Fast Calculation of Robust Stabilizing Proportional Controllers

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Abstract— **In this paper a fast algorithm to determine all stabilizing proportional controllers for single-input single-output systems with multiplicative uncertainty is considered.**

I. INTRODUCTION

STABILITY is the most important property in the design of all dynamical systems. A reasonable approach to controller design is to find the set of all stabilizing compensators and then using a member of this set to satisfy further design criteria. A complete parameterization of all stabilizing controllers for a given system was suggested by Youla et. al. [1],[2]. An important disadvantage of this parameterization is that the order of the controller cannot be fixed. Therefore, in the last few years computation of all stabilizing controllers of a given order is examined by several researchers [3]-[11].

This paper presents a fast algorithm to find all stabilizing gains and the idea behind this study can be traced back to the papers of Söylemez et. al. [7],[12]. The main idea is related with the Nyquist stability criterion that without actually drawing the polar plot decision on the stabilizing gain intervals is made. Here, the determination of the real axis crossing directions is done very fast, so the algorithm has become faster. The idea and the new algorithm is used in calculation of all robustly stabilizing gains for a single-input single output linear time-invariant system with multiplicative uncertainty.

II. STABILIZING GAINS

A. SISO LTI Model without Uncertainty

Consider a simple single-input single-output control system, as shown in Fig.1, which has a transfer function as below

$$G(s)=\frac{N(s)}{D(s)}=\frac{a_m s^m + a_{m-1}s^{m-1}+..+a_1 s+a_0}{s^n+b_{n-1}s^{n-1}+....b_1 s+b_0} \qquad (1)$$

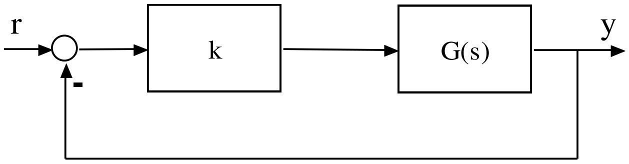

Fig. 1. Closed-loop control system with proportional control

After substituting $s = jw$, we may represent (1) as follows,

$$G(jw)=\mathrm{Re}\{G(jw)\}+\mathrm{Im}\{G(jw)\} \qquad (2)$$

where $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$ are for the real and the imaginary parts of $G(jw)$, respectively. To find all nominal or H_∞ robust stabilizing gains k, all critical frequency values should be known. Critical frequencies are the frequencies w, that satisfy the equation,

$$\mathrm{Im}\{G(jw)\}=0\,. \qquad (3)$$

In other words, these are the frequencies, where the Nyquist plot of $G(jw)$ crosses the real axis. If we call these frequencies w^*, and substitute them in $\mathrm{Re}\{G(jw)\}$, we can find the x values; the $x-$intercepts (real-axis intercepts) of the Nyquist plot of $G(jw)$. Note that, it is of special importance that we don't actually need to draw the Nyquist plot. The gains k, related to the critical frequencies, can be computed by,

$$k_i=-\frac{1}{x_i} \qquad (4)$$

where the x_i $(i=1,2,...)$ denote the values Nyquist plot of $G(jw)$ crosses the real axis, as mentioned before. To decide if the corresponding gain k^* stabilizes the system, we may use the following algorithm:

Step 1. Find all critical frequencies.

Step 2. Find all $x-$intercepts.

Step 3. Find all critical gains.

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Step 4. Sort the gains in ascending order.

Step 5. Sort all frequencies in ascending order.

Step 6. Find the sign of $\mathrm{Im}\{G(jw)\}$ for the last frequency interval, which is the sign of the ratio of leading coefficients of the numerator and denominator polynomials.

Step 7. Build the sign table for $\mathrm{Im}\{G(jw)\}$ by checking the multiplicities of the critical frequencies only. If the algebraic multiplicity of the critical frequency is an odd number, there will be a sign change. A change in sign from negative to positive (from positive to negative) indicates that two complex conjugate closed loop transfer function poles become "unstable" ("stable"). For $w^* = 0$ only one pole is moving from left to right or vice versa.

Step 8. Calculate the number of all unstable closed loop poles for a gain in any critical gain interval.

