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On the Lyapunov and Bohl exponent of time-varying discrete linear system

Adam Czornik, Aleksander Nawrat, Michal Niezabitowski and Aneta Szyda

Abstract— **This note studies the problem of bounds for the Lyapunov exponent of a parameter perturbed system when the perturbation has finite average value. Such a bound is presented in terms of Bohl exponent of the unperturbed system. In particular, it has been shown that the Lyapunov exponent of perturbed system is not greater than the Bohl exponent of the unperturbed system if the average value of perturbations is zero. The obtained result is illustrated by a numerical example.**

I. INTRODUCTION

CONSIDER a dynamic system described by the following linear time-varying difference equation

$$x(n+1) = A(n)x(n), \tag{1}$$

where $A(n)$ is a bounded sequence of s-by-s real matrices. Along with (1), we consider a perturbed system of the form

$$y(n+1) = (A(n) + \Delta(n))\, y(n), \tag{2}$$

where $\Delta(n)$, $n = 0, 1, ...$ is a sequence of s-by-s real matrices which model the parameters perturbations. One of important problem is to construct bounds for the Lyapunov exponents of the perturbed system (2) when the perturbation belongs to some smallness class. A particular case of this problem is the question about stability radius. The question is how large the perturbation may be without destroying stability assuming that the original system (1) is stable. More precisely, we are looking for the largest bound r such that stability is preserved for all perturbations $\Delta(n)$ of operator norm strictly less than r in a given normed perturbation set. This largest bound is called the stability radius. First time the problem was formally formulated for the continuous time-invariant system in the famous paper of Hinrichsen and Pritchard [11] and since then many results have been obtained [12].

The general problem of the variation of the characteristic exponents under perturbation of coefficients is much better understood for continuous-time system than for discrete one. This is very transparent when one compares the review of the results for continuous-time system from Chapter V of [1], or from [6] with the results contained in monograph of Barreira and Pesin [5] devoted to Lyapunov exponents of discrete-time systems. Some recent results for discrete-time systems are presented in [7]-[10].

In the paper, we present an upper bound for the Lyapunov exponent of (2) for the perturbations satisfying the following condition

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} \|\Delta(k)\|}{n} = \widehat{\Delta} < \infty. \tag{3}$$

A. Czornik, A. Nawrat, M. Niezabitowski and A. Szyda are with Faculty of Automatic Control, Electronics and Computer Science, Silesian University of Technology, 16 Akademicka Street, 44-101 Gliwice, Poland, (e-mail: Adam.Czornik@polsl.pl, Aleksander.Nawrat@polsl.pl, Michal.Niezabitowski@polsl.pl, Aneta.Szyda@polsl.pl)

II. PRELIMINARY RESULTS

Consider the following time-varying discrete linear system

$$z(n+1) = B(n)z(n), \tag{4}$$

where $B(n)$, $n = 0, 1 ...$is a sequence of real s-by-s matrices. The evolution matrix associated with this system is defined as

$$\Phi_B(k,k) = I,\ \Phi_B(l,k) = B(l-1)...B(k) \text{ for } l > k.$$

To characterize exponential stability the concepts of Lyapunov and Bohl exponents have been introduced ([1], [3], [6]). The definitions are as follows.

Definition 1: The Lyapunov exponent of system (4) is defined as

$$\lambda_B = \limsup_{n\to\infty} \|\Phi_B(n,0)\|^{1/n}$$

and the Bohl exponent as

$$\mu_B = \limsup_{n, n-k\to\infty} \|\Phi_B(n,k)\|^{1/(n-k)}.$$

In the literature devoted to continuous-time system the Lyapunov exponent is usually defined as logarithm of λ_B, however in discrete-time case the evolution matrix may be zero and therefore it is more convenient to define the exponent as in our definition. The Bohl exponent for discrete-time system has been investigated in [13], where it has been named generalized spectral radius. In this paper many different formulas for the Bohl exponents have been obtained. From the definition it is clear that inequality

$$\mu_B < 1$$

is a necessary and sufficient condition for the uniform exponential stability, while

$$\lambda_B < 1$$

is a necessary and sufficient condition for the exponential stability (see [2] for the definition of these stability concepts).

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Moreover μ_B is finite if, and only if, $B(n)$, $n = 0, 1...$ is bounded. It is also clear that, in general

$$\lambda_B \leq \mu_B,$$

and the inequality may be strict. It is also well known (see for example [5]) that

$$\lambda_B = \max_{z(0)} \left(\limsup_{n\to\infty} \|z(n)\|^{1/n} \right). \tag{5}$$

In our further considerations we will use the following discrete version of Gronwall inequality (see [2]).

Theorem 1: Suppose that for two sequences $u(n)$ and $f(n)$, $n = 0, 1, ...$ of nonnegative real numbers the following inequality

$$u(n) \leq p + q \sum_{i=0}^{n-1} u(i) f(i)$$

holds for certain $p, q \in R$ and all $n = 0, 1, ...$ Then the following inequality is true

$$u(n) \leq p \prod_{i=0}^{n-1} (1 + q f(i)) \tag{6}$$

for all $n = 0, 1, ...$.

