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Determination of an Unknown Input Distribution Matrix for Non-Linear Discrete-Time Stochastic Systems

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***Abstract*— The paper deals with the problem of estimating an unknown input distribution matrix for non-linear discrete-time stochastic systems. In particular, it is shown how to use the unscented Kalman filter as an unknown input filter. Subsequently, an analysis of the impact of unknown input decoupling on the fault detection is performed and a suitable fault detection condition is developed. Based on the achieved results, a numerical optimisation-based approach is proposed that can be used to estimate the unknown input distribution matrix. The final part of the paper presents an illustrative example with an induction motor, which confirms the performance of the proposed approach.**

I. Introduction

As can be observed in the literature, observers (or filter in a stochastic framework) are commonly used in both control and fault diagnosis schemes of non-linear systems (see, e.g., [1], [2], [3], [4] and the references therein). Undoubtedly, the most common approach is to use robust observers, such as the Unknown Input Observer (UIO) [5], [2], which can tolerate a degree of model uncertainty and hence increase the reliability of fault diagnosis. Although the origins of UIOs can be traced back to the early 1970's (cf. the seminal work of Wang et al. [6]) the problem of designing such observers is still of paramount importance both from the theoretical and practical viewpoints. A large amount of knowledge on using these techniques for model-based fault diagnosis has been accumulated through the literature for the last three decades (see [2] and the references therein). Generally, the design problems regarding UIOs for non-linear systems can be divided into the three distinct categories:

- *nonlinear state-transformation-based techniques:* apart from a relatively large class of systems for which they can be applied, even if the nonlinear transformation is possible it leads to another nonlinear system and hence the observer design problem remains open (see [5] and the references therein).
- *linearization-based techniques:* such approaches are based on a similar strategy as that for the Extended Kalman filter (EKF) [1]. In [2] the author proposed an extended unknown input observer for non-linear systems. He also proved that the proposed observer is convergent under certain conditions.
- *observers for particular classes of nonlinear systems:* for example Unknown Input Observers for polynomial and bilinear systems or UIOs for Lipschitz systems [7], [8], [2]).

Generally, the design problems regarding UIOs (undertaken within the framework of this paper) can be divided into the two distinct categories:

1) How to determine the the unknown input distribution matrix, which will not decouple the effect of faults from the residual?
2) How to develop the possibly simple and reliable design procedure of the UIO for non-linear stochastic systems?

Concerning the first question, the partial answer can be found in the book of [9]. Indeed, the authors concentrate on the determination of the unknown input distribution matrix for linear systems but they do not answer the question when this matrix will cause the fault decoupling effect. To tackle this problem within the framework of this paper, a numerical optimisation-based approach is proposed that can be used to estimate the unknown input distribution matrix, which does not cause the fault decoupling effect. As an answer to the second question, this work presents an alternative Unknown Input Filter (UIF) for non-linear systems, which is based on the general idea of the Unscented Kalman Filter (UKF) [10]. This approach is based on the similar idea as that proposed by [2] but the structure of the scheme is different and instead of the EKF, the UKF is employed.

II. Unknown input decoupling

Let us consider a non-linear stochastic system given by the following equations:

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{E}\boldsymbol{d}_k + \boldsymbol{L}\boldsymbol{f}_k + \boldsymbol{w}_k, \quad (1)$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}, \quad (2)$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ is a system state, $\boldsymbol{y}_k \in \mathbb{R}^m$ is a system output, $\boldsymbol{E} \in \mathbb{R}^{n\times q}$ is an unknown input distribution matrix, $\boldsymbol{d}_k \in \mathbb{R}^q$ $(q \leq m)$ stands for an unknown input, $u_k \in \mathbb{R}^r$ is a system input, $\boldsymbol{f}_k$ is the fault, $\boldsymbol{L} \in \mathbb{R}^{n\times s}$ stands for the fault distribition matrix, $\boldsymbol{g}\left(\cdot\right)$ and $\boldsymbol{h}(\cdot)$ are non-linear functions and $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are process and measurement noise, respectively.