Step 9. Go to the next critical gain, check the sign change using the sign table. If the sign changes from negative to positive add 2 to the number of unstable poles. If the sign changes from positive to negative subtract 2. If no sign change is observed (algebraic multiplicity of critical frequency is an even number), the number of unstable poles remains the same. Note that, for $w^* = 0$ add one or subtract one. Repeat this step until all gain intervals are visited.

Step 10. Call those intervals stabilizing gain intervals if the algorithm ended with a zero unstable pole count in that interval. □

The algorithm is explained by the following example.

B. *Example 1*

Consider a system with the transfer function,

$$G(s) = \frac{2s^4 - 12s^3 + 24s^2 - 108s + 30}{s^5 + 11.8s^4 + 183.81s^3 + 1497.9s^2 + 2862.4s + 5579.6} \quad (5)$$

The Nyquist plot of the system at different scales is given in Fig 2. The imaginary part of $G(jw)$ is obtained as

$$\frac{jw\left(-2w^8 + 533.22w^6 - 2941.5w^4 + 302940w^2 - 688470\right)}{w^{10} - 228.38w^8 + 4160.5w^6 + 1323100w^4 - 8522000w^2 + 31132000} \quad (6)$$

Using (3) and (6) the critical frequencies are calculated as $\{13.97097, 7.67535, 3.04765, 1.79531, 0\}$. Corresponding real axis intercepts, x_i's, and critical gains, k_i's, are given as $\{0.43954, -0.13323, 0.00279, -0.030408, 0.00538\}$ and $\{-2.27513, 7.50591, -359.00759, 32.89473, -185.98667\}$, respectively.

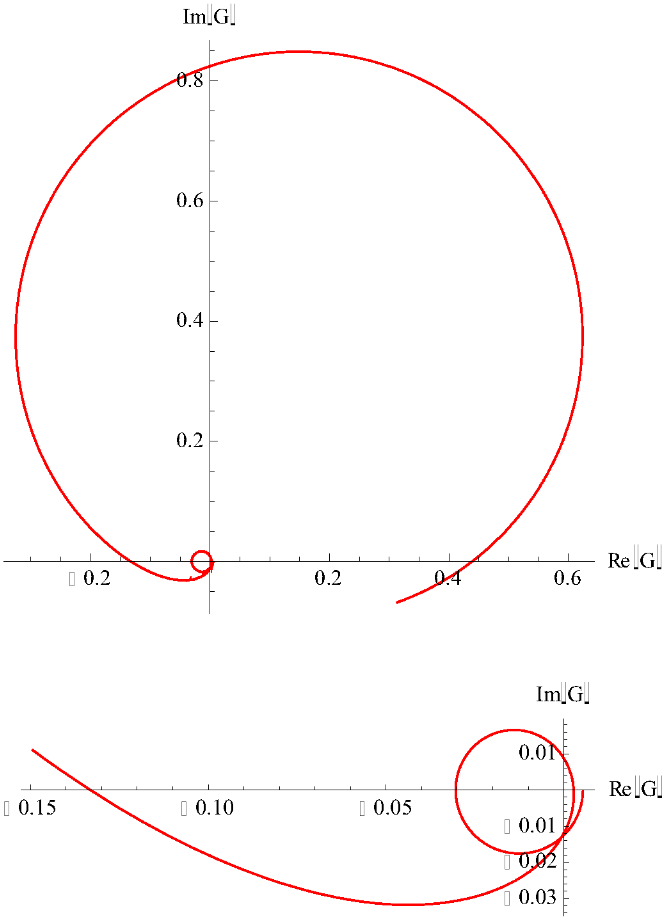

Fig 2. Nyquist Plot for Example 1 at different scales

The sign of the ratio of leading coefficients of numerator and denominator polynomials in $\mathrm{Im}\{G(jw)\}$ is $sign(-2) = -1$. We may represent the change of sign as in Table I, starting with -1. Recall that the algebraic multiplicities of the roots are odd, therefore at each critical frequency there will be a sign change.

TABLE I
SIGN TABLE FOR Im{G(jw)}

$w-$ Interval	Sign of Im{G(jw)}	Multiplicity of the root
(13.97097, ∞)	-	odd
(7.67535,13.97097)	+	odd
(3.04765, 7.67535)	-	odd
(1.7953, 3.04765)	+	odd
(0, 1.79531)	-	odd

Let us reorder the critical gains. Using the facts presented in sign table above we can immediate determine the stabilizing gain intervals. The corresponding gains and critical frequencies are given in Table II.

