}_f$ and $\mathcal{D}(t_f) = \mathcal{D}_f$ exist on the interval of optimization $[t_0,t_f]$.
The development that follows is to present the application of uncertain performance variable to risk-averse decision making for the decision problem of the random variable (8) described by probability distribution of the generalized chi-squared type. The knowledge of decision making here is dealt by a new concept of risk-value aware performance index which naturally contains some tradeoffs between performance values and risks for the subject class of stochastic control problems.

On $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ the performance index with risk-value considerations in the optimal statistical control is subsequently defined as follows.

Definition 2: Risk-Value Aware Performance Index.
Fix $k \in \mathbb{N}$ and the sequence of scalar coefficients $\mu = \{\mu_r \ge 0\}_{r=1}^k$ with $\mu_1 > 0$. Then for the given z_0, the risk-value

aware performance index

$$\phi_0 : \{t_0\} \times (\mathbb{R}^{2n\times 2n})^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$$

pertaining to the statistical optimal control of the stochastic system with low sensitivity over $[t_0, t_f]$ is defined by

$$\phi_0\left(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)\right) \triangleq \underbrace{\mu_1\kappa_1}_{\text{Value Measure}} + \underbrace{\mu_2\kappa_2 + \cdots + \mu_k\kappa_k}_{\text{Risk Measures}}$$
$$= \sum_{r=1}^{k} \mu_r \left[z_0^T \mathcal{H}_r(t_0) z_0 + \mathcal{D}_r(t_0)\right] \quad (27)$$

where additional design freedom by means of μ_r's utilized by the control designer with risk-averse attitudes are sufficient to meet and exceed different levels of performance-based reliability requirements, for instance, mean (i.e., the average of performance measure), variance (i.e., the dispersion of values of performance measure around its mean), skewness (i.e., the anti-symmetry of the density of performance measure), kurtosis (i.e., the heaviness in the density tails of performance measure), etc., pertaining to closed-loop performance variations and uncertainties while the supporting solutions $\{\mathcal{H}_r(\tau)\}_{r=1}^k$ and $\{\mathcal{D}_r(\tau)\}_{r=1}^k$ evaluated at $\tau = t_0$ satisfy the dynamical equations (25)-(26).

Next, the optimization statement over $[t_0, t_f]$ is stated.

Definition 3: Optimization Problem.

Fix $k \in \mathbb{N}$ and the sequence of scalar coefficients $\mu = \{\mu_r \geq 0\}_{r=1}^k$ with $\mu_1 > 0$. The optimization problem of the statistical optimal control over $[t_0, t_f]$ is given by

$$\min_{K(\cdot)\in\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}} \phi_0(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)) \quad (28)$$

subject to the dynamical equations (25)-(26) for $\tau \in [t_0, t_f]$.

Theorem 4: Property of Risk-Value Performance Index.

If the compact and convex $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ is nonempty, then the performance index (27) is strictly convex in K.

Proof: Indeed, the set of feedback strategies $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ is nonempty, compact and convex. What remains is to show the continuous function $\phi_0(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0))$ is strictly convex in K. Such a case is illustrated by aggregating the equations (22)-(23)

$$\frac{d}{d\tau}\Lambda(\tau) = -F_a^T(\tau)\Lambda(\tau) - \Lambda(\tau)F_a(\tau) - \mu_1 N(\tau) \quad (29)$$
$$-\sum_{r=2}^{k}\mu_r \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s(\tau)G_a(\tau)WG_a^T(\tau)\mathcal{H}_{r-s}(\tau)$$

whereby $\Lambda(\tau) \triangleq \sum_{r=1}^k \mu_r\mathcal{H}_r(\tau)$ and $\Lambda(t_f) = \mu_1 N_f$. The fundamental theorem of calculus and stochastic differential rule applied to $z^T(\tau)\Lambda(\tau)z(\tau)$ yield the result

