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Analytical Tuning Rules of Digital PID Controllers for Integrating Processes via the Symmetrical Optimum Criterion

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Abstract— **Explicit tuning rules for digital PID regulators are presented regarding the control of integrating processes. Controller parameters are determined analytically as a function of the process parameters and the sampling time of the controller. The derivation of the proposed PID control law lies in the principle of the Symmetrical Optimum criterion. The performance of the proposed control law is compared with the conventional tuning of the PID regulator via the Symmetrical Optimum criterion when controlling the same process. Simulation examples show improvement of output disturbance rejection of up to 71.05% decrease of settling time.**

I. Introduction

The demanding problem of controlling integrating processes comes up over many industry applications quite frequently, i.e. vector control of induction motor drives. For coping with this issue, many researchers have exploited the principle of well known control methods such as the PID control law, the IMC principle and the Smith predictor. Further to the aforementioned control schemes, it is evident that the PID control law proves still to be an effective and efficient method when comes the critical question of the real time implementation issue regarding the control of such processes [1]. Careful review of the current state of the art regarding the tuning of the PID controller for controlling integrating processes shows that, in principle, for developing a PID control law, simple process models are employed, (FOPDT)[1]. The approach of designing a controller based on model reduction techniques leads often to poor tuning and instability of the control loop. This can be justified by the fact that the controller is often tuned based on two, or maximum three basic parameters of the process[2]. For that reason and further to the proposed theory, in this work a general process model in the frequency domain will be adopted. Taking into account the analytical PID control law that has recently been proposed in [2], [3], scope of this work is to present an explicit solution for the P, I, and D parameters regardless of the process complexity and when the controller is implemented digitally. For achieving this target, the same line as proposed in [2] will be followed. For that reason, the modelling of the process takes place in the frequency domain and consists of the plant's dc gain k_p, n poles plus unknown time delay d. The basis of the proposed control law is again the Symmetrical Optimum criterion [4], see Appendix

Taking into account that the performance of the control law proposed in [2], [3] shows promising results regarding the control of integrating processes compared to the conventional PID tuning, in this work, sampling time T_s is considered as one more parameter in the control loop. Since in many industry applications the PID controller is nowadays implemented digitally, after the proof of the proposed theory, control engineers are given the opportunity to make 1) extensive and accurate investigation of the affect of the sampling time to the closed loop performance 2) investigate how model uncertainties affect the stability of the control loop. To that end and for the sake of a clear presentation of the proposed theory, this work is organized as follows. In section II the conventional tuning of the PID controller via the Symmetrical Optimum criterion is presented. Based on this principle, in section III the proposed theory for controlling integrating processes is presented. The potential of the proposed theory is justified in section IV via simulation examples for two benchmark processes met in various industry applications.

II. The Conventional Symmetrical Optimum Criterion

The closed loop system of Fig.1 is considered. Let the integrating process $G(s)$ met in many industry applications be defined by

$$G(s) = \frac{1}{sT_m(1+sT_{p_1})(1+sT_{\Sigma p})} \tag{1}$$

where T_m is the integrator's plant time constant, T_{p_1} the plant's dominant time constant and $T_{\Sigma p}$ the process parasitic time constant, [5]. For controlling (1), the PID control defined by

$$C(s) = \frac{(1+sT_n)(1+sT_v)}{sT_i(1+sT_{\Sigma c})} \tag{2}$$

is applied. Time constant $T_{\Sigma c}$ stands for the controller parasitic dynamics. If $T_n = T_v = 0$, I control cannot be applied, because it is easily proved that the closed loop transfer function is unstable[3]. If the dominant time constant T_{p_1} is evaluated, PI control through (2) by setting $T_n = T_{p_1}$ and

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[1]First Order Plus Dead Time.

[2]k_p plant's dc gain, T_d time delay of the process output, dominant time constant of the process.

[3]Intermediate terms of s^j from the denominator polynomial at the final closed loop transfer function are missing, see Appendix VI-B

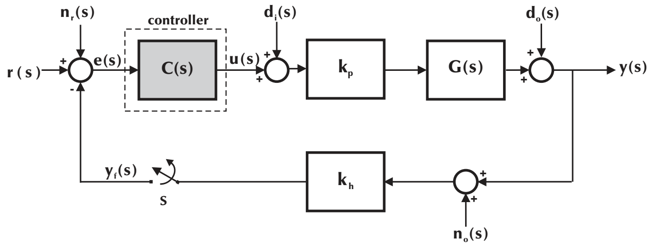
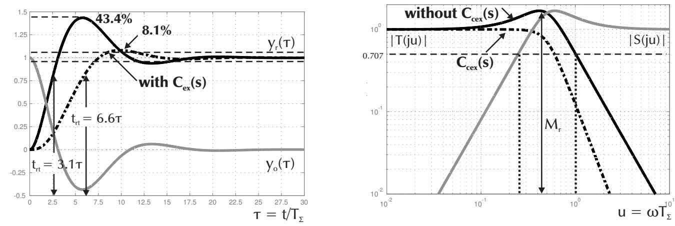


Fig. 1. Block diagram of the closed loop control system. $G(s)$ is the plant transfer function, $C(s)$ is the controller transfer function, $r(s)$ is the reference signal, $d_o(s)$ and $d_i(s)$ are the output and input disturbance signals respectively and $n_r(s)$, $n_o(s)$ are the noise signals at the reference input and process output respectively. k_p is the plant's dc gain and k_h is the feedback path.