We will also use the following result from [13].

Theorem 2: For any bounded sequence $B(n)$, $n = 0, 1, ...$ and $\varepsilon > 0$ there exists $C_\varepsilon \geq 1$ such that for all $n \geq k \geq 0$ we have

$$\|\Phi_B(n,k)\| \leq C_\varepsilon (\mu_B + \varepsilon)^{n-k}. \tag{7}$$

III. MAIN RESULT

The next theorem contains the main result of this note.

Theorem 3: For any bounded sequence $A(n)$, $n = 0, 1, ...$ and any sequence of perturbations $\Delta(n)$, $n = 0, 1, ...$ satisfying (3), the Lyapunov exponent $\lambda_{A+\Delta}$ of (2) satisfies

$$\lambda_{A+\Delta} \leq (\mu_A + \varepsilon) e^{\frac{\widehat{\Delta} C_\varepsilon}{\mu_A + \varepsilon}}, \tag{8}$$

where $C_\varepsilon \geq 1$ is the constant defined by (7).

Proof: We can rewrite (2) in the form

$$y(n+1) = A(n)y(n) + \Delta(n)y(n).$$

Then for $y(0) = y_0$ its solution is given by

$$y(n) = \Phi_A(n,0)y_0 + \sum_{i=0}^{n-1} \Phi_A(n, i+1)\Delta(i)y(i).$$

Hence by (7)

$$\|y(n)\| \leq C_\varepsilon (\mu_A + \varepsilon)^n \|y_0\| + C_\varepsilon \sum_{i=0}^{n-1} (\mu_A + \varepsilon)^{n-i-1} \|\Delta(i)\| \|y(i)\|.$$

Multiplying this inequality by $(\mu_A + \varepsilon)^{-n}$ yields

$$(\mu_A + \varepsilon)^{-n} \|y(n)\| \leq C_\varepsilon \|y_0\| + \frac{C_\varepsilon}{\mu_A + \varepsilon} \sum_{i=0}^{n-1} (\mu_A + \varepsilon)^{-i} \|y(i)\| \|\Delta(i)\|.$$

Applying Gronwall's inequality (6) with

$$u(n) = (\mu_A + \varepsilon)^{-n} \|y(n)\|,$$

$$p = C_\varepsilon \|y_0\|,$$

$$q = \frac{C_\varepsilon}{\mu_A + \varepsilon},$$

$$f(i) = \|\Delta(i)\|$$

we obtain

$$(\mu_A + \varepsilon)^{-n} \|y(n)\| \leq C_\varepsilon \|y_0\| \prod_{i=0}^{n-1} \left(1 + \frac{C_\varepsilon}{\mu_A + \varepsilon} \|\Delta(i)\|\right).$$

By virtue of the relation $1 + u \leq e^u$ valid for each $u \in R$, we have

$$(\mu_A + \varepsilon)^{-n} \|y(n)\| \leq C_\varepsilon \|y_0\| \exp\left(\frac{C_\varepsilon}{\mu_A + \varepsilon} \sum_{i=0}^{n-1} \|\Delta(i)\|\right)$$

and

$$\|y(n)\|^{1/n} \leq (C_\varepsilon \|y_0\|)^{1/n} (\mu_A + \varepsilon) \cdot \exp\left(\frac{C_\varepsilon}{n(\mu_A + \varepsilon)} \sum_{i=0}^{n-1} \|\Delta(i)\|\right).$$

Taking $\limsup_{n\to\infty}$ on both sides we obtain

$$\limsup_{n\to\infty} \|y(n)\|^{1/n} \leq (\mu_A + \varepsilon) \exp\left(\frac{C_\varepsilon}{\mu_A + \varepsilon} \widehat{\Delta}\right).$$

The last inequality implies the conclusion of the Theorem, because of (5). ■

Exactly in the same way one can show the following result which does not refer to the Bohl exponent.

Theorem 4: For sequence $A(n)$, $n = 0, 1, ...$ such that

$$\|\Phi_A(n,k)\| \leq C\mu^{n-k} \tag{9}$$

and any sequence of perturbations $\Delta(n)$, $n = 0, 1, ...$ satisfying (3), the Lyapunov exponent $\lambda_{A+\Delta}$ of (2) satisfies

$$\lambda_{A+\Delta} \leq \mu e^{\frac{\widehat{\Delta} C}{\mu}}, \tag{10}$$

where $C_\varepsilon \geq 1$ is the constant defined by (7).

A drawback of the estimations (8) and (10) is that the right side involve constants C_ε or C, which are difficult to calculate. This problem disappear in case of $\widehat{\Delta} = 0$ as it is stated in the next corollary.