The main problem is to design a filter which is insensitive to the influence of the unknown input (external disturbances and modeling errors). The necessary condition for the existence of a solution to the unknown input de-coupling problem is as follows:

$$\text{rank}(\boldsymbol{C}\boldsymbol{E}) = \text{rank}(\boldsymbol{E}) = q, \quad (3)$$

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(see [2] for a comprehensive explanation). If condition (3) is satisfied, then it is possible to calculate $\boldsymbol{H} = (\boldsymbol{CE})^{+} = \left[(\boldsymbol{CE})^T \boldsymbol{CE}\right]^{-1} (\boldsymbol{CE})^T$. Thus, by inserting (1) into (2) for $\boldsymbol{x}_{k+1}$ and then multiplying (2) by $\boldsymbol{H}$ it is straightforward to show that

$$\boldsymbol{d}_k = \boldsymbol{H}\Big[\boldsymbol{y}_{k+1} - \boldsymbol{C}\left[\boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \boldsymbol{L}\boldsymbol{f}_k + \boldsymbol{w}_k\right] - + \boldsymbol{v}_{k+1}\Big]. \tag{4}$$

Substituting (4) into (1) for $\boldsymbol{d}_k$ gives

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1} + \bar{\boldsymbol{L}}\boldsymbol{f}_k + \bar{\boldsymbol{w}}_k, \tag{5}$$

where:

$$\bar{\boldsymbol{g}}\left(\cdot\right) = \boldsymbol{G}\boldsymbol{g}\left(\cdot\right), \bar{\boldsymbol{h}}(\cdot) = \boldsymbol{G}\boldsymbol{h}(\cdot)$$
$$\bar{\boldsymbol{E}} = \boldsymbol{E}\boldsymbol{H}, \bar{\boldsymbol{w}}_k = \boldsymbol{G}\boldsymbol{w}_k - \boldsymbol{E}\boldsymbol{H}\boldsymbol{v}_{k+1},$$

and

$$\boldsymbol{G} = \boldsymbol{I} - \boldsymbol{E}\boldsymbol{H}\boldsymbol{C}.$$

Consequently, the general unknown input observer (filter in the stochastic case) structure is:

$$\hat{\boldsymbol{x}}_{k+1} = \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1} + \boldsymbol{K}(\cdot), \tag{6}$$

where $\hat{\boldsymbol{x}}_k$ stands for the state estimate, and $\boldsymbol{K}(\cdot)$ is the state correction term of any form.

Let us define a residual as a difference between the output of the system and the estimated output:

$$\boldsymbol{z}_{k+1} = \boldsymbol{y}_{k+1} - \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1} = = \boldsymbol{C}\left(\bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) - \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) - \boldsymbol{K}(\cdot)\right) + \bar{\boldsymbol{f}}_k + \boldsymbol{C}\bar{\boldsymbol{w}}_k + \boldsymbol{v}_{k+1}, \tag{7}$$

where

$$\bar{\boldsymbol{f}}_k = \boldsymbol{C}\bar{\boldsymbol{L}}\boldsymbol{f}_k = = \boldsymbol{C}\left[\boldsymbol{I}_n - \boldsymbol{E}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{C}\right]\boldsymbol{L}\boldsymbol{f}_k \tag{8}$$

As can be easily found in the literature [1], [2], the above-defined residual is the main mean used for fault diagnosis. Indeed, it should be as close as possible to zero in fault-free case and significantly different then zero otherwise.

The objective of the subsequent part of this paper is to provide an answer to the following questions:

- How to determine (estimate) the unknown input distribution matrix $\boldsymbol{E}$?
- How to determine the conditions under which the UIF will not decouple the fault from the residual? In other words, how to check if $\bar{\boldsymbol{f}}_k$ defined by (8) will be different than zero when a fault occurs?

III. Unscented Kalman filter

Let us consider a non-linear, discrete-time fault-free system, i.e. (1)–(2) for $\boldsymbol{f}_k = \boldsymbol{0}$:

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{h}(\boldsymbol{u}_k) + \bar{\boldsymbol{d}}_k + \boldsymbol{w}_k, \tag{9}$$
$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1} + \boldsymbol{v}_{k+1}. \tag{10}$$

where $\bar{\boldsymbol{d}}_k = \boldsymbol{E}\boldsymbol{d}_k$.

The UKF [10] can be perceived a derivative-free alternative to the extended Kalman filter in the framework of state-estimation. The UKF calculates the mean and covariance of a random variable, which undergoes a non-linear transformation by utilizing a deterministic "sampling" approach. Generally, $2L+1$ (L is the state dimension, i.e. $L = n$), *sigma* points are chosen based on a square-root decomposition of the prior covariance. These sigma points are propagated through the true nonlinearity, without any approximation, and then a weighted mean and covariance are taken.