(a) (b)

Fig. 2. Type-II closed loop control system. The effect of the two degree of freedom controller to the (a) step and (b) frequency response of the closed loop control system. (a) Step and (b) frequency response (solid black), filtered step response (dotted black). $y_o(s) = S(s)\, d_o(s)$ if $r(s) = n_r(s) = d_i(s) = n_o(s) = 0$, $y_r(s) = T(s)\, r(s)$ if $d_o(s) = n_r(s) = d_i(s) = n_o(s) = 0$.

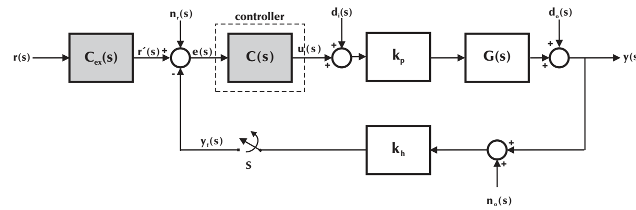


Fig. 3. Block diagram of a two degree of freedom controller. Controller $C_{ex}(s)$ filters the reference input towards the overshoot decrease.

$T_v = 0$ cannot still be applied[4], because the closed loop transfer function becomes unstable again as for the case of I control. Assuming again that the dominant time constant T_{p_1} is accurately measured, $T_v = T_{p_1}$ is set. The closed loop transfer function becomes

$$T(s) = \frac{k_p T_n s + k_p}{s^3 T_i T_m T_\Sigma + s^2 T_i T_m + s k_h k_p T_n + k_h k_p} \tag{3}$$

for which $T_\Sigma = T_{\Sigma c} + T_{\Sigma p}$ and $T_{\Sigma c} T_{\Sigma p} \approx 0$ has been set. The magnitude of (3) is given by

$$|T(j\omega)| = \sqrt{\frac{k_p k_h \left[1 + (\omega T_n)^2\right]}{(k_p k_h - T_i T_m \omega^2)^2 + \omega^2 \begin{pmatrix} k_p k_h T_n - \\ T_i T_m T_\Sigma \omega^2 \end{pmatrix}^2}} \tag{4}$$

The denominator of (4) is equal to

$$D(\omega) = \sqrt{\begin{pmatrix} (T_i T_m T_\Sigma)^2 \omega^6 + T_i T_m (T_i T_m - 2 k_p k_h T_n T_\Sigma)\, \omega^4 \\ + \left[(k_p k_h T_n)^2 - 2 k_p k_h T_i T_m\right] \omega^2 + k_p^2 k_h^2 \end{pmatrix}} \tag{5}$$

and becomes minimum in the lower frequency range when

$$k_h = 1, \quad T_n = 4T_\Sigma, \quad T_i = 8 k_p k_h \frac{T_\Sigma^2}{T_m}, \quad T_v = T_{p_1}. \tag{6}$$

see [2], [3]. Using (6) along with (3) results in

$$T(s) = \frac{1 + 4T_\Sigma s}{8T_\Sigma^3 s^3 + 8T_\Sigma^2 s^2 + 4T_\Sigma s + 1}. \tag{7}$$

Normalizing the time by setting $s' = sT_\Sigma$, (7) becomes

$$T(s') = \frac{1 + 4s'}{8s'^3 + 8s'^2 + 4s' + 1}. \tag{8}$$

The respective step and frequency response of (8) are shown in Fig.2(a), 2(b). It is clear that the step response of the closed loop control system exhibits an undesired overshoot of 43.4%. In order to overcome that obstacle, the reference input is filtered by adding an external controller $C_{ex}(s)$, Fig.3. The great overshoot of the step response in (7) is owed to the numerator of the transfer function, $N(s) = 1 + 4T_\Sigma$. In that, if an external controller of the form

$$C_{ex}(s) = \frac{r'(s)}{r(s)} = \frac{1}{1 + 4T_\Sigma s}, \tag{9}$$

is chosen, the overshoot decreases from 43.4% to 8.1%. Let it be noted that the rise time increases from $t_{rt} = 3.1T_\Sigma$ to $t_{rt} = 6.6T_\Sigma$. The choice of the external filter parameters is not that critical regarding output disturbance rejection $S_o(s)$, since it does not participate into $S_o(s) = \left.\frac{y(s)}{d_o(s)}\right|_{r(s)=0}$.