Corollary 5: For any bounded sequence $A(n)$, $n = 0, 1, ...$ and any sequence of perturbation $\Delta(n)$, $n = 0, 1, ...$ satisfying (3) with $\widehat{\Delta} = 0$, the Lyapunov exponent $\lambda_{A+\Delta}$ of (2) satisfies

$$\lambda_{A+\Delta} \leq \mu_A. \tag{11}$$

One may ask if the Bohl exponent μ_A in (11) can be replaced by the Lyapunov exponent λ_A. The following example shows that in general case it is impossible.

Example 1: Consider system (1) with

$$A(n)=\begin{cases}\begin{bmatrix}1&0\\0&0\end{bmatrix} & \text{for } n=0\\[2ex] \begin{bmatrix}0&0\\0&2\end{bmatrix} & \text{for } n>0\end{cases}.$$

Then $\mu_A=2$ and $\lambda_A=0$, whereas for the perturbation

$$\Delta(n)=\begin{cases}\begin{bmatrix}0&0\\0&\varepsilon\end{bmatrix} & \text{for } n=0\\[2ex] \begin{bmatrix}0&0\\0&0\end{bmatrix} & \text{for } n>0\end{cases}$$

we have $\lambda_{A+\Delta}=2$ for any $\varepsilon\neq 0$ and in fact the inequality (11) does not hold when μ_A is replaced by λ_A.

This simple example also shows, the well known in continuous-time case (see, [6]), fact that the Lyapunov exponent is not a continuous function with respect to the supremum norm of perturbation. Finally, let us note the following stability result which is a consequence of Corollary 5. In the statement of this result we use different types of stability of linear time-varying system and we refer the reader to the book [2] for the definitions.

Corollary 6: If the unperturbed system (1) is uniformly asymptotically stable then the perturbed system (1) is exponentially stable for any perturbation $\Delta(n),\ n=0,1,\ldots$ satisfying (3) with $\widehat{\Delta}=0$. ■

Proof: If the unperturbed system (1) is uniformly asymptotically stable then it is uniformly exponentially stable (see [2], Theorem 5.5.1) and it implies that $\mu_A<1$ (see [13]). By (11) $\lambda_{A+\Delta}<1$ what means that the perturbed system is exponentially stable (see [5]). ■

Observe that the Example 1 shows also that the last Corollary is not longer true when the uniform asymptotic stability of (1) is replaced by exponential stability.

IV. AN EXAMPLE

Consider a three-dimensional stationary system of the form (1)

$$A(n)=A=\begin{bmatrix}0&1&0\\0&0&1\\0.1&-0.1&-0.5\end{bmatrix}\quad\text{for all } n\geq 0.$$

It is well known that for time-invariant system Bohl exponent and Lyapunov exponent are equal and coincide with the spectral radius of A. To find constants μ and C such that (9) is satisfied we will use the following result.

Theorem 7: [4] If for certain s-by-s matrices A,Q,H, $Q=Q^T>0$, $H=H^T>0$ the following discrete Lyapunov equation

$$H-A^THA=Q \tag{12}$$

is satisfied, then (9) holds with spectral norm $\|\cdot\|_2$,

$$C=\sqrt{\frac{\|H\|_2}{\|H^{-1}\|_2}}$$

and

$$\mu=\sqrt{1-\frac{\lambda\min(Q)}{\|H\|_2}},$$

where $\lambda\min(Q)$ is the smallest eigenvalues of Q.

Let us solve the Lyapunov equation (12) with $Q=I_3$. The solution is

$$H=\begin{bmatrix}1.0388&-5.6122\times10^{-2}&-0.17347\\-5.6122\times10^{-2}&2.1122&0.20408\\-0.17347&0.20408&3.8776\end{bmatrix}$$

and according to Theorem 7 the constants are

$$C=\sqrt[2]{\frac{3.912}{0.97422}}=2.0039$$

and

$$\mu_A=0.86277.$$

According to (10) we have the following bound for the Lyapunov exponent of perturbed system

$$\lambda_{A+\Delta}\leq 0.86277e^{\widehat{\Delta}2.3226}.$$

The last inequality implies that the perturbed system is exponentially stable for all perturbation such that

$$\widehat{\Delta}<0.064.$$

The same example has been considered in [14] and it was shown that for much smaller class of perturbations namely for the structured perturbations of the form

$$\Delta(n)=\begin{bmatrix}0\\0\\1\end{bmatrix}u(n)\begin{bmatrix}0&1&0\end{bmatrix}$$

the system remains stable for $|u(n)|<0.6$.

V. CONCLUSIONS

In this note we presented a bound for the Lyapunov exponent of a parameter perturbed system when the perturbation has finite average value. The bound is in terms of Bohl exponent of unperturbed system and upper limit of the arithmetic mean of perturbation operator norm. In particular, it has been shown that the Lyapunov exponent of perturbed system is not greater than the Bohl exponent of the unperturbed system if the average value of perturbation is zero.

This implies that for such perturbation the perturbed system is exponentially stable and provided that the original system is uniformly asymptotically stable. By an example we showed that this class of perturbations do not preserve exponential stability.

The obtained result is illustrated on a numerical example.

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