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TABLE II

CALCULATION OF GAIN INTERVAL

k_{i-1}	k_i	$\# C_+$	w^*	#c
-∞	-359.00759	5	3.04765	-2
-359.00759	-185.98667	3	0	-1
-185.98667	-2.27513	2	13.97097	-2
-2.27513	7.50591	**0**	7.67535	2
7.50591	32.89473	2	1.79531	2
32.89473	∞	4	∞	1

On the table above, #c and $\# C_+$ denote number of crossings and number of unstable poles, respectively. We conclude that he stabilizing gain interval is $k \in (-2.27513,\ 7.50591)$.

C. SISO LTI System with Multiplicative Uncertainty

Consider again $G(s)$ in (1) with multiplicative uncertainty as shown in Fig 3.

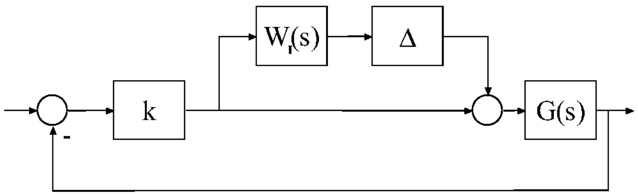

Fig 3. Closed-loop control of an uncertainty system

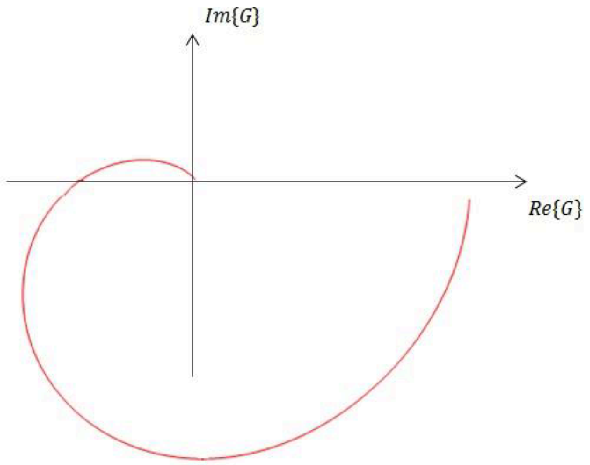

Fig 4. Nyquist Plot of the nominal system.

Assume the nominal system $G(s)$ has a frequency response as is Fig 4. We define a vector as

$$G(jw^*) = \begin{pmatrix} \mathrm{Re}\{G(jw^*)\} \\ \mathrm{Im}\{G(jw^*)\} \end{pmatrix}. \tag{7}$$

This vector points the value on Nyquist plot at the given frequency $w = w^*$. The frequency response of a system $G(jw)$ with multiplicative uncertainty at a given frequency w^* can be represented with a disk centered $G(jw^*)$ and a radius $r = |W_I G(jw^*)|$. To find the farthest points from system's Nyquist plot that uncertainty causes, firstly, we define a unit vector to the Nyquist, prensented with $\vec{a}$ in Fig 5, which is tangent at the point $G(jw^*)$.

$$\vec{a} = \begin{pmatrix} \mathrm{Re}\{G'(jw^*)\} \\ \mathrm{Im}\{G'(jw^*)\} \end{pmatrix} \frac{1}{|G'(jw^*)|} \tag{8}$$

