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Observability/detectability analysis for nonlinear systems with unknown inputs - application to biochemical processes

Jaime A. Moreno, Edmundo Rocha-Cózatl and Alain Vande Wouwer

Abstract— **Determination of the observability/detectability properties of a nonlinear system is fundamental to assess the possibility of constructing observers and their expected properties as convergence assignability. For linear systems this task can be solved by well-known techniques, for the unperturbed and also the case with perturbations. However, for nonlinear systems this study is usually a very hard task, in particular when perturbations are present. In this paper a general method to study these properties will be described, and its capabilities and feasibility will be assessed by means of a few case studies related to the culture of phytoplankton in the chemostat.**

I. INTRODUCTION

As it is well-known, the possibility of constructing an observer is tied to the observability/detectability properties of the system model. When only the initial conditions are unknown, observability corresponds to the (theoretical) possibility of estimating the state in a finite time-horizon, whereas if the system is only detectable the state estimation can only be attained asymptotically. In a more realistic case, besides the uncertainty in the initial conditions, also model parameters or even input uncertainties are usually present. In these cases, the concepts of observability/detectability have to be modified in order to consider the given uncertainties. Observability would then correspond to the possibility of reconstructing the state in a finite-horizon despite of the uncertainties acting on the system, while detectability would allow this asymptotically.

A general state-space model of biochemical reaction systems is generally obtained from mass and energy balances [1], [2] and can be written in a compact and generalized form as:

$$\Sigma_R : \begin{cases} \dot{x} = K\varphi(x) - D(t)x - Q(x) + F(t) \ , \\ y = h(x) \ . \end{cases} \tag{1}$$

where $y \in \mathbb{R}^m$ is the output vector, the state $x \in \mathbb{R}^n$ consists of component concentrations, volumes and temperatures, $K \in \mathbb{R}^{n\times q}$ is the constant stoichiometric coefficient matrix, $\varphi \in \mathbb{R}^q$ is the reaction rate vector, D is the (matrix) dilution rate, Q is the outflow rate vector, F is the feedrate vector.

An important question about system (1) is on the possibility of estimating the state x given the knowledge of the inputs and the measurement of the variables y. When the model and the inputs are assumed to be perfectly known, there are some classical tools to analyse the observability properties, as for example the observability map or the Kalman Observability matrix [1], [3], [4], [5]. The question becomes more difficult to answer when some inputs are not known, since the observability/detectability concept becomes more delicate, and there are less tools available to study nonlinear systems. A similar situation occurs when the model is uncertain, i.e., there are some uncertain parameters or part of the dynamical model is unknown.

Our objective in this paper is to propose a method to systematically study the observability/detectability properties of a nonlinear model, and in particular of the generalized reaction model (1). The method allows this study under very general conditions, and it can be used for observability as well as for detectability analysis, when there are unknown inputs or when uncertainties are considered in the model. We will present the basic idea of the method and apply it to the study of a few bioprocess examples related to the culture of phytoplankton in the chemostat, in order to illustrate its ability to determine the observability properties when other methods would fail. It is clear that the observability analysis of a general system, when uncertainties and unknown inputs are present, can be a very difficult task. However, the non trivial examples show that for realistic and reasonable examples the proposed method is applicable with reasonable efforts.

II. OBSERVABILITY AND DETECTABILITY CONCEPTS FOR SYSTEMS WITH UNKNOWN INPUTS

Observability and Detectability analysis is a classical topic in the control literature. To review some of the classical methods to analyse these properties let us consider a nonlinear system

$$\Sigma_{NL} : \begin{cases} \dot{x}(t) & = f(x(t), u(t), w(t)), x(0) = x_0 \\ y(t) & = h(x(t)), \end{cases} \tag{2}$$

where $x \in R^n$ is the state, $y \in \mathbb{R}^p$ is the measured variable, $u \in \mathbb{R}^m$ is the measured (or known) input and $w \in \mathbb{R}^q$ is the unmeasured (or unknown) input. $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ is a smooth vector field and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a smooth function.

A. State Observability and Detectability

A basic concept for systems without unknown inputs is that of indistinguishable states [4]. Roughly speaking, two states are said to be indistinguishable if their trajectories are different although both the input and the output of the system

J. A. Moreno is with Instituto de Ingeniería, Universidad Nacional Autónoma de México, Cd. Universitaria, CP 04510, Mexico, D.F., Mexico. JMorenoP@ii.unam.mx

E. Rocha-Cózatl is with Departamento de Ingeniería Mecatrónica, Universidad Nacional Autónoma de México, Cd. Universitaria, CP 04510, Mexico, D.F., Mexico. e.rocha.cozatl@comunidad.unam.mx

A. Vande Wouwer is with Service d'Automatique, Université de Mons, Boulevard Dolez 31, B-7000 Mons, Belgium. Alain.VandeWouwer@umons.ac.be

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are identical. The importance of this definition comes from the fact that the observer existence for the system strongly relies on the existence (and the type) of such states. For systems with unknown inputs (2), similar concepts can be introduced, that are in general input dependent.