III. Optimal Control Law for Digital PID Controller

Let the integrating process in series with its constant gain k_p[5] at steady state in Fig.3 be defined by

$$k_p G(s) = \frac{k_p}{s \begin{pmatrix} s^n p_n + s^{n-1} p_{n-1} + \cdots + s^5 p_5 + \\ s^4 p_4 + s^3 p_3 + s^2 p_2 + s p_1 + 1 \end{pmatrix}} e^{-sT_d}. \tag{10}$$

The proposed PID type controller is given by

$$C(s) = C^*(s) C_{ZOH}(s) = \left(\frac{1 + sX + s^2 Y}{s^2 T_i^2}\right)^* \frac{(1 - e^{-sT_s})}{sT_s} \tag{11}$$

where $C_{ZOH}(s)$ stands for the zero order hold transfer function T_s stands for the controller sampling period. $C^*(s)$ represents the PID controller and the problem is to determine analytically its parameters T_i, X, Y. In similar fashion with (2), $X = T_n + T_v$, $Y = T_n T_v$, allowing its zeros to become conjugate complex. In the sequel, all time constants are

[4]The proposed theory is able to control (1) if no pole-zero cancellation occurs.

[5]slope of the step response.

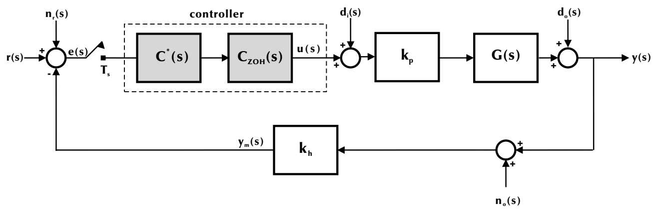


Fig. 4. Closed loop control system including the digital PID type controller.

normalized in the frequency domain with the sampling period T_s, where $s' = sT_s$. The resulting expressions (10) and (11) take the form

$$k_pG(s') = \frac{k_p}{\left(\begin{array}{l} s'^n r_n + s'^{n-1} r_{n-1} + \cdots + s'^5 r_5 + s'^4 r_4 + \\ s'^3 r_3 + s'^2 r_2 + s' r_1 + 1 \end{array}\right)} e^{-s'd} \tag{12}$$

and

$$C(s') = C^*(s')C_{ZOH}(s') = \left(\frac{1 + s'x + s'^2 y}{s'^2 t_i^2}\right)^* \frac{(1 - e^{-s'})}{s'} \tag{13}$$

$$x = \frac{X}{T_s}, \quad y = \frac{Y}{T_s^2}, \quad t_i = \frac{T_i}{T_s}, \tag{14}$$

$$d = \frac{T_d}{T_s}, \quad r_j = \frac{p_j}{T_s^j}, \quad \forall\ j = 1, \ \ldots n. \tag{15}$$

respectively. We will make the transition from the Laplace domain to the **z** domain by utilizing the following equation

$$\frac{1}{s'} = \frac{z'}{z' - 1} = \frac{e^{s'}}{e^{s'} - 1} \tag{16a}$$

$$\frac{1}{s'^2} = \frac{T_s z'}{(z' - 1)^2} = \frac{T_s e^{s'}}{(e^{s'} - 1)^2} \tag{16b}$$

In that, the digital PID type controller takes the form

$$C(s') = C^*(s')C_{ZOH}(s') = \frac{1}{t_i^2}\left(\frac{1}{s'^2} + \frac{x}{s'} + y\right)^* \frac{(1 - e^{-s'})}{s'} \tag{17}$$

or finally

$$\begin{aligned} C(s') &= \frac{1}{t_i^2}\left[\frac{(x+y)e^{2s'} - (x + 2y - T_s)e^{s'} + y}{(e^{s'} - 1)^2}\right] \\ &= \frac{T_s}{t_i^2}\left[\frac{(\frac{x}{T_s} + \frac{y}{T_s})e^{2s'} - (\frac{x}{T_s} + 2\frac{y}{T_s} - 1)e^{s'} + \frac{y}{T_s}}{(e^{s'} - 1)^2}\right] \end{aligned} \tag{18}$$

By making the following transformation

$$\hat{x} = \frac{x}{T_s} + 2\frac{y}{T_s} - 1, \quad \hat{y} = \frac{x}{T_s} + \frac{y}{T_s}. \tag{19}$$

results in

$$\frac{x}{T_s} = 2\hat{y} - \hat{x} - 1, \quad \frac{y}{T_s} = \hat{x} - \hat{y} + 1. \tag{20}$$