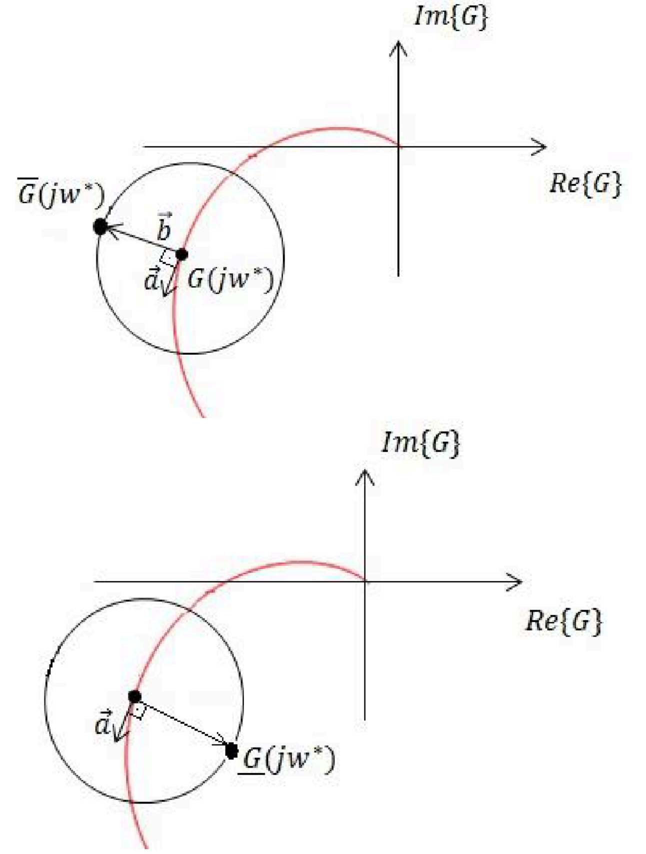

Fig 5. Nyquist Plot of the nominal system with uncertainty disk

In (8), $G'(jw)$ is the first derivative of $G(jw)$ with respect to w and $|G'(jw)|$ is the magnitude value of $G'(jw)$. To reach the point $\overline{G}(jw^*)$, we need to rotate the vector by an angle of 90° clockwise and multiply it by the radius of the uncertainty disk $r = |W_I G(jw^*)|$. We define the vector $\vec{b}$ in Fig 5 as follows

$$\vec{b} = \begin{pmatrix} \operatorname{Im}\{G'(jw^*)\} \\ -\operatorname{Re}\{G'(jw^*)\} \end{pmatrix} \frac{1}{|G'(jw^*)|} |W_I G(jw^*)| \tag{9}$$

As a result, we may identify the point $\overline{G}(jw^*)$ with (10).

$$\overline{G}(jw^*) = \begin{pmatrix} \operatorname{Re}\{G\} \\ \operatorname{Im}\{G\} \end{pmatrix} + \begin{pmatrix} \operatorname{Im}\{G'\} \\ -\operatorname{Re}\{G'\} \end{pmatrix} \frac{1}{|G'|} |W_I G| \tag{10}$$

If we calculate $\overline{G}(jw^*)$ for all frequency values w, we get a new Nyquist plot, that shows the $G(jw)$ under multiplicative uncertainty with the maximum uncertainty value for $\|\Delta\| \le 1$. Also we can find $\underline{G}(jw^*)$ by rotating the unit vector $\vec{a}$ by 90° counterclockwise.

$$\underline{G}(jw^*) = \begin{pmatrix} \operatorname{Re}\{G\} \\ \operatorname{Im}\{G\} \end{pmatrix} + \begin{pmatrix} -\operatorname{Im}\{G'\} \\ \operatorname{Re}\{G'\} \end{pmatrix} \frac{1}{|G'|} |W_I G| \tag{11}$$

The stabilizing gains k for the system with the multiplicative uncertainty can be calculated by the same way in Part A. For $\overline{G}(jw)$ the imaginary and real parts can be defined as,

$$\operatorname{Im}\{\overline{G}(jw)\} = \operatorname{Im}\{G(jw)\} - \operatorname{Re}\{G'(jw)\} \left| \frac{W_I G}{G'} \right| \tag{12}$$

$$\operatorname{Re}\{\overline{G}(jw)\} = \operatorname{Re}\{G(jw)\} + \operatorname{Im}\{G'(jw)\} \left| \frac{W_I G}{G'} \right| \tag{13}$$

respectively. Also for $\underline{G}(jw)$ the imaginary and real parts are,

$$\operatorname{Im}\{\underline{G}(jw)\} = \operatorname{Im}\{G(jw)\} + \operatorname{Re}\{G'(jw)\} \left| \frac{W_I G}{G'} \right| \tag{14}$$

$$\operatorname{Re}\{\underline{G}(jw)\} = \operatorname{Re}\{G(jw)\} - \operatorname{Im}\{G'(jw)\} \left| \frac{W_I G}{G'} \right| \tag{15}$$

As the frequency approaches infinity and the Nyquist plot to the origin for proper systems, all disks accumulate around $\{0,0\}$. Therefore, we can immediately conclude that for high frequencies the method does not work. Fig.6 visualizes this phenomenon that is observable at high frequencies.