Definition 1. (UI Observability and detectability) Consider for system Σ_{NL} (2) an input u, an initial state x_0 and an unknown input w.

- If $\bar{x}_0 \neq x_0$ is such that $y(t,x_0,u,w) = y(t,\bar{x}_0,u,\bar{w})$, $\forall t \in [0,\infty)$ and for some $w,\bar{w} \in \mathscr{W}$, then $\bar{x}_0$ is a *strongly u-indistinguishable* state of x_0. Denote by $\mathscr{I}^{UI}_{(u,x_0)}$ the set of strongly u-indistinguishable states of x_0.
- Σ_{NL} is *strongly u-observable* if for every x_0, $\mathscr{I}^{UI}_{(u,x_0)} = \{x_0\}$
- Σ_{NL} is *strongly u-detectable* if for every x_0 and every $\bar{x}_0 \in \mathscr{I}^{UI}_{(u,x_0)}$ and any pair of signals w and $\bar{w}$ that renders $\bar{x}_0$ indistinguishable, i.e. $y(t,\bar{x}_0,u,\bar{w}) = y(t,x_0,u,w)$, it follows that $x(t,\bar{x}_0,u,\bar{w}) \to x(t,x_0,u,w)$.
- Σ_{NL} is *strongly u-asydetectable* if $y(t,\bar{x}_0,u,\bar{w}) \to y(t,x_0,u,w)$ implies $x(t,\bar{x}_0,u,\bar{w}) \to x(t,x_0,u,w)$.
- Σ_{NL} is *strongly observable [detectable, asydetectable]* if it is strongly u-observable [u-detectable, u-asydetectable] for every $u \in \mathscr{U}$.

These concepts are generalizations of the ones introduced by [6] for LTI systems, where strong asydetectability has been called strong* detectability. For systems without unknown inputs they reduce to the usual concepts of *(u-)observability* and *(u-)detectability*.

Note that the presence of an unknown input requires the introduction of two different detectability notions: strong detectability and strong asydetectability. For continuous time LTI systems with unknown inputs, it has been shown in [6] that they are indeed different properties, but they become equivalent concepts when the inputs are known.

Remark 2. It is clear that strong (u-)asydetectability implies strong (u-)detectability, but the converse is not true. Moreover, strong (u-)observability implies strong (u-)detectability, but it does not necessarily imply strong (u-)asydetectability.

These properties are indeed related to the existence of Unknown Input Observers (UIO).

Definition 3. (UI Observer) A system

$$\Omega : \begin{cases} \dot{z} = \vartheta(z,u,y)\ , \quad z(0) = z_0 \\ \hat{x} = \chi(z,u,y)\ , \end{cases} \tag{3}$$

where $z \in \mathbb{R}^r$ is the state vector and ϑ, χ are functions defined in $(z,u,y) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^m$, is an *unknown input observer (UIO)* for the class of input/output signals $(u,y) \in (\mathscr{U} \times \mathscr{Y})_c \subseteq \mathscr{U} \times \mathscr{Y}$ for system (2) if $\exists z_0 \in \mathbb{R}^r$ such that $\forall x_0 \in \mathbb{R}^n, \forall w \in \mathscr{W}$, and $\forall u \in \mathscr{U}$ such that $(u,y) \in (\mathscr{U} \times \mathscr{Y})_c$, it is satisfied that $\hat{x}(t,z_0,u,y(t,x_0,u,w)+\delta(t)) \to x(t,x_0,u,w)$ for every asymptotically vanishing $\delta(t)$, i.e. $\delta(t) \to 0$. It will be said that Ω is an UIO for u if it is an UIO for $\{u\} \times \mathscr{Y}_u$, where $\mathscr{Y}_u$ represents the set of output signals of Σ_U generated by input u and $\forall x_0 \in \mathbb{R}^n, \forall w \in \mathscr{W}$.

Note that in this definition convergence is not required for every initial condition z_0 of the observer, and it can also depend on the input. This latter phenomenon is a natural consequence of the input dependence of the observability/detectability. If the observer is LTI it follows from its internal stability that a converging input produces a converging output, that is, for every $\delta_1(t)$ and $\delta_2(t)$ such that $\delta_1 \to 0$, $\delta_2 \to 0$ the state estimate for $(u+\delta_1, y+\delta_2)$ converges to the same signal $\hat{x}$. For a nonlinear or time-varying UI observer this property does not always follow from the internal stability. Since this is a reasonable robustness property for a useful observer a converging input converging output property has been imposed on the observer.

For LTI systems [6] strong asydetectability is equivalent to the existence of an UIO, so that neither strong detectability nor strong observability are sufficient for the existence of an UIO. The following Lemma generalizes partially this result for nonlinear systems (for a proof see [10]).