By utilizing (20), (18) becomes

$$C(s') = \frac{T_s}{t_i^2}\left[\frac{(1 - e^{s'})\hat{x} + (e^{2s'} - 1)\hat{y} + 1}{(e^{s'} - 1)^2}\right]. \tag{21}$$

In addition, the respective open and closed loop transfer functions, see Fig.4 become

$$F_{ol}(s') = \frac{y_m(s)}{r(s)} = k_h k_p C(s')G(s') \tag{22}$$

or

$$k_h k_p C(s')G(s') = k_h k_p \frac{T_s}{t_i^2} \frac{\left[(1 - e^{s'})\hat{x} + (e^{2s'} - 1)\hat{y} + 1\right]}{\left(s'^n r_n + \cdots + s'^3 r_3 + s'^2 r_2 + s' r_1 + 1\right) e^{s'd} \left(e^{s'} - 1\right)^2} \tag{23}$$

and $T(s') = \frac{F_{fp}(s')}{1 + F_{ol}(s')}$ where $F_{fp} = k_p C^*(s') C_{ZOH}(s') G(s')$[6]

$$\begin{aligned} T(s') &= \frac{k_p C(s')G(s')}{1 + k_h k_p C(s')G(s')} = \frac{N(s')}{D(s')} = \frac{N(s')}{D_1(s') + k_h N(s')} \\ &= \frac{\left[\ k'_p\left[(1 - e^{s'})\hat{x} + (e^{2s'} - 1)\hat{y} + 1\right]\ \right]}{\left[\begin{array}{l} t_i^2(s'^n r_n + \cdots + s'^2 r_2 + s' r_1 + 1)e^{s'd}(e^{s'} - 1)^2 \\ + k_h k'_p\left((1 - e^{s'})\hat{x} + (e^{2s'} - 1)\hat{y} + 1\right) \end{array}\right]} \end{aligned} \tag{24}$$

where $D_1(s') = t_i^2(s'^n r_n + \cdots + s'^3 r_3 + s'^2 r_2 + s' r_1 + 1)e^{s'd}(e^{s'} - 1)^2$ and $k'_p = k_p T_s$. Approximating the delay time constant $e^{s'}$ by the series

$$e^{s'} - 1 = s' + \frac{1}{2!}s'^2 + \frac{1}{3!}s'^3 + \frac{1}{4!}s'^4 + \frac{1}{5!}s'^5 + \frac{1}{6!}s'^6 + \cdots \tag{25}$$

and respectively by $e^{2s'}$[7]

$$e^{2s'} - 1 = 2s' + \frac{4}{2!}s'^2 + \frac{8}{3!}s'^3 + \frac{16}{4!}s'^4 + \frac{32}{5!}s'^5 + \frac{64}{6!}s'^6 + \cdots \tag{26}$$

results in

$$\begin{aligned} (e^{s'} - 1)^2 &= (s' + \frac{1}{2!}s'^2 + \frac{1}{3!}s'^3 + \frac{1}{4!}s'^4 + \frac{1}{5!}s'^5 + \frac{1}{6!}s'^6 + \cdots)^2 \\ &= \cdots + 0.0861 s'^6 + 0.25 s'^5 + 0.5833 s'^4 + 1 s'^3 + s'^2 \end{aligned}. \tag{27}$$

In addition, $e^{s'd}$ is equal to

$$\begin{aligned} e^{s'd} &= 1 + ds' + \frac{1}{2!}d^2 s'^2 + \frac{1}{3!}d^3 s'^3 + \frac{1}{4!}d^4 s'^4 + \frac{1}{5!}d^5 s'^5 \\ &+ \frac{1}{6!}d^6 s'^6 + \cdots \end{aligned} \tag{28}$$

Additionally from (27) and (28) we have

$$e^{s'd}(e^{s'} - 1)^2 = s'^2 + d_3 s'^3 + d_4 s'^4 + d_5 s'^5 + d_6 s'^6 + \cdots \tag{29}$$

where

$$\begin{bmatrix} d_3 \\ d_4 \\ d_5 \\ d_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 + d \\ 0.5833 + d + \frac{1}{2}d^2 \\ 0.25 + 0.5833d + \frac{1}{2}d^2 + \frac{1}{6}d^3 \\ 0.0861 + 0.25d + \frac{0.5833}{2}d^2 + \frac{1}{6}d^3 + \frac{1}{24}d^4 \\ \vdots \end{bmatrix} \tag{30}$$

[6]Forward path transfer function.
[7]We substitute with $2s'$ where s' in (25).