First, let us assume that $\bar{\boldsymbol{d}}_k = \boldsymbol{0}$ for which the standard UKF will be presented, and then it will be extended to the case $\bar{\boldsymbol{d}}_k \neq \boldsymbol{0}$. The UKF involves a recursive application of these sigma points to the state-space equations. The standard UKF implementation for state-estimation uses the following variable definitions:

- $W_0^m = \lambda/(L+\lambda)$,
- $W_0^c = \lambda/(L+\lambda) + (1-\alpha^2+\beta)$,
- $W_i^m = W_i^c = 1/\{2(L+\lambda)\}$,
- $\lambda = L(\alpha^2 - 1)$,
- $\eta = \sqrt{(L+\lambda)}$,

where W_i $(i = 1, \ldots, 2L)$ is a set of scalar weights, λ and η are scaling parameters. The constant α determines the spread of sigma points around $\hat{x}$ and is usually set to $1e-4 \leq \alpha \leq 1$. β is used to incorporate prior knowledge of the distribution (for Gaussian distribution $\beta = 2$ is an optimal choice). The UKF algorithm is as follows:

Initialize with:

$$\hat{\boldsymbol{x}}_0 = \mathcal{E}[\boldsymbol{x}_0] \quad \boldsymbol{P}_0 = \mathcal{E}[(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)(\boldsymbol{x}_0 - \hat{\boldsymbol{x}}_0)^T], \tag{11}$$

For $k \in \{1, \ldots, \infty\}$
Calculate $2L+1$ sigma points:

$$\hat{\boldsymbol{X}}_{k-1} = [\hat{\boldsymbol{x}}_{k-1} \quad \hat{\boldsymbol{x}}_{k-1} + \eta S(1), \ldots, \hat{\boldsymbol{x}}_{k-1} + \eta S(n), \hat{\boldsymbol{x}}_{k-1} - \eta S(1), \ldots, \hat{\boldsymbol{x}}_{k-1} - \eta S(n)], \tag{12}$$

where $\boldsymbol{S} = \sqrt{\boldsymbol{P}_{k-1}}$ and $\boldsymbol{S}(j)$ stands for the jth column of $\boldsymbol{S}$.

Time update equations:

$$\hat{\boldsymbol{X}}_{i,k|k-1} = \boldsymbol{g}\left(\hat{\boldsymbol{X}}_{i,k-1}\right) + \boldsymbol{h}(\boldsymbol{u}_k),\ i = 0,\ldots,2L, \quad (13)$$

$$\hat{\boldsymbol{x}}_{k,k-1} = \sum_{i=0}^{2L} W_i^{(m)} \hat{\boldsymbol{X}}_{i,k|k-1} \quad (14)$$

$$\boldsymbol{P}_{k,k-1} = \sum_{i=0}^{2L} W_i^{(c)} [\hat{\boldsymbol{X}}_{i,k|k-1} - \\ - \hat{\boldsymbol{x}}_{k,k-1}][\hat{\boldsymbol{X}}_{i,k|k-1} - \hat{\boldsymbol{x}}_{k,k-1}]^T + \boldsymbol{Q} \quad (15)$$

$$\boldsymbol{Y}_{i,k|k-1} = \boldsymbol{C}\hat{\boldsymbol{X}}_{i,k|k-1}, \qquad i = 0,\ldots,2L \quad (16)$$

$$\hat{\boldsymbol{y}}_{k,k-1} = \sum_{i=0}^{2L} W_i^{(m)} \boldsymbol{Y}_{i,k|k-1} \quad (17)$$