$$E\left\{z^T(t_f)\mu_1 N_f z(t_f)\right\} - z_0^T\Lambda(t_0)z_0 =$$
$$E\left\{\int_{t_0}^{t_f} d\left[z^T(\tau)\Lambda(\tau)z(\tau)\right]\right\}$$
$$= E\Big\{\int_{t_0}^{t_f}\left[dz^T(\tau)\Lambda(\tau)z(\tau) + z^T(\tau)\Lambda(\tau)dz(\tau)\right]$$
$$z^T(\tau)\frac{d}{d\tau}\Lambda(\tau)z(\tau)d\tau + dz^T(\tau)\Lambda(\tau)dz(\tau)\Big\}.$$

After some manipulations, it follows that

$$E\left\{z^T(t_f)\mu_1 N_f z(t_f)\right\} - z_0^T\Lambda(t_0)z_0$$
$$= E\Big\{\int_{t_0}^{t_f} z^T(\tau)F_a(\tau)^T\Lambda(\tau)z(\tau)d\tau$$
$$+ z^T(\tau)\frac{d}{d\tau}\Lambda(\tau)z(\tau)d\tau + \int_{t_0}^{t_f} z^T(\tau)\Lambda(\tau)F_a(\tau)z(\tau)d\tau\Big\}$$
$$+ \int_{t_0}^{t_f} \text{Tr}\left\{\Lambda(\tau)G_a(\tau)WG_z^T(\tau)\right\}d\tau. \quad (30)$$

Notice that the solution of (24) is written by an integral form

$$\mathcal{D}_r(t_0) = \int_{t_0}^{t_f}\text{Tr}\left\{\mathcal{H}_r(\tau)G_a(\tau)WG_a^T(\tau)\right\}d\tau, \quad 1 \leq r \leq k$$

In view of the definition of $\Lambda(\cdot)$, it is then easy to see that

$$\sum_{r=1}^{k}\mu_r\mathcal{D}_r(t_0) = \int_{t_0}^{t_f}\text{Tr}\left\{\Lambda(\tau)G_a(\tau)WG_a^T(\tau)\right\}d\tau.$$

Henceforth, the performance index (27) is now rewritten as

$$\phi_0(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)) = z_0^T\Lambda(t_0)z_0$$
$$+ \int_{t_0}^{t_f}\text{Tr}\left\{\Lambda(\tau)G_a(\tau)WG_a^T(\tau)\right\}d\tau. \quad (31)$$

Replacing the results (29) and (31) into (30), it yields

$$\phi_0(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)) = E\left\{z^T(t_f)\mu_1 N_f z(t_f)\right\}$$
$$+ E\Bigg\{\int_{t_0}^{t_f} z^T(\tau)\Big[\mu_1 N(\tau) + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}$$
$$\mathcal{H}_s(\tau)G_a(\tau)WG_a^T(\tau)\mathcal{H}_{r-s}(\tau)\Big]m(\tau)d\tau\Bigg\} \quad (32)$$

which leads to

$$\phi_0(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0)) = \text{Tr}\left\{\mu_1 N_f P(t_f)\right\}$$
$$+ \text{Tr}\Bigg\{\int_{t_0}^{t_f}\Big[\mu_1 N(\tau) + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}$$
$$\mathcal{H}_s(\tau)G_a(\tau)WG_a^T(\tau)\mathcal{H}_{r-s}(\tau)\Big]P(\tau)d\tau\Bigg\} \quad (33)$$

where the positive-definite solution $P(\cdot) \triangleq E\{m(\cdot)m^T(\cdot)\}$ is satisfying the forward-in-time matrix-valued differential equation

$$\frac{d}{d\tau}P(\tau) = P(\tau)F_a^T(\tau) + F_a(\tau)P(\tau)$$
$$+ G_a(\tau)WG_a^T(\tau), \quad P(t_0) = z_0 z_0^T. \quad (34)$$

Notice that within the integrand of (33), $N(\cdot)$ is strictly convex in K while other factors are positive semi-definite. Henceforth, the generalized performance index (33), or equivalently risk-value aware performance index (27) is strictly convex in K. ■