In Fig.6 the red curve is a part of the nominal system Nyquist plot. The blue and magenta curves are the expected boundary curves. As can be seen from the figure real axis intersection points cannot be always obtained as easy as mentioned before. To overcome this difficulty let us consider any uncertainty disk (circle) equation;

$$(x - \operatorname{Re}[G])^2 + (y - \operatorname{Im}[G])^2 = (|W_I G|)^2 \tag{16}$$

The circle intersects with the real-axis (y=0) at at most two points. These points are the solution to the parametric quadratic polynomial equation given below,

$$(x - \operatorname{Re}[G])^2 + (\operatorname{Im}[G])^2 - (|W_I G|)^2 = 0\,. \tag{17}$$

Note that, (17) has exactly two solutions if and only if

$$(|W_I G|)^2 - (\operatorname{Im}[G])^2 > 0 \tag{18}$$

for some w. It is easy to show that, the critical frequencies for the extreme x-intercepts are calculated by the following equation

$$-\operatorname{Re}\{G'\} \pm \frac{|W_I G| \cdot (|W_I G|)' - \operatorname{Im}\{G\} \cdot (\operatorname{Im}\{G\})'}{0.5\sqrt{(|W_I G|)^2 - (\operatorname{Im}\{G\})^2}} = 0\,. \tag{19}$$

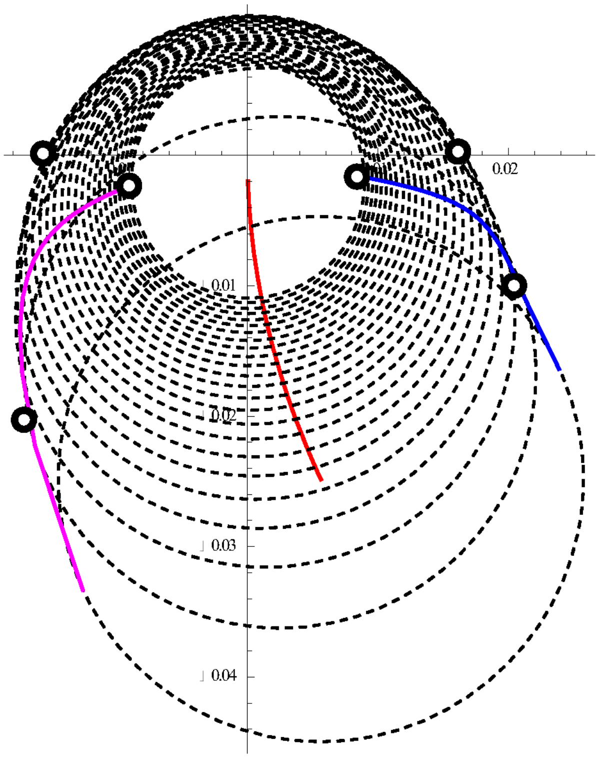

Fig 6. Nyquist plot with uncertainty disks for high frequencies

However, if we are not interested in the high frequency behavior then equating (12) and (14) to zero and simultaneously solving both equations is sufficient to find the critical frequencies.

After all the critical frequencies are obtained, using (17) the real-axis intersection points of the uncertainty disks are determined. Recall that (17) has two solutions each time, choose the minimum (maximum) ones the find the left (right)

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most intersection points.

D. Main Result

For the given system with multiplicative uncertainty, all H_∞ robust stabilizing gains k, can be found by subtracting the gain sets of the system with uncertainty from the gain set of the nominal system.

E. Example 2

Consider again the system given in Example 1 and assume that there is a multiplicative uncertainty in the system model describes by a weight transfer function with 0.1% uncertainty in the model for low frequencies, and 300% uncertainty for high frequencies; $W_I(s)=\dfrac{3s+0.2}{s+200}$.