Measurement update equations:

$$\boldsymbol{P}_{\tilde{y}_k\tilde{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\boldsymbol{Y}_{i,k|k-1} - \hat{\boldsymbol{y}}_{k,k-1}] \cdot \\ \cdot [\boldsymbol{Y}_{i,k|k-1} - \hat{\boldsymbol{y}}_{k,k-1}]^T + \boldsymbol{R} \quad (18)$$

$$\boldsymbol{P}_{x_k y_k} = \sum_{i=0}^{2L} W_i^{(c)} [\hat{\boldsymbol{X}}_{i,k|k-1} - \hat{\boldsymbol{x}}_{k,k-1}][\boldsymbol{Y}_{i,k|k-1} - \hat{\boldsymbol{y}}_{k,k-1}]^T \quad (19)$$

$$\boldsymbol{K}_k = \boldsymbol{P}_{x_k y_k} \boldsymbol{P}_{\tilde{y}_k\tilde{y}_k}^{-1} \quad (20)$$

$$\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_{k,k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k,k-1}) \quad (21)$$

$$\boldsymbol{P}_k = \boldsymbol{P}_{k,k-1} - \boldsymbol{K}_k \boldsymbol{P}_{\tilde{y}_k\tilde{y}_k} \boldsymbol{K}_k^T \quad (22)$$

where $\boldsymbol{Q}$ is the covariance of $\boldsymbol{w}_k$ while $\boldsymbol{R}$ is the covariance of $\boldsymbol{v}_k$.

The unknown input filter (6) can be designed with the application of the unscented Kalaman filter (11)–(22). Indeed, replacing (13) by

$$\hat{\boldsymbol{X}}_{i,k|k-1} = \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{X}}_{i,k-1}\right) + \bar{\boldsymbol{h}}(\boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1},\ i = 0,\ldots,2L, \quad (23)$$

an Unknown Input Unscented Kalman Filter (UIUKF) is developed.

IV. Preventing fault decoupling

The following theorem provides a simple rule for checking if the proposed UIUKF will not decouple the effect of a fault from the residual. It relates the fault and unknown input distribution matrices denoted by $\boldsymbol{E}$ and $\boldsymbol{L}$, respectively. Moreover, let us assume that the following rank condition is satisfied:

$$\text{rank}(\boldsymbol{CL}) = \text{rank}(\boldsymbol{L}) = s. \quad (24)$$

Theorem 1. The fault $\boldsymbol{f}_k$ will not be decoupled from the residual (7) if and only if the matrix

$$[\boldsymbol{CE}\,\boldsymbol{CL}] \quad (25)$$

is a full-rank one.

Proof. Let us suppose (theoretically) that $\text{rank}(\boldsymbol{C}\bar{\boldsymbol{L}}) = s$ then it can be shown that

$$\boldsymbol{f}_k = (\boldsymbol{C}\bar{\boldsymbol{L}})^+ \bar{\boldsymbol{f}}_k \quad (26)$$

which means that there exists a unique relationship between $\boldsymbol{f}_k$ and $\bar{\boldsymbol{f}}_k$ and hence the fault will not be decoupled from the residual. Unfortunately, the subsequent part of the proof shows that this is not always possible to attain. Indeed, (8) can be written into an equivalent form

$$\bar{\boldsymbol{f}}_k = \left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\boldsymbol{f}_k. \quad (27)$$

Moreover, it can be observed that

$$\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]^2 \\ = \boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T, \quad (28)$$

which means that $\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T$ is an idempotent matrix. One of the fundamental properties of an idempotent matrix is that its rank is equal to the trace, i.e.:

$$\begin{aligned} &\text{rank}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right) \\ &= \text{trace}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right) \\ &= \text{trace}\left(\boldsymbol{I}_m\right) - \text{trace}\left(\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right) \\ &= m - \text{trace}\left(\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CE}\right) = m - q \end{aligned} \quad (29)$$

Thus, from (24) it is clear that

$$\begin{aligned} &\text{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \leq \\ &\leq \min(m-q, s) \end{aligned} \quad (30)$$

On the other hand,

$$\begin{aligned} &\text{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \geq \\ &\geq \text{rank}\left(\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right) + \\ &+ \text{rank}\,(\boldsymbol{CL}) - m = s - q. \end{aligned} \quad (31)$$

Finally,

$$\begin{aligned} &\max(s-q, 0) \leq \\ &\leq \text{rank}\left(\left[\boldsymbol{I}_m - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\right]\boldsymbol{CL}\right) \leq \\ &\leq \min(m-q, s). \end{aligned} \quad (32)$$

Thus, it is necessary to find an alternative condition under which

$$\begin{aligned} \bar{\boldsymbol{f}}_k &= \boldsymbol{CL}\boldsymbol{f}_k - \boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CL}\boldsymbol{f}_k = \\ &= \boldsymbol{CL}\boldsymbol{f}_k - \boldsymbol{CL}\boldsymbol{f}_k = \boldsymbol{0}. \end{aligned} \quad (33)$$