IV. LOW SENSITIVITY CONTROL WITH RISK AVERSION

Opposite to the spirit of the earlier work by the authors [3] and [4] relative to the traditional approach of dynamic programming to the optimization problem of Mayer form, the problem (28) of finding extremals may, however, be recast as that of minimizing the fixed-time optimization problem in Bolza form, that is,

$$\phi_0(t_0, \mathcal{X}(t_0)) = \operatorname{Tr}\left\{\mathcal{X}(0) z_0 z_0^T\right\} + \int_{t_0}^{t_f} \operatorname{Tr}\left\{\mathcal{X}(t) G_a(t) W G_a^T(t)\right\} dt \quad (35)$$

subject to

$$\frac{d}{d\tau}\mathcal{X}(\tau) = -F_a^T(\tau)\mathcal{X}(\tau) - \mathcal{X}(\tau)F_a(\tau) - \mu_1 N(\tau) - \sum_{r=2}^{k}\mu_r \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s(\tau)G_a(\tau)WG_a^T(\tau)\mathcal{H}_{r-s}(\tau)\,, \quad (36)$$

wherein $\mathcal{X}(\tau) \triangleq \mu_1\mathcal{H}_1(\tau) + \cdots + \mu_k\mathcal{H}_k(\tau)$ and $\{\mathcal{H}_r(\tau)\}_{r=1}^{k}$ are satisfying the dynamical equations (22)-(24) with the initial-value condition $\mathcal{M}(t_f) = \mu_1 N_f$ for all $\tau \in [t_0, t_f]$.

Furthermore, the transformation of problem (35) and (36) into the framework required by the matrix minimum principle [5] that make it possible to apply the Pontryagin's results directly to the problems whose state variables are most conveniently regarded as matrices is complete if further changes of variables are introduced, that is, $t_f + t_0 - t = \tau$ and $\mathcal{X}(t_f + t_0 - t) = \mathcal{M}(t)$. Thus, the aggregate equation (36) with the initial-value condition $\mathcal{M}(t_0) = \mu_1 N_f$ is rewritten

$$\frac{d}{dt}\mathcal{M}(t) = F_a^T(t)\mathcal{M}(t) + \mathcal{M}(t)F_a(t) + \mu_1 N(t) + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s(t)G_a(t)WG_a^T(t)\mathcal{H}_{r-s}(t). \quad (37)$$

Now the aggregate matrix coefficients $F_a(t)$ and $N(t)$ for $t \in [t_0, t_f]$ of the composite dynamics (7) with trajectory sensitivity consideration are next partitioned to conform with the n-dimensional structure of (1) by means of $I_0^T \triangleq \begin{bmatrix} I & 0 \end{bmatrix}$ and $I_1^T \triangleq \begin{bmatrix} 0 & I \end{bmatrix}$, whereby I is an $n \times n$ identity matrix and

$$\begin{aligned} F_a(t) &= I_0(A(t,\zeta) + B(t,\zeta)K(t))I_0^T \\ &\quad + I_1(A_\zeta(t,\zeta_0) + B_\zeta(t,\zeta_0)K(t))I_0^T \\ &\quad + I_1(A(t,\zeta_0) + B(t,\zeta_0)K(t))I_1^T \end{aligned} \quad (38)$$

$$N(t) = I_0(Q(t) + K^T(t)R(t)K(t))I_0^T + I_1 S(t) I_1^T\,. \quad (39)$$

Assume that $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ is nonempty and convex in $\mathbb{R}^{m\times n}$. For all $(t, K) \in [t_0, t_f] \times \mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$, the maps $h(\mathcal{M})$ and $q(t, \mathcal{M}(t, K))$ having the property of twice continuously differentiable, as defined from the risk-value aware performance index (35)

$$\phi_0(K(\cdot)) = h(\mathcal{M}(t_f)) + \int_{t_0}^{t_f} q(t, \mathcal{M}(t, K(t)))dt = \operatorname{Tr}\left\{\mathcal{M}(t_f)z_0 z_0^T\right\} + \int_{t_0}^{t_f}\operatorname{Tr}\left\{\mathcal{M}(t)G_a(t)WG_a^T(t)\right\}dt \quad (40)$$

are supposed to have all partial derivatives with respect to $\mathcal{M}$ up to order 2 being continuous in $(\mathcal{M}, K)$ with appropriate growths.