Lemma 4. *If system (2) has an UIO for the input $u \in \mathscr{U}$, then it is strongly u-asydetectable.*

B. State Observability tests

We will consider some usual observability tests

1) Known inputs: In the usual case when there are no unknown inputs, i.e. $w = 0$, there are some standard tests of observability. For example, when there are no known inputs, i.e. $u = 0$ observability is checked using the injectivity of the observability map

$$\mathbb{O}(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{4}$$

or the local observability through the full rank of the observability matrix

$$d\mathbb{O}(x) = \begin{bmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \end{bmatrix}. \tag{5}$$

When there is a known input and a single output, it is well-known that the system is uniformly observable for every input [7], [5] if it can be transformed to the triangular form

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} g_1(x_1,u) \\ g_2(x_1,x_2,u) \\ \vdots \\ g_n(x,u) \end{bmatrix}, y = x_1 \tag{6}$$

or in the case without (known) inputs a (slightly) generalized version is

$$\dot{x} = \begin{bmatrix} g_1(x_1,x_2) \\ g_2(x_1,x_2,x_3) \\ \vdots \\ g_n(x) \end{bmatrix}, y = x_1 \tag{7}$$

where

$$\frac{\partial g_i(x_1,\cdots,x_{i+1})}{\partial x_{i+1}} \neq 0\,, i=1,2,\cdots,n-1\,. \tag{8}$$

2) Unknown inputs: In the case where there are unknown inputs, the classical tests are for Linear Time Invariant systems obtained by [6], i.e.

$$\begin{aligned} f(x,u,w) &= Ax+Bu+Dw \\ h(x) &= Cx \end{aligned}$$

and in this case if

$$\operatorname{rank}\begin{bmatrix} sI-A & D \\ C & 0 \end{bmatrix} = n+\operatorname{rank}\{\mathrm{D}\} \tag{9}$$

for all $s\in\mathbb{C}$, then the system is strongly observable; if (9) is satisfied for all $s\in\mathbb{C}^-$ then the system is strongly detectable. Morover, if

$$\operatorname{rank}\{CD\} = \operatorname{rank}\{D\} \tag{10}$$

then system is asyobservable.

C. Unknown Input Observability and Detectability

For system (2), with sufficiently smooth inputs, the UI-Observability (Detectability) is related to the possibility of determining uniquely in finite-time (asymptotically) the unknown inputs $w(t)$, having as information the known inputs and the outputs. When both state and UI have to be estimated, the appropriate indistinguishability concept is as follows:

Consider for system Σ_{NL} (2) an input u, an initial state x and an unknown input $w\in\mathscr{W}$. If $(\bar{x},\bar{w}(t))\neq(x,w(t))$ is such that $y(t,x,u,w)=y(t,\bar{x},u,\bar{w})$, $\forall t\in[0,\infty)$, then $(\bar{x},\bar{w}(t))$ is an *UI u-indistinguishable* pair of $(x,w(t))$. The absence of indistinguishable pairs corresponds to UI-Observability, whereas the asymptotic convergence of indistinguishable trajectories corresponds to the UI-Detectability.

III. A dynamical interpretation of strong observability and detectability concepts

A dynamical interpretation of the concepts introduced previously will be useful to derive sufficient conditions for the existence of robust global state observers for the uncertain system (2).

A. Dynamic Characterization of Observability and Detectability

Definition 1 is based on the comparison of two different trajectories of Σ_{NL} generated under particular conditions. Therefore a dynamical interpretation of the concepts can be obtained considering two identical systems Σ_{NL} (2)

$$\begin{aligned} &\dot{x}_i = f(x_i,u,w_i)\ ,\ x_i(0)=x_{i0}\ , \\ &y_i = h(x_i)\ , \qquad\qquad i=1,2\ , \end{aligned}$$

that generate the required trajectories, but with different initial conditions and unknown inputs. Introducing the variables $x=x_1$, $y=y_1$, $w=w_1$, $\tilde{x}=x_1-x_2$, $\tilde{y}=y_1-y_2$, $\tilde{w}=w_1-w_2$, $F(\tilde{x},\tilde{w};x,u,w)\triangleq f(x,u,w)-f(x-\tilde{x},u,w-\tilde{w})$, $H(\tilde{x};x)\triangleq h(x)-h(x-\tilde{x})$, the *Error Dynamics* of the plant is defined as the cascade connection of the plant Σ_{NL} (2) and

$$\Xi:\begin{cases} \dot{\tilde{x}} = F(\tilde{x},\tilde{w};x,u,w)\ , \\ \tilde{y} = H(\tilde{x},x)\ , \qquad \tilde{x}(0)=\tilde{x}_0\ . \end{cases} \tag{11}$$