According to the above, polynomial $D_1(s')$ can be rewritten in the form of

$$\begin{aligned} D_1(s') &= t_i^2(s'^n r_n + \cdots + s'^3 r_3 + s'^2 r_2 + s' r_1 + 1)e^{s'd}(e^{s'}-1)^2 \\ &= t_i^2(\cdots + q_7 s'^7 + q_6 s'^6 + q_5 s'^5 + q_4 s'^4 + q_3 s'^3 + s'^2) \end{aligned} \tag{31}$$

where $q_n = \sum_{j=0}^{n} r_j d_{n-j}$, $d_0 = d_1 = 0$, $d_2 = 1$ and $r_0 = 1$. Since $(1-e^{s'})\hat{x} + (e^{2s'}-1)\hat{y} + 1$ is equal to

$$\begin{aligned} &(1-e^{s'})\hat{x} + (e^{2s'}-1)\hat{y} + 1 = \\ &= 1 + (2\hat{y}-\hat{x})s' + (2\hat{y}-\frac{1}{2}\hat{x})s'^2 + (\frac{4}{3}\hat{y}-\frac{1}{6}\hat{x})s'^3 + \\ &+ (\frac{2}{3}\hat{y}-\frac{1}{24}\hat{x})s'^4 + (\frac{32}{5!}\hat{y}-\frac{1}{5!}\hat{x})s'^5 + (\frac{64}{6!}\hat{y}-\frac{1}{6!}\hat{x})s'^6 + \cdots \end{aligned} \tag{32}$$

polynomial $k_h N(s')$ takes the form

$$k_h N(s') = k_h k_p' \left[\begin{array}{l} \cdots + (y_7\hat{y} - x_7\hat{x})\, s'^7 + (y_6\hat{y} - x_6\hat{x})\, s'^6 \\ + (y_5\hat{y} - x_5\hat{x})\, s'^5 + (y_4\hat{y} - x_4\hat{x})\, s'^4 + (y_3\hat{y} - x_3\hat{x})\, s'^3 \\ + (y_2\hat{y} - x_2\hat{x})\, s'^2 + (2\hat{y} - x_1\hat{x})\, s' + 1 \end{array} \right] \tag{33}$$

where $x_n = \frac{1}{n!}$, $(n \in [1,7],\ n \in \mathbf{Z})$ and the y_j $(j \in [1,7],\ j \in \mathbf{Z})$ coefficients are given by $y_2 = 2$, $y_3 = \frac{4}{3}$, $y_4 = \frac{2}{3}$, $y_5 = \frac{32}{5!}$, $y_6 = \frac{64}{6!}$, $y_7 = \frac{128}{7!}$. Finally, the corresponding polynomials $N(s')$, $D(s')$ for both the numerator and denominator of the closed loop transfer function are given by

$$N(s') = \sum_{j=0}^{m} \left[k_p' \, (y_j\hat{y} - x_j\hat{x})\, s'^j \right] \tag{34}$$

where $y_0 = x_0 = 0$, $y_1 = 2$, $x_1 = 1$ and

$$D(s') = \sum_{j=0}^{n} \left[\left(q_j t_i^2 + k_h k_p' (y_j\hat{y} - x_j\hat{x}) \right) s'^j \right] \tag{35}$$

where $q_2 = 1$, $q_1 = 0$, $y_1 = 2$, $x_1 = 1$ and $q_0 = y_0 = x_0 = 0$. Therefore, the resulting transfer function of the closed loop control system is given by

$$\begin{aligned} T(s') &= \frac{N(s')}{D_1(s') + k_h N(s')} \\ &= \frac{\sum_{j=0}^{m} \left[k_p' \, (y_j\hat{y} - x_j\hat{x})\, s'^j \right]}{\sum_{j=0}^{n} \left[\left(q_j t_i^2 + k_h k_p' (y_j\hat{y} - x_j\hat{x}) \right) s'^j \right]} \end{aligned} \tag{36}$$

For determining the optimal control law, the principle proved in the Appendix VI will be applied.

Optimization Condition 1: $a_0 = b_0$.

By applying the first optimization condition to the closed loop transfer function results in

$$k_h = 1. \tag{37}$$

which implies that the final closed loop control system exhibits steady state position and velocity error. From, (36), it is apparent that if $k_h = 1$ then the respective terms s^0, s^1, of $N(s') = \cdots + k_p'(2\hat{y}-\hat{x})\, s' + k_p'$ and $D(s') = \cdots + k_h k_p'(2\hat{y}-\hat{x})\, s' + k_h k_p'$ are equal.

Optimization Condition 2: $a_1^2 - 2a_2a_0 = 0$.

By making use of $a_1^2 - 2a_2a_0 = b_1^2 - 2b_2b_0$ we end up with $t_i = 0$. For that reason, we set $a_1^2 - 2a_2a_0 = 0$ as another means of optimizing the magnitude of (56), [2], [3]. This results in,

$$t_i^2 = \frac{1}{2} k_h k_p' [(2\hat{y}-\hat{x})^2 - 2y_2\hat{y} + 2x_2\hat{x}] \tag{38}$$

Optimization Condition 3: $a_2^2 - 2a_3a_1 + 2a_4a_0 = b_2^2 - 2b_3b_1 + 2b_4b_0$.