Moreover, any admissible feedback gain $K^* \in \mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ minimizing the risk-value aware performance index (40) is called optimal strategies with risk aversion of the optimization problem (28). The corresponding state process $\mathcal{M}^*(\cdot)$ is called an optimal state process. Further denote $\mathcal{P}(t)$ by the co-state matrix associated with $\mathcal{M}(t)$ for each $t \in [t_0, t_f]$. The scalar Hamiltonian function for the optimization problem (37) and (40) is thus defined by

$$\mathcal{V}(t, \mathcal{M}, K) \triangleq \operatorname{Tr}\left\{\mathcal{M}G_aWG_a^T\right\} + \operatorname{Tr}\Big\{\Big[F_a^T\mathcal{M} + \mathcal{M}F_a + \mu_1 N + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_sG_aWG_a^T\mathcal{H}_{r-s}\Big]\mathcal{P}^T(t)\Big\}. \quad (41)$$

whereby in view of (38)-(39), the matrix variables $\mathcal{M}$, F_a and N shall be considered as $\mathcal{M}(t, K)$, $F_a(t, K)$ and $N(t, K)$, respectively.

Using the matrix minimum principle [5], the set of first-order necessary conditions for K^* to be an extremizer is composed of

$$\begin{aligned} \frac{d}{dt}\mathcal{M}^*(t) &= \left.\frac{\partial\mathcal{V}}{\partial\mathcal{P}}\right|_* \\ &= (F_a^*)^T(t)\mathcal{M}^*(t) + \mathcal{M}^*(t)F_a^*(t) + \mu_1 N^*(t) \\ &\quad + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(t)G_a(t)WG_a^T(t)\mathcal{H}_{r-s}^*(t) \end{aligned} \quad (42)$$

and

$$\begin{aligned} \frac{d}{dt}\mathcal{P}^*(t) &= -\left.\frac{\partial\mathcal{V}}{\partial\mathcal{M}}\right|_* \\ &= -F_a^*(t)\mathcal{P}^*(t) - \mathcal{P}^*(t)(F_a^*)^T(t) - G_a(t)WG_a^T(t) \end{aligned} \quad (43)$$

whereby $\mathcal{M}^*(t_0) = \mu_1 N_f$ and $\mathcal{P}^*(t_f) = z_0 z_0^T$.

Moreover, if K^* is a local extremum of (41), it implies

$$\mathcal{V}(t, \mathcal{M}^*(t), K) - \mathcal{V}(t, \mathcal{M}^*(t), K^*(t)) \geq 0 \quad (44)$$

for all $K \in \mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}$ and $t \in [t_0, t_f]$. That is,

$$\min_{K\in\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu}} \mathcal{V}(t, \mathcal{M}^*(t), K) = \mathcal{V}(t, \mathcal{M}^*(t), K^*(t)) = 0\,, \qquad \forall\, t \in [t_0, t_f]\,. \quad (45)$$

Equivalently, it follows that

$$\begin{aligned} 0 \equiv \left.\frac{\partial\mathcal{V}}{\partial K}\right|_* &= -2[B^T(t,\zeta)I_0^T + B_\zeta^T(t,\zeta_0)I_1^T]\mathcal{M}^*(t)\mathcal{P}^*(t)I_0 \\ &\quad - 2\mu_1 R(t)KI_0^T\mathcal{P}^*(t)I_0\,. \end{aligned} \quad (46)$$

Furthermore, the second-order sufficient conditions that ensure the Hamiltonian functional (41) achieving its local minimum, require the following Hessian matrix to be positive definite; in particular,

$$\left.\frac{\partial^2\mathcal{V}}{\partial K^2}\right|_* = 2\mu_1 R(t) \otimes I_0^T\mathcal{P}^*(t)I_0 \quad (47)$$

wherein $\otimes$ stands for the Kronecker matrix product operator.