Note that for all (x,u,w) it follows that $F(0,0;x,u,w)=0$ and $H(0;x)=0$. This error dynamics (Σ_{NL},Ξ) represents all possible pairs of plant trajectories and deviations from it that can be obtained when the known input signals are the same. The subset of those trajectories with the same output can be selected. This can be done setting $\tilde{y}(t)=0$ for all $t\geq 0$ in (Σ_{NL},Ξ). This leads to the following Differential-Algebraic (DA) system

$$\Xi_c:\begin{cases} \dot{\tilde{x}} = F(\tilde{x},\tilde{w};x,u,w)\ ,\ \tilde{x}(0)=\tilde{x}_0\ , \\ 0 = H(\tilde{x},x)\ . \end{cases} \tag{12}$$

The cascade connection of the plant (2) and (12), i.e. system (Σ_{NL},Ξ_c), will be called the *Strongly Indistinguishable Dynamics* of the plant, and characterizes dynamically all possible strong indistinguishable trajectories of Σ_{NL}. The following result is a simple consequence of the definitions.

Lemma 5. *For system Σ_{NL} (2)*

1) *Two state trajectories are* strongly u-indistinguishable *if and only if they are of the form $x(t,x_0,u,w)$ and $x(t,x_0,u,w)+\tilde{x}(t,\tilde{x}_0,u,y,\tilde{w})$, where $x(t)$ and $y(t)$ is a solution of (2) and $\tilde{x}(t)$ is a solution of (12).*
2) Σ_{NL} *is* strongly u-detectable *if and only if the constrained system (12) has $\tilde{x}=0$ as a globally attractive equilibrium point for every $\tilde{w}$, i.e. for every $y(t,x_0,u,w)$, solution of (2), every $\tilde{x}_0$ and every $\tilde{w}$ such that (12) is satisfied, $\tilde{x}\to 0$.*
3) Σ_{NL} *is* strongly u-observable *if and only if the constrained system (12) is trivial,i.e. the only solution is $\tilde{x}(t)=0$.*
4) Σ_{NL} *is* strongly u-asydetectable *if and only if for the error dynamics (2, 11) $\tilde{y}(t)\to 0$ implies that $\tilde{x}(t)\to 0$.*

Remark 6. Recall that a usual characterization of the zeros (or zero dynamics) of a system $\dot{\xi}=\phi(\xi,\mu)$, $\eta=\chi(\xi)$ [8] corresponds to the set of pairs (ξ_0,μ) of initial conditions and inputs, such that the output of the system is zero for all the time, i.e. $\chi(\xi)\equiv 0$. For the error dynamics of the plant, i.e. system (Σ_{NL},Ξ), with state vector $(x,\tilde{x})$, input vector $(u,w,\tilde{w})$ and output $\tilde{y}$, the *zero dynamics* is given by the strongly indistinguishable dynamics, i.e. system (Σ_{NL},Ξ_c). This means that the strong indistinguishable trajectories of Σ_{NL} correspond to the trajectories of the zero dynamics, that strong observability is equivalent to the absence of strong indistinguishable trajectories, i.e to the absence of zeros, and strong detectability coincides with the attractivity of the invariant manifold $(x,\tilde{x})=(x,0)$ of the zero dynamics for every input.

Lemma 5 gives a dynamical interpretation of the (strong) observability/detectability concepts for the specific case considered in the paper. It is clear that this idea can be used for more general systems, although the obtained indistinguishability dynamics systems are, in general, not so simple as here. Compared to the usual observability criteria, that are based on the construction of the observability map with the vector fields [3], [7], this characterization has several advantages: (i) The approach is not local neither in time nor in the state space. (ii) It allows to determine detectability, what is usually impossible with the other criteria. (iii) The dynamical interpretation is appealing. (iv) Several nonlinear tools can be used for the analysis, as for example the characterization of the zero dynamics in geometric control. (v) Lyapunov functions can be used for the characterization of the properties. (vi) It is of very general nature. No special smoothness or structural properties are necessary. (vii) For systems with unknown inputs there is no observability test based on the system vector fields in the literature.

A similar (dynamical) characterization can be given for the Unknown Input/State observability or detectability concepts, that cannot be presented here because of lack of space.

B. State Observability tests

We will show that the usual observability tests presented in the previous section are special cases of Lemma 5.

1) Known inputs: When there are no inputs, i.e. $u = 0$, $w = 0$, the indistinguishable dynamics (Σ_{NL}, Ξ_c) becomes

$$\begin{array}{l} \Sigma_{NL}: \\ \\ \Xi_c: \end{array} \begin{cases} \dot{x}(t) = f(x(t)), x(0) = x_0 \\ y(t) = h(x(t)), \\ \dot{\tilde{x}} = f(x) - f(x-\tilde{x}), \ \tilde{x}(0) = \tilde{x}_0, \\ 0 = h(x) - h(x-\tilde{x}). \end{cases} \tag{13}$$

Taking successive time derivatives of the last equation $0 = h(x) - h(x-\tilde{x})$ one concludes that $\mathbb{O}(x) = \mathbb{O}(x-\tilde{x})$ and if the observability map $\mathbb{O}(x)$ (4) is injective, it follows that $\tilde{x} = 0$, i.e. according to Lemma 5 the system is observable. When only the observability matrix $d\mathbb{O}(x)$ (5) is full rank at the point x, then the equation $\mathbb{O}(x) = \mathbb{O}(x-\tilde{x})$ implies, using the classical rank theorem, that locally in a neighborhood of x the only solution is $\tilde{x} = 0$, that is, the system is locally observable.