Indeed, any vector $\boldsymbol{CL}\boldsymbol{f}_k \in \text{col}(\boldsymbol{CE})$, where

$$\text{col}(\boldsymbol{CE}) = \{\boldsymbol{\alpha} \in \mathbb{R}^m :\ \boldsymbol{\alpha} = \boldsymbol{CE}\boldsymbol{\beta} \quad \text{for some} \quad \boldsymbol{\beta} \in \mathbb{R}^q\}, \quad (34)$$

can be written as

$$\boldsymbol{CL}\boldsymbol{f}_k = \boldsymbol{CE}\tilde{\boldsymbol{f}}_k, \quad (35)$$

for some non-zero vector $\tilde{\boldsymbol{f}}_k$. As a consequence:

$$\begin{aligned} &\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CL}\boldsymbol{f}_k = \\ &\boldsymbol{CE}\left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T\boldsymbol{CE}\tilde{\boldsymbol{f}}_k = \boldsymbol{CE}\tilde{\boldsymbol{f}}_k = \boldsymbol{CL}\boldsymbol{f}_k. \end{aligned} \quad (36)$$

From the above discussion, it is clear that the proposed unknown input observer will not decouple the fault effect from the residual iff $\boldsymbol{CLf}_k \notin \text{col}(\boldsymbol{CE})$, which is equivalent to:

$$\text{rank}\left([\boldsymbol{CE}\ \boldsymbol{CLf}_k]\right) = q+1, \tag{37}$$

for all $f_{i,k} \neq 0$, $i = 1, \ldots, s$. It is clear that (37) is equivalent to the fact that the only solution to (for all $f_{i,k} \neq 0$, $i = 1, \ldots, s$)

$$\alpha_1(\boldsymbol{CE})_1 + \alpha_2(\boldsymbol{CE})_2 + \cdots + \alpha_q(\boldsymbol{CE})_q + \alpha_{q+1}\boldsymbol{CLf}_k = 0, \tag{38}$$

is for $\alpha_i = 0$, $\imath = 1, \ldots, q+1$. By further expanding (38) to

$$\begin{aligned}&\alpha_1(\boldsymbol{CE})_1 + \cdots + \alpha_q(\boldsymbol{CE})_q + \alpha_{q+1}f_{1,k}(\boldsymbol{CL})_1 + \ldots \\ &\cdots + \alpha_{q+1}f_{s,k}(\boldsymbol{CL})_s = 0,\end{aligned} \tag{39}$$

it can be seen that the zero-valued solution to (39) is equivalent to the existance of a full-rank matrix (25), which completes the proof.

V. Determination of $\boldsymbol{E}$

As a result of the deliberations presented in the previous section, the matrix $\boldsymbol{E}$ should satisfy the following conditions:

$$\text{rank}(\boldsymbol{CE}) = \text{rank}(\boldsymbol{E}) = q, \tag{40}$$

and the matrix

$$[\boldsymbol{CE}\ \boldsymbol{CL}] \tag{41}$$

should be a full rank one. The subsequent part of this section presents a numerical algorithm that can be used for estimating the unknown input distribution matrix $\boldsymbol{E}$ based on a set of input-output measurements $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$, where n_t is the number of data points.

To settle the problem of numerical estimation of $\boldsymbol{E}$, the following optimization criterion is assumed

$$\hat{\boldsymbol{E}} = \arg \min_{\boldsymbol{E} \in \mathbb{R}^{n \times q}} J(\boldsymbol{E}), \tag{42}$$

with

$$J(\boldsymbol{E}) = \frac{1}{mn_t} \sum_{k=1}^{n_t} \boldsymbol{z}_k^T \boldsymbol{z}_k, \tag{43}$$

where $\boldsymbol{z}_k$ stands for the residual defined by (7) and $\hat{\boldsymbol{E}}$ is an estimate of $\boldsymbol{E}$.

It is important to underline that the computation of (43) requires the run of the proposed UIUKF for a given instance of the unknown input distribution matrix $\boldsymbol{E}$. The computation of the cost function (43) is a definitely most time consuming part of the proposed algorithm. On the other hand, the computation time and the resulting computational burden are not of paramount importance since the proposed algorithm performs off-line. Indeed, only the result of the proposed algorithm being an estimate of the unknown input distribution matrix $\boldsymbol{E}$ is utilised on-line for unknown input decoupling.