For the calculation of critical frequencies (12) and (14) or (19) can be used. Note that (12) and (14) do not result in critical high frequencies. The critical frequencies for the uncertain system are calculated as

$$w_{1,2}=0,\ w_3=1.79543,\ w_4=1.79551,\ w_5=3.04754,$$
$$w_6=3.0478,\ w_7=7.4331,\ w_8=8.05372,\ w_9=13.60898,$$
$$w_{10}=14.43633,\ w_{11}=127.96696,\ w_{12}=147.93065.$$

For example, let us concentrate on $w_5=3.04754$ and $w_6=3.0478$. Substituting w_5 and w_6 in (17) respectively result in the points $Q_{1,2}=\{0.00265795, 0.00291253\}$ and $P_{1,2}=\{0,00265845, 0.00291303\}$ on the x-axis (real-axis). We choose the left most and right most points among these points, those are $Q_1=\{0.00265795\}$ and $P_2=\{0.00291303\}$ respectively.

To decide the H_∞ robust stabilizing gains k, the crossing direction of Nyquist plot at critical frequencies real axis intersection points are calculated again, to simplify the results only critical gain intervals are given in Table III. From Table III, it is clear that the H_∞ robust stabilizing gain interval of the system with multiplicative uncertainty is $k\in(-1.704332, 6.05519)$. It is obvious that the robust stabilizing gain interval is a subset of nominal stabilizing gain interval.

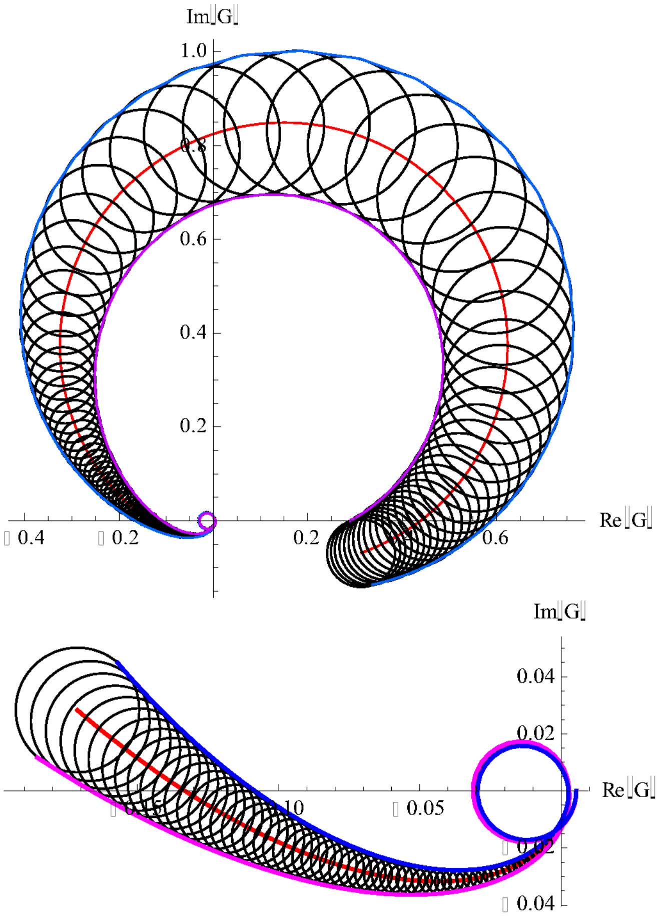

Fig 7. Nyquist plot with uncertainties for Example 2 at different scales

TABLE III

UNCERTAINTY GAIN INTERVALS

k_{i-1}	k_i
-∞	-376.23007
-376.23007	-343.28506
-343.28506	-186.17284
-186.17284	-185.80087
-185.80087	-44.82967
-44.82967	-3.28392
-3.28392	-1.704332
-1.704332	6.05519
6.05519	9.07275
9.07275	32.03157
32.03157	33.80579
33.80579	54.00063
54.00063	∞

III. CONCLUSION

The paper presented a fast algorithm to find all stabilizing gains for nominal and uncertain single-input single-output systems. It has been shown that without drawing any (polar) plots, or calculating real axis crossings all critical frequencies can be computed, and all stabilizing gains can be determined.

The future work is directed into consideration of all stabilizing controllers for two-input two output systems.

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