When there is a known input and there is a single output, if the system can be brought to the triangular form (6), then the indistinguishable dynamics (Σ_{NL}, Ξ_c) becomes

$$\begin{array}{l} \Sigma_{NL}: \\ \\ \\ \\ \Xi_c: \end{array} \begin{cases} \dot{x}(t) = \begin{bmatrix} g_1(x_1,u) + x_2 \\ g_2(x_1,x_2,u) + x_3 \\ \vdots \\ g_n(x,u) \end{bmatrix}, \ x(0) = x_0 \\ y(t) = x_1(t), \\ \dot{\tilde{x}} = \begin{bmatrix} g_1(x_1,u) + x_2 \\ g_2(x_1,x_2,u) + x_3 \\ \vdots \\ g_n(x,u) \end{bmatrix}, \ \tilde{x}(0) = \tilde{x}_0 \\ \quad - \begin{bmatrix} g_1(x_1-\tilde{x}_1,u) + x_2 - \tilde{x}_2 \\ g_2(x_1-\tilde{x}_1, x_2-\tilde{x}_2, u) + x_3 - \tilde{x}_3 \\ \vdots \\ g_n(x-\tilde{x},u) \end{bmatrix} \\ 0 = \tilde{x}_1(t). \end{cases} \tag{14}$$

From the last equation it follows that $\tilde{x}_1(t) = 0$ and $\dot{\tilde{x}}_1(t) = 0$ so that from the first equation for Ξ_c, i.e.

$$\dot{\tilde{x}}_1(t) = g_1(x_1,u) + x_2 - [g_1(x_1-\tilde{x}_1,u) + x_2 - \tilde{x}_2]$$

it follows that $\tilde{x}_2(t) = 0$ and therefore $\dot{\tilde{x}}_2(t) = 0$. Using the second equation for Ξ_c, i.e.

$$\dot{\tilde{x}}_2(t) = g_2(x_1,x_2,u) + x_3 - [g_2(x_1-\tilde{x}_1, x_2-\tilde{x}_2, u) + x_3 - \tilde{x}_3]$$

it follows that $\tilde{x}_3(t) = 0$ and therefore $\dot{\tilde{x}}_3(t) = 0$. Using the same argument recursively one concludes that $\tilde{x}(t) = 0$, so that the system is observable for every input u, i.e. uniformly observable.

In a similar way, if the system can be brougth to the form (7), then the indistinguishable dynamics (Σ_{NL}, Ξ_c) becomes

$$\begin{array}{l} \Sigma_{NL}: \\ \\ \\ \\ \Xi_c: \end{array} \begin{cases} \dot{x}(t) = \begin{bmatrix} g_1(x_1,x_2) \\ g_2(x_1,x_2,x_3) \\ \vdots \\ g_n(x) \end{bmatrix}, \ x(0) = x_0 \\ y(t) = x_1(t), \\ \dot{\tilde{x}} = \begin{bmatrix} g_1(x_1,x_2) \\ g_2(x_1,x_2,x_3) \\ \vdots \\ g_n(x) \end{bmatrix}, \ \tilde{x}(0) = \tilde{x}_0 \\ \quad - \begin{bmatrix} g_1(x_1-\tilde{x}_1, x_2-\tilde{x}_2) \\ g_2(x_1-\tilde{x}_1, x_2-\tilde{x}_2, x_3-\tilde{x}_3) \\ \vdots \\ g_n(x-\tilde{x}) \end{bmatrix} \\ 0 = \tilde{x}_1(t). \end{cases} \tag{15}$$

From the last equation it follows that $\tilde{x}_1(t) = 0$ and $\dot{\tilde{x}}_1(t) = 0$ so that from the first equation for Ξ_c, i.e.

$$\dot{\tilde{x}}_1(t) = g_1(x_1,x_2) - g_1(x_1-\tilde{x}_1, x_2-\tilde{x}_2) = 0$$

it follows that $g_1(x_1,x_2) = g_1(x_1, x_2-\tilde{x}_2)$. The condition (8) implies that the function $g_1(x_1,x_2)$ is invertible in its second argument (x_2), so that $\tilde{x}_2(t) = 0$, and therefore $\dot{\tilde{x}}_2(t) = 0$. Applying this argument in a recursive manner one concludes that $\tilde{x}(t) = 0$, so that the system is observable.

2) Unknown inputs: The classical observability test for linear systems with Unknown Inputs (9) and (10) can be easily derived from Lemma 5. The detailed calculation has been done in [9].