By the matrix variation of constants formula [6], the matrix solutions of the cumulant-generating equations (22)-(23) and the co-state equation (43) can be rewritten in the integral forms, for each $\tau \in [t_0, t_f]$

$$\mathcal{H}_1^*(\tau) = \Phi^T(t_f,\tau)N_f\Phi(t_f,\tau) + \int_\tau^{t_f} \Phi(t_f,t)N^*(t)\Phi(t_f,t)dt \tag{48}$$

$$\mathcal{H}_r^*(\tau) = \int_\tau^{t_f} \Phi(t_f,t)\cdot \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(t)G_a(t)WG_a^T(t)\mathcal{H}_{r-s}^*(t)\Phi(t_f,t)dt \tag{49}$$

$$\mathcal{P}^*(\tau) = \Phi^T(t_f,\tau)z_0z_0^T\Phi(t_f,\tau) + \int_\tau^{t_f} \Phi(t_f,t)G_a(t)WG_a^T(t)\Phi(t_f,t)dt \tag{50}$$

provided that $\frac{d}{dt}\Phi(t,t_0) = F_a^*(t)\Phi(t,t_0)$ and $\Phi(t_0,t_0) = I$.

It can easily be verified that the following matrix inequalities hold for all $t \in [t_0, t_f]$

$$N_f \geq 0; \quad N^*(t) > 0; \quad z_0z_0^T \geq 0; \quad G_a(t)WG_a^T(t) > 0$$

$$\sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(t)G_a(t)WG_a^T(t)\mathcal{H}_{r-s}^*(t) \geq 0\,.$$

Therefore, it implies that $\{\mathcal{H}_r^*(t)\}_{r=1}^k$ and thus $\mathcal{M}^*(t)$, as well as $\mathcal{P}^*(t)$ with the integral forms (48)-(50) are positive definite on $[t_0, t_f]$. Subsequently, the following matrix inequality is shown to be valid

$$I_0^T\mathcal{P}^*(t)I_0 > 0\,, \qquad \forall t \in [t_0, t_f]\,. \tag{51}$$

In view of (51), it is concluded that the Hessian matrix (47) is indeed positive definite, in addition with the fact of $R(t)$ being positive definite for all $t \in [t_0, t_f]$. As the result, the local extremizer K^* formed by the first order necessary condition (46) becomes a local minimizer.

Notice that the result (42) is coupled by the forward-in-time and backward-in-time matrix valued differential equations. Putting the corresponding state and co-state equations together, the following optimality system with risk aversion and low sensitivity is summarized as follows.

Theorem 5: Risk-Averse Control with Low Sensitivity. Let (A, B) be uniformly stablizable and $u(\cdot) = K(\cdot)x(\cdot)$. Then, the low sensitivity feedback strategy with risk aversion supported by the optimal feedback gain $K^*(\cdot)$ is given by

$$K^*(t) = -R^{-1}(t)[B^T(t,\zeta)I_0^T\sum_{r=1}^k \hat{\mu}_r\mathcal{H}_r^*(t)\mathcal{P}^*(t_0+t_f-t)I_0 + B_\zeta^T(t,\zeta_0)I_1^T\sum_{r=1}^k \hat{\mu}_r\mathcal{H}_r^*(t)\mathcal{P}^*(t_0+t_f-t)I_0]\cdot [I_0^T\mathcal{P}^*(t_0+t_f-t)I_0]^{-1} \tag{52}$$