The application of (60) to (36)

$$t_i^2 = 2k_h k_p' \left[(x_2 - q_3)\hat{x} + (2q_3 - y_2)\hat{y} - q_4 \right] \tag{39}$$

Optimization Condition 4: $a_3^2 + 2a_1a_5 - 2a_6a_0 - 2a_4a_2 = b_3^2 + 2b_1b_5 - 2b_6b_0 - 2b_4b_2$.

Finally the application of (61) to (36) leads to

$$t_i^2 = L \left[\begin{array}{l} \left(\begin{array}{c} q_4y_2 - q_3y_3 \\ -2q_5 + y_4 \end{array} \right)\hat{y} + \left(\begin{array}{c} q_3x_3 - q_4x_2 \\ +q_5 - x_4 \end{array} \right)\hat{x} \\ +q_6 \end{array} \right]. \tag{40}$$

where $L = \frac{2k_h k_p'}{(q_3^2 - 2q_4)}$. For determining the optimal controller parameters we will manipulate (38), (39) and (39), (40) together. Therefore, from (38), (39) we have

$$(2\hat{y}-\hat{x})^2 - 2\,(4q_3 - y_2)\,\hat{y} + 2\,(2q_3 - x_2)\,\hat{x} + 4q_4 = 0. \tag{41}$$

From (39), (40) we find

$$\begin{aligned} &\left[(q_3^2 - 2q_4)(x_2 - q_3) - (q_3x_3 - q_4x_2 + q_5 - x_4) \right]\hat{x} \\ &+ \left[(q_3^2 - 2q_4)(2q_3 - y_2) - (q_4y_2 - q_3y_3 - 2q_5 + y_4) \right]\hat{y} \\ &= q_4(q_3^2 - 2q_4) + q_6. \end{aligned} \tag{42}$$

Let us now make the following substitutions

$$A = 4q_3 - y_2 \tag{43}$$

$$B = 2q_3 - x_2 \tag{44}$$

$$C = 4q_4 \tag{45}$$

$$D = \left[\begin{array}{l} (q_3^2 - 2q_4)(x_2 - q_3) - \\ (q_3x_3 - q_4x_2 + q_5 - x_4) \end{array} \right] \tag{46}$$

$$E = \left[\begin{array}{l} (q_3^2 - 2q_4)(2q_3 - y_2) - \\ (q_4y_2 - q_3y_3 - 2q_5 + y_4) \end{array} \right] \tag{47}$$

$$Z = \left(q_3^2 - 2q_4\right) q_4 + q_6 \tag{48}$$

Substituting (43)-(48) back into (41)-(42) results in

$$(2\hat{y}-\hat{x})^2 - 2A\hat{y} + 2B\hat{x} + C = 0 \tag{49}$$

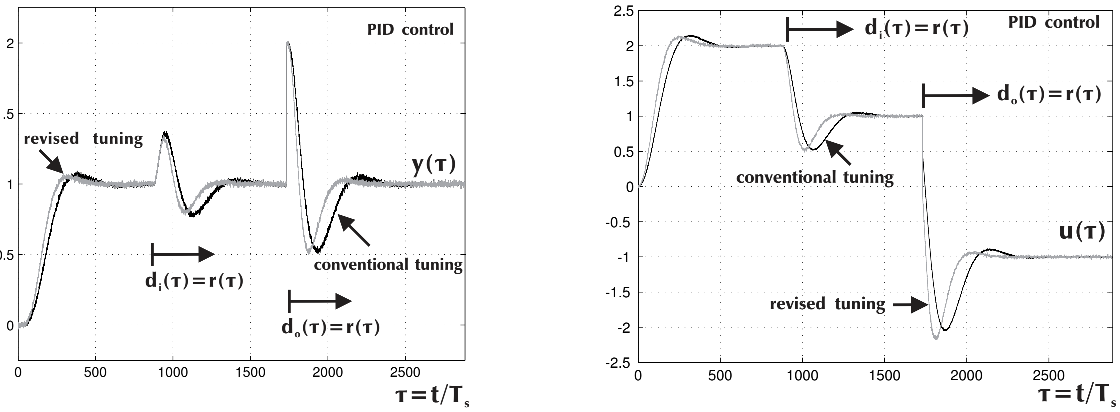
$$D\hat{x} + E\hat{y} = Z \tag{50}$$

respectively. From (50) it is apparent that $\hat{x}$ is equal to

$$\hat{x} = \frac{Z - E\hat{y}}{D}. \tag{51}$$

Substituting (51) into (49) results in

$$\hat{y}^2 - 2\frac{[(2D+E)Z + D(AD+BE)]}{(2D+E)^2}\hat{y} + \frac{D(2BZ+CD)+Z^2}{(2D+E)^2} = 0. \tag{52}$$



(a) Output $y(\tau)$ of the control loop at the presence of input and output disturbances. Settling time of output disturbance is at $t = 1731\tau + 576.9$ (conventional tuning) and ($t = 1731\tau + 425.19$) (revised tuning).