The previous paragraphs show that the characterization of observability/detectability given in Lemma 5 is very general and encompasses many well known tests in the literature, but it goes well beyond these classical tests. In [10] this technique has been used to characterize the observability/detectability properties of the General Reactor Model (1) when the reaction rates are considered as unknown inputs. This leads to easy to verify conditions and it is at the heart of the well-known design of Asymptotic Observers [1], [11], [2]. The same method has been used in [12] to study state observability/detectability for an induction machine, when all inputs are known.

IV. Application to phytoplankton cultures

A. Phytoplankton culture: Droop Model

Droop model [13] is a simple and widely used model of phytoplankton culture. The mass balance equations are given by

$$\Sigma_p \begin{cases} \dot{X}(t) &= -D(t)X(t) + \mu(Q)X(t) \\ \dot{Q}(t) &= \rho(S) - \mu(Q)Q(t) \\ \dot{S}(t) &= D(t)[S_{in}(t) - S(t)] - \rho(S)X(t) \end{cases} \tag{16}$$

where X is the biovolume (i.e., the volume of cells in a unit volume of culture medium), Q is the internal quota, which is defined as the quantity of nitrogen per unit of biovolume, $D(t)$ represents the dilution rate, S is the substrate (inorganic nitrogen) concentration and S_{in} is the input substrate concentration. Function $\rho(S) = \rho_m \frac{S(t)}{S(t)+k_S}$ is the uptake rate, and $\mu(Q) = \bar{\mu}\left(1 - \frac{k_Q}{Q(t)}\right)$ is the growth rate; constants k_S and ρ_m represent a half-saturation constant for the substrate and the maximum uptake rate, respectively; constant $\bar{\mu}$ is the theoretical maximum growth rate, obtained for an infinite internal quota and k_Q is the minimum internal quota allowing growth.

Our objective is to analyse observability/detectability of the unmeasured state variables Q, S and the unknown input D, from the knowledge of the dynamic model (16) and the known input S_{in} and the output variable X. (State/Unknown input) Observability corresponds to the distinguishability of two different unmeasured state/unknown input trajectories $(D_1(t), Q_1(t), S_1(t))$ and $(D_2(t), Q_2(t), S_2(t))$ from the corresponding measured variables $(S_{in1}(t), X_1(t))$ and $(S_{in2}(t), X_2(t))$, i.e. if $S_{in1}(t) = S_{in2}(t)$, $X_1(t) = X_2(t)$ for an interval of time $t \in [0, T]$ the unmeasured variables are identical $D_1(t) = D_2(t)$, $Q_1(t) = Q_2(t)$, $S_1(t) = S_2(t)$ during the same time interval. Detectability corresponds to the validity of this property in an asymptotic manner, i.e. $\lim_{t\to\infty}(D_1(t) - D_2(t)) = 0$, $\lim_{t\to\infty}(Q_1(t) - Q_2(t)) = 0$ and $\lim_{t\to\infty}(S_1(t) - S_2(t)) = 0$ [10]. Checking the observability property when all inputs are known can be done using the Observability map (or its Jacobian for the local property). However, observability with unknown inputs or detectability is a property much harder to evaluate.

The idea underlying the analysis is to consider two identical systems, one is the plant Σ_p (16), and a copy of it (with states x_1, x_2, x_3, unknown input d and known input u) given by

$$\Sigma_x \begin{cases} \dot{x}_1(t) &= -d(t)x_1(t) + \mu(x_2)x_1(t) \\ \dot{x}_2(t) &= \rho(x_3) - \mu(x_2)x_2(t) \\ \dot{x}_3(t) &= d(t)[u(t) - x_3(t)] - \rho(x_3)x_1(t) \end{cases} . \tag{17}$$

If we assume that the measured variables of systems Σ_p and Σ_x are identical for a time interval, i.e. $S_{in}(t) = u(t)$, $X(t) = x_1(t)$, (state and unknown input) observability is equivalent to the fact that the only possible solutions of equations (16) and (17) under these restrictions is given by $Q(t) = x_2(t)$, $S(t) = x_3(t)$ and $D(t) = d(t)$, during the same time interval, whereas detectability is equivalent to satisfying these equalities asymptotically. If we introduce the state and unknown input error variables $\varepsilon_1 = X - x_1$, $\varepsilon_2 = Q - x_2$, $\varepsilon_3 = S - x_3$, and $\varepsilon_d = D - d$, respectively, equation (17) can be replaced by