where the normalized parametric design of freedom $\hat{\mu}_r \triangleq \frac{\mu_r}{\mu_1}$ and the optimal state solutions $\{\mathcal{H}_r^*(\cdot)\}_{r=1}^k$ supporting all the statistics for performance robustness and risk-averse decisions are governed by the forward-in-time matrix valued differential equations with the terminal-value conditions $\mathcal{H}_1^*(t_0) = N_f$ and $\mathcal{H}_r^*(t_0) = 0$ for $2 \leq r \leq k$

$$\frac{d}{dt}\mathcal{H}_1^*(t) = (F_a^*)^T(t)\mathcal{H}_1^*(t) + \mathcal{H}_1^*(t)F_a^*(t) + N^*(t) \tag{53}$$

$$\frac{d}{dt}\mathcal{H}_r^*(t) = (F_a^*)^T(t)\mathcal{H}_r^*(t) + \mathcal{H}_r^*(t)F_a^*(t) + \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(t)G_a(t)WG_a^T(t)\mathcal{H}_{r-s}^*(t) \tag{54}$$

and the optimal co-state solution $\mathcal{P}^*(\cdot)$ satisfies the backward-in-time matrix valued differential equation

$$\frac{d}{dt}\mathcal{P}^*(t) = -F_a^*(t)\mathcal{P}^*(t) - \mathcal{P}^*(t)(F_a^*)^T(t) - G_a(t)WG_a^T(t)$$

$$\mathcal{P}^*(t_f) = z_0z_0^T\,. \tag{55}$$

Remarks. The results herein are certainly viewed as the generalization of those obtained from [3]. With respect to the subject of performance robustness, the previously developed work has fundamentally focused on the higher-order assessment of performance variations. Hence, the work therein [3] has not yet addressed the system trajectory sensitivity with respect to constant parameter variables.

V. CONCLUSIONS

The optimal risk-averse control strategy with low sensitivity has been obtained. A two-point boundary value problem involving matrix differential equations must be solved. Moreover, the states $\{\mathcal{H}_r(\cdot)^*\}_{r=1}^k$ and co-state $\mathcal{P}^*(\cdot)$ play an important role in the determination of feedback strategies with risk aversion. It is important to note that the feedback gain $K^*(\cdot)$ depends on the mathematical statistics associated with performance uncertainty; in particular, mean, variance, skewness, etc. These statistics serve not only as feedback information for future risk-averse decisions and but also as an influence mechanism for the low sensitivity controller.

Finally, it is conjectured that an observer or a dynamical compensator could be used to implement a low sensitivity control with risk aversion. The corresponding problem would then be cast into an output feedback form and thus, will be the emerging subject of future research investigation.

REFERENCES

[1] H. D'Angelo, M. L. Moe, and T. C. Hendricks, "Trajectory Sensitivity of an Optimal Control Systems," 4th Allerton Conference on Circuit and Systems Theory, pp. 489-498, 1966

[2] S. J. Kahne, "Low-Sensitivity Design of Optimal Linear Control Systems," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-4, No. 3, pp. 374-379, 1968

[3] K. D. Pham, M. K. Sain, and S. R. Liberty, "Cost Cumulant Control: State-Feedback, Finite-Horizon Paradigm with Application to Seismic Protection," Special Issue of *Journal of Optimization Theory and Applications,* Edited by A. Miele, Kluwer Academic/Plenum Publishers, New York, Vol. 115, No. 3, pp. 685-710, 2002

[4] K. D. Pham, "New Risk-Averse Control Paradigm for Stochastic Two-Time-Scale Systems and Performance Robustness," *Journal of Optimization Theory and Applications,* Edited by A. Miele, Kluwer Academic/Plenum Publishers, Vol. 146, No. 2, pp. 511-537, 2010

[5] M. Athans, "The Matrix Minimum Principle," *IEEE Transactions on Information and Control,* Vol. 11, pp. 592-606, 1968

[6] R. W. Brockett, *Finite Dimensional Linear Systems.* New York: Wiley, 1970