(b) Control signal $u(\tau)$ of the control loop at the presence of input and output disturbances.

Fig. 5. Comparison between the conventional and the proposed control law. Control of an integrating process with equivalent time constants. Sampling time is $\frac{T_{p1}}{T_s} = 20$ than the dominant time constant. Input disturbance is applied at $\tau = 865.42$ and output disturbance is applied at $\tau = 1731$.

Finally, the optimal control law is given by

$$\begin{bmatrix} 1 & -2k_h k_p' (x_2 - q_3) & -2k_h k_p' (2q_3 - y_2) \\ 0 & D & E \\ 0 & 0 & -2\frac{[(2D+E)Z+D(AD+BE)]}{(2D+E)^2} \end{bmatrix} \begin{bmatrix} t_i^2 \\ \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} -2k_h k_p' q_4 \\ Z \\ -\frac{D(2BZ+CD)+Z^2}{(2D+E)^2} - \hat{y}^2 \end{bmatrix} \tag{53}$$

IV. Evaluation Results

For justifying the potential of the proposed optimal control law a comparison between the conventional analog PID tuning, (Section II) and the proposed digital control law (Section III) will be performed when controlling the same process. In both cases all time constants have been normalized with sampling time T_s, $s' = sT_s$. Controller unmodelled dynamics have been chosen equal to $T_{\Sigma c} = 0.1T_{p_1}$. Special attention is drawn on the output $y(s)$ and the control input $u(s)$ regarding reference tracking $r(s)$ and at the presence of input $d_i(s)$ and output $d_o(s)$ disturbances, see Fig.1. For handling the issue with the great overshoot (in both cases, conventional and revised control law) as mentioned in Section II, an external filter of the form $C_{ex}(s') = \frac{1}{1+s'x+s'^2y}$ is added to the reference signal $r(s)$, where x, y are the zeros of the corresponding PID controller.

A. An integrating process with three equivalent time constants

In the first example, the process defined by

$$G(s') = \frac{0.5}{\begin{bmatrix} s(1+20s')(1+16.58s')(1+10.8s')(1+8.8s') \\ (1+1.56s') \end{bmatrix}} \tag{54}$$

is adopted. From Fig.5(a) it is apparent that both input and output disturbance rejection has been improved without paying this cost in the control effort Fig.5(b) as far as the overshoot of the control signal $u(\tau)$ is concerned. Output disturbance rejection has been decreased by 26.29%.

B. An integrating process with time delay three times greater than the dominant time constant

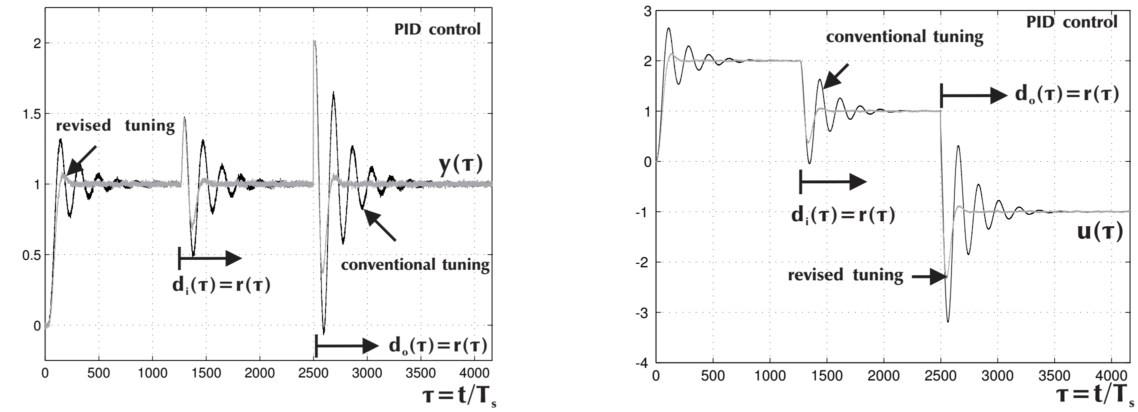
In the second example the integrating process defined by

$$G(s') = \frac{0.5}{(1+4s')^5} e^{-12s'} \tag{55}$$

is employed which involves a time delay three times greater than its dominant constant. From Fig.6(a) it is apparent that the conventional control law leads to unacceptable performance regarding reference tracking at the presence of input and output disturbance rejection. The revised control law has led to a reduction of settling time regarding output disturbance rejection of 71.05%, $(832.4\tau \rightarrow 240.97\tau)$. Effort of the control signal is acceptable since $u(\tau)$ does not oscillate, see Fig.6(b).