$$\begin{aligned} \dot{\varepsilon}_1 &= -\varepsilon_d X - d\varepsilon_1 + \bar{\mu}\left(1 - \frac{K_Q}{Q}\right)X \\ &\quad -\bar{\mu}\left(1 - \frac{K_Q}{Q-\varepsilon_2}\right)(X - \varepsilon_1) \\ \dot{\varepsilon}_2 &= \rho_m\left[\frac{S}{S+K_S} - \frac{S-\varepsilon_3}{S-\varepsilon_3+K_S}\right] - \bar{\mu}\varepsilon_2 \\ \dot{\varepsilon}_3 &= \varepsilon_d[S_{in} - S] - d\varepsilon_3 \\ &\quad -\rho_m\left[\frac{S}{S+K_S}X - \frac{S-\varepsilon_3}{S-\varepsilon_3+K_S}(X - \varepsilon_1)\right] \end{aligned} \tag{18}$$

and observability requires that under the restrictions $S_{in}(t) = u(t)$ and $\varepsilon_1(t) = 0$ the only solutions of system (16) and (18) are $\varepsilon_2(t) = 0$, $\varepsilon_3(t) = 0$ and $\varepsilon_d(t) = 0$, or asymptotic convergence in case of detectability. This Differential-Algebraic (DA) system, consisting of the differential equations (16) and (18) and the algebraic restrictions $S_{in}(t) = u(t)$ and $\varepsilon_1(t) = 0$, will be next reduced to a simpler differential equation system. From $\varepsilon_1(t) = 0$ (during a time interval) it follows that also $\dot{\varepsilon}_1(t) = 0$ and, from the first equation of (18), that

$$\varepsilon_d = -\bar{\mu}K_Q\frac{\varepsilon_2}{Q(Q - \varepsilon_2)} . \tag{19}$$

Equations (18) therefore reduce to

$$\begin{aligned} \dot{\varepsilon}_2 &= -\bar{\mu}\varepsilon_2 + \frac{\rho_m K_S}{(S+K_S)(S-\varepsilon_3+K_S)}\varepsilon_3 \\ \dot{\varepsilon}_3 &= \frac{-\bar{\mu}K_Q[S_{in}-S]}{Q(Q-\varepsilon_2)}\varepsilon_2 - \left[d + \rho_m K_S\frac{X}{(S+K_S)(S-\varepsilon_3+K_S)}\right]\varepsilon_3 . \end{aligned} \tag{20}$$

It can be easily proved that $\varepsilon_2 \to 0$ and $\varepsilon_3 \to 0$, provided that the state variables are positive and the substrate concentration is lower than the inlet concentration (which is meaningful from a biological point of view). Moreover, from (19) it also follows that $\varepsilon_d \to 0$. We conclude that the model (16) is state and unknown input detectable, and so the measurement of X and S_{in} provide sufficient information to estimate asymptotically the two unmeasured states Q, S and the unknown input D.

B. A more detailed model with light influence

In [14], a more detailed model, taking the influence of the incident light intensity is developed, as

$$\begin{aligned} &\dot{C}(t) = -D(t)C(t) - \lambda C(t) + a(I)L(t) \\ &\dot{L}(t) = -D(t)L(t) + \gamma(I)N(t)\frac{L(t)}{C(t)} - \beta L(t) \\ &\dot{N}(t) = -D(t)N(t) + \rho_m\frac{S(t)}{S(t)+K_S}C(t) - \gamma(I)N(t)\frac{L(t)}{C(t)} + \beta L(t) \\ &\dot{S}(t) = D(t)(S_{in} - S(t)) - \rho_m\frac{S(t)}{S(t)+K_S}C(t) \end{aligned} \tag{21}$$

Variable C represents the particulate carbon concentration (carbon biomass), L, the chlorophyllian nitrogen concentration, and N, the internal nitrogen concentration. β is the coefficient of chlorophyll degradation, λ is the factor of respiration, and functions γ and a describe the influence of the light intensity I in the process

$$a(I) = \frac{\alpha I(t)}{K_I + I(t)}, \quad \gamma(I) = \frac{\alpha K_L I(t)}{K_I + I(t)}\frac{K_C}{K_C + I(t)}.$$

We would like to analyze observability/detectability in the case where C and L can be measured. Consider the plant Σ_p (21), and a copy of it Σ_x, with states $x_1 = C$, $x_2 = L$, $x_3 = N$, $x_4 = S$, known inputs $u_1 = D$, $u_2 = S_{in}$ and $w_1 = a(I)$, $w_2 = \gamma(I)$ as the unknown inputs:

$$\dot{x} = \begin{bmatrix} -\lambda x_1 \\ -\beta x_2 \\ \rho_m \frac{x_4}{x_4+K_S} x_1 + \beta x_2 \\ -\rho_m \frac{x_4}{x_4+K_S} x_1 \end{bmatrix} + \begin{bmatrix} -u_1 x_1 \\ -u_1 x_2 \\ -u_1 x_3 \\ u_1 (u_2 - x_4) \end{bmatrix} + \begin{bmatrix} w_1 x_2 \\ w_2 x_3 \frac{x_2}{x_1} \\ -w_2 x_3 \frac{x_2}{x_1} \\ 0 \end{bmatrix} \tag{22}$$