V. Conclusions and Outlook

Analytical expressions for the digital PID controller tuning have been presented regarding the control of integrating processes. The explicit control law takes into account all modelled process parameters (model of n poles plus unknown time delay d) plus the controller's sampling time T_s. Basis of the proposed theory is the Symmetrical Optimum criterion. The conventional PID tuning is thoroughly revised and is replaced by the proposed control law. Comparison between the conventional and the revised PID tuning shows promising results in terms of reference tracking and disturbance rejection. Future work deals with improving the control law by introducing a process involving n poles, m zeros plus unknown time delay d.



(a) Output $y(\tau)$ of the control loop at the presence of input $d_i(s)$ and output $d_o(s)$ disturbances.

(b) Control signal $u(\tau)$ of the control loop at the presence of input and output disturbances.

Fig. 6. Comparison between the conventional and the proposed control law. Control of an integrating process with time delay three times greater than its dominant time constant. Step response of the closed loop control system. Input disturbance $d_i(s)=r(s)$ is applied at $\tau=1248$ and output disturbance $d_o(s)=r(s)$ is applied at $\tau=2497$. Sampling time is $\frac{T_{p1}}{T_s}=4$ than the dominant time constant.

VI. Acknowledgment

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Appendix

A. Proof of Optimization Conditions

Let the closed loop transfer function be defined by (56),

$$T(s)=\frac{s^m b_m+s^{m-1}b_{m-1}+\cdots+s^2b_2+sb_1+b_0}{s^n a_n+s^{n-1}a_{n-1}+\cdots+s^2a_2+sa_1+a_0}=\frac{N(s)}{D(s)} \tag{56}$$

where $m\leq n$. Target of the Symmetrical Optimum criterion is to maintain $|T(s)|\simeq 1$ in the wider possible frequency range. Thus, by setting $s=j\omega$ into (56) and squaring $|T(j\omega)|$ leads to

$$|T(j\omega)|^2=\frac{|N(j\omega)|^2}{|D(j\omega)|^2}. \tag{57}$$

By making equal the terms of $\omega^j,(j=1,2,\ldots,n)$ in polynomials $|D(j\omega)|^2$, $|N(j\omega)|^2$ it is easily proved that conditions

$$a_0=b_0 \tag{58}$$

$$a_1^2-2a_2a_0=b_1^2-2b_2b_0 \tag{59}$$

$$a_2^2-2a_3a_1+2a_4a_0=b_2^2-2b_3b_1+2b_4b_0 \tag{60}$$

$$\begin{pmatrix} a_3^2+2a_1a_5-2a_6a_0 \\ -2a_4a_2 \end{pmatrix}=\begin{pmatrix} b_3^2+2b_1b_5-2b_6b_0 \\ -2b_4b_2 \end{pmatrix} \tag{61}$$

$$\cdots=\cdots$$

have to hold by.

B. The Conventional Symmetrical Optimum Criterion

Let the integrating process be defined by (1). If for controlling (1), I control of the form

$$C(s)=\frac{1}{sT_i(1+sT_{\Sigma c})}, \tag{62}$$

is applied, then the closed loop transfer function is given by

$$T(s)=\frac{k_p}{s^2T_iT_m(1+sT_{p_1})(1+sT_\Sigma)+k_hk_p} \tag{63}$$

where $T_{\Sigma p}T_{\Sigma c}\approx 0$ and $T_\Sigma=T_{\Sigma p}+T_{\Sigma c}$. From (63) it is evident

$$T(s)=\frac{k_p}{s^4T_iT_mT_{p_1}T_\Sigma+s^3T_iT_m(T_{p_1}+T_\Sigma)+s^2T_iT_m+k_hk_p} \tag{64}$$

According to (64), it is evident that $T(s)$ is unstable since the term of s is missing. In similar fashion, if PI control of the form

$$C(s)=\frac{1+sT_n}{sT_i(1+sT_{\Sigma c})}, \tag{65}$$

is employed, then for determining controller parameter T_n via the conventional Symmetrical Optimum criterion, pole-zero cancellation must take place, $T_n=T_{p_1}$. Therefore, $T(s)$ becomes

$$T(s)=\frac{k_p}{s^3T_iT_mT_\Sigma+s^2T_iT_m+k_hk_p} \tag{66}$$

which is unstable again for the same reason as stated for (64). Finally, PID control by canceling two real or conjugate complex poles of $G(s)$ cannot be applied, since it is proved that $T(s)$ becomes unstable for the same reason as for (64).

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