This problem corresponds to the situation where the models of $a(I)$ and $\gamma(I)$ are not well known and/or the light intensity is unknown (if only the light intensity is unknown then the problem could be simplified to only one unknown input). If we introduce the state and unknown input error variables $\varepsilon_1 = C - x_1$, $\varepsilon_2 = L - x_2$, $\varepsilon_3 = N - x_3$, $\varepsilon_4 = S - x_4$, and $\varepsilon_{w_1} = a(I) - w_1$, $\varepsilon_{w_2} = \gamma(I) - w_2$ respectively, equation (22) can be replaced by

$$\dot{\varepsilon} = \begin{bmatrix} -\lambda \varepsilon_1 \\ -\beta \varepsilon_2 \\ \rho_m \frac{S}{S+K_S} C - \rho_m \frac{S-\varepsilon_4}{S-\varepsilon_4+K_S}(C-\varepsilon_1) + \beta \varepsilon_2 \\ -\rho_m \frac{S}{S+K_S} C + \rho_m \frac{S-\varepsilon_4}{S-\varepsilon_4+K_S}(C-\varepsilon_1) \end{bmatrix} + \begin{bmatrix} -u_1 \varepsilon_1 \\ -u_1 \varepsilon_2 \\ -u_1 \varepsilon_3 \\ -u_1 \varepsilon_4 \end{bmatrix} + \begin{bmatrix} \varepsilon_{w_1} x_2 + w_1 \varepsilon_2 \\ \gamma N \frac{L}{C} - (\gamma - \varepsilon_{w_2})(N - \varepsilon_3) \frac{L-\varepsilon_2}{C-\varepsilon_1} \\ -\gamma N \frac{L}{C} + (\gamma - \varepsilon_{w_2})(N - \varepsilon_3) \frac{L-\varepsilon_2}{C-\varepsilon_1} \\ 0 \end{bmatrix} \tag{23}$$

Observability requires that under the restrictions $u_1 = D$, $u_2 = S_{in}$ and $\varepsilon_1(t) = 0$, $\varepsilon_2(t) = 0$ the only solutions of system (21) and (23) are $\varepsilon_3(t) = 0$, $\varepsilon_4(t) = 0$ and $\varepsilon_{w_1}(t) = 0$, $\varepsilon_{w_2}(t) = 0$ or asymptotic convergence in case of detectability. This Differential-Algebraic system reduces to

$$\begin{array}{l} 0 = \varepsilon_{w_1} L \\ 0 = \gamma N \frac{L}{C} - (\gamma - \varepsilon_{w_2})(N - \varepsilon_3) \frac{L}{C} \\ \dot{\varepsilon}_3 = \rho_m \frac{S}{S+K_S} C - \rho_m \frac{S-\varepsilon_4}{S-\varepsilon_4+K_S} C - u_1 \varepsilon_3 \\ -\gamma N \frac{L}{C} + (\gamma - \varepsilon_{w_2})(N - \varepsilon_3) \frac{L}{C} \\ \dot{\varepsilon}_4 = -\rho_m \frac{S}{S+K_S} C + \rho_m \frac{S-\varepsilon_4}{S-\varepsilon_4+K_S} C - u_1 \varepsilon_4 \end{array} \tag{24}$$

The first equation of (24) has a trivial solution $L(t) = 0$. The solution of interest is therefore $\varepsilon_{w_1} = 0$ for any $L(t) > 0$, implying that the first unknown input is observable.

Using the second equation of (24), the third equation can be simplified to

$$\dot{\varepsilon}_3 = \rho_m \frac{S}{S+K_S} C - \rho_m \frac{S-\varepsilon_4}{S-\varepsilon_4+K_S} C - u_1 \varepsilon_3 \tag{25}$$

Further, this equation and the fourth of (24) can be rewritten as

$$\begin{array}{l} \dot{\varepsilon}_4 = (\frac{\rho_m K_S C}{(S+K_S)(S-\varepsilon_4+K_S)} + u_1) \varepsilon_4 \\ (\dot{\varepsilon}_3 + \dot{\varepsilon}_4) = -u_1 (\varepsilon_3 + \varepsilon_4) \end{array} \tag{26}$$

The analysis of the first expression shows that so that $\varepsilon_4 \to 0$, and the analysis of the second one shows that, provided $u_1(t) = D(t) \neq 0$ (batch operation is excluded), $\varepsilon_3 + \varepsilon_4 \to 0$, implying detectability of $N(t)$ and $C(t)$.

If $\varepsilon_3 \to 0$, provided that L(t), N(t), C(t) do not vanish (which again is physically meaningful), the second equation of (24) gives $\varepsilon_{w_2} \to 0$, so that the second unknown input is detectable.

V. Conclusions

In this paper, a general method to study observability/detectability of nonlinear systems with unknown inputs has been described. Potentatiality of the method has been demonstrated with a few case studies related to the culture of phytoplankton in the chemostat.

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