The outline of the proposed algorithm:

Step 0: Obtain the fault-free input-output data set from the system $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$.

Step 2: Initialise the algorithm with some initial value of $\boldsymbol{E}$ satisfying (40) and (41).

Step 3: Use an optimisation strategy to find an estimate of $\boldsymbol{E}$ for which (43) reaches its minimum and conditions (40) and (41) are satisfied.

Similarly as in the case of (8) it can be shown that the fault-free residual is:

$$\begin{aligned}&\boldsymbol{z}_{k+1} = \boldsymbol{y}_{k+1} - \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1} = \\ &= \boldsymbol{C}\left(\bar{\boldsymbol{g}}\left(\boldsymbol{x}_k\right) - \bar{\boldsymbol{g}}\left(\hat{\boldsymbol{x}}_k\right) - \boldsymbol{K}(\cdot)\right) + \tilde{\boldsymbol{d}}_k + \boldsymbol{C}\bar{\boldsymbol{w}}_k + \boldsymbol{v}_{k+1},\end{aligned} \tag{44}$$

where

$$\tilde{\boldsymbol{d}}_k = \boldsymbol{C}\left[\boldsymbol{I}_n - \hat{\boldsymbol{E}}\left[(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\hat{\boldsymbol{E}}\right]^{-1}(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\right]\bar{\boldsymbol{d}}_k. \tag{45}$$

Alternatively, assuming $\bar{\boldsymbol{d}}_k = \boldsymbol{E}\boldsymbol{d}_k$, it can be expressed by

$$\tilde{\boldsymbol{d}}_k = \boldsymbol{C}\left[\boldsymbol{I}_n - \hat{\boldsymbol{E}}\left[(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\hat{\boldsymbol{E}}\right]^{-1}(\boldsymbol{C}\hat{\boldsymbol{E}})^T\boldsymbol{C}\right]\boldsymbol{E}\boldsymbol{d}_k, \tag{46}$$

Following the same line of reasoning as in the proof of Theorem 1, it can be shown can be shown that for any vector $\boldsymbol{CE}\boldsymbol{d}_k \in \text{col}(\boldsymbol{C}\hat{\boldsymbol{E}})$ the effect of an unknown input $\tilde{\boldsymbol{d}}_k$ will be decoupled from the residual, i.e. $\tilde{\boldsymbol{d}}_k = \boldsymbol{0}$.

Based on the above deliberations, it seems that an alternative approach is:

Step 0: Obtain the fault-free input-output data set from the system $\{(\boldsymbol{u}_k, \boldsymbol{y}_k)\}_{k=1}^{n_t}$.

Step 1: Estimate $\bar{\boldsymbol{d}}_k$ for $k = 1, \ldots, n_t$ with, e.g., an augmented UKF.

Step 2: Find a basis of $[\bar{\boldsymbol{d}}_1, \ldots, \bar{\boldsymbol{d}}_{n_t}]$ (e.g. an orthonormal basis), which will constitute an estimate of $\boldsymbol{E}$.

Apart from the unquestionable appeal of the above algorithm it does not take into account that conditions (40) and (41) must be satisfied. On the other hand, it was empirically proven that due to the process and measurement noise, an accurate estimation of $\bar{\boldsymbol{d}}_k$ (for $k = 1, \ldots, n_t$) is impossible, and hence, Step 2 of the above algorithm cannot be realised with expected results.

Thus, the only fruitful conclusion is that an estimate of $\boldsymbol{E}$ is not unique, which will undoubtedly facilitate the performance of the optimisation-based approach presented in the subsequent part of this section. The search process of the Adaptive Random Search algorithm (ARS) [11], [2] can be split into two phases. The first phase (variance-selection phase) consists in selecting an element from the sequence

$$\{\sigma^{(i)}\}, \qquad i = 1, \ldots, i_{max} \tag{47}$$

where $\sigma^{(1)}$ stands for an initial standard deviation selected by the designer (forming the covariance matrix $\Sigma = \sigma \boldsymbol{I}_{n \times q}$, where $n \times q$ is the number of elements of $\boldsymbol{E}$), and

$$\sigma^{(i)} = 10^{(-i+1)}\sigma^{(1)}. \tag{48}$$

In this way, the range of σ ensures both proper exploration properties over the search space and a sufficient accuracy of optimum localization. Larger values of σ decrease the possibility of getting stuck in a local minimum. The second

phase (variance-exploration phase) is dedicated to exploring the search space with the use of σ obtained from the first phase and consists in repetitive random perturbation of the best point obtained in the first phase. The scheme of the ARS algorithm used in further simulations is as follows:

0. Input data:

- $\sigma^{(1)}$ – the initial standard deviation;
- $j_{\max}$ – the number of iterations in each phase;
- $i_{\max}$ – the number of standard deviations (σ^i) changes;
- $k_{\max}$ – the global number of algorithm runs;
- $\boldsymbol{E}^{(0)}$ – the initial value of the unknown input distribution matrix;

1. Initialize

(1.1) Generate $\boldsymbol{E}_{\text{best}} \rightarrow \boldsymbol{E}^0$, satisfying (40) and (41), $k \rightarrow 1$, $i \rightarrow 1$.

2. Variance-selection phase

(2.1) $j \rightarrow 1, \boldsymbol{E}^{(j)} \rightarrow \boldsymbol{E}^{(0)}$ and $\sigma^{(i)} \rightarrow 10^{(-i+1)}\sigma^{(1)}$.
(2.2) Perturb $\boldsymbol{E}^{(j)}$ to get a new trial point $\boldsymbol{E}_+^{(j)}$ satisfying (40) and (41).
(2.3) If $J(\boldsymbol{E}_+^{(j)}) \leq J(\boldsymbol{E}^{(j)})$ then $\boldsymbol{E}^{(j+1)} \rightarrow \boldsymbol{E}_+^{(j)}$ else $\boldsymbol{E}^{(j+1)} \rightarrow \boldsymbol{E}^{(j)}$.
(2.4) If $J(\boldsymbol{E}_+^{(j)}) \leq J(\boldsymbol{E}_{\text{best}})$ then $\boldsymbol{E}_{\text{best}} \rightarrow \boldsymbol{E}_+^{(j)}, i_{\text{best}} \rightarrow i$.
(2.5) If $(j \leq j_{\max}/i)$ then $j \rightarrow j+1$ and go to (2.2).
(2.6) If $(i < i_{\max})$ then set $i \rightarrow i+1$ and go to (2.1).

3. Variance-exploration phase

(3.1) $j \rightarrow 1, \boldsymbol{E}^{(j)} \rightarrow \boldsymbol{E}_{\text{best}}, i \rightarrow i_{\text{best}}$ and $\sigma^{(i)} \rightarrow 10^{(-i+1)}\sigma^{(1)}$.
(3.2) Perturb $\boldsymbol{E}^{(j)}$ to get a new trial point $\boldsymbol{E}_+^{(j)}$ satisfying (40) and (41).
(3.3) If $J(\boldsymbol{E}_+^{(j)}) \leq J(\boldsymbol{E}^{(j)})$ then $\boldsymbol{E}^{(j+1)} \rightarrow \boldsymbol{E}_+^{(j)}$ else $\boldsymbol{E}^{(j+1)} \rightarrow \boldsymbol{E}^{(j)}$.
(3.4) If $J(\boldsymbol{E}_+^{(j)}) \leq J(\boldsymbol{E}_{\text{best}})$ then $\boldsymbol{E}_{\text{best}} \rightarrow \boldsymbol{E}_+^{(j)}$
(3.5) If $(j \leq j_{\max})$ then $j \rightarrow j+1$ and go to Step 3.2.
(3.6) If $(k \rightarrow k_{\max})$ then STOP.
(3.7) $k \rightarrow k+1, \boldsymbol{E}^{(0)} \rightarrow \boldsymbol{E}_{\text{best}}$ and resume from (2.1).

VI. Illustrative example

The numerical example considered here is a fifth-order two-phase non-linear model of an induction motor, which has already been the subject of a large number of various control design applications [12], [2]. A detailed description of the above non-linear system can be found in [12], [2] and the references therein. A complete discrete-time model of this system is as follows:

$$\begin{aligned}
x_{1,k+1} =& x_{1,k} + h(-\gamma x_{1,k} + \frac{K}{T_r}x_{3,k} + Kpx_{5,k}x_{4,k} \\
&+ \frac{1}{\sigma L_s}u_{1,k}) + E_{1,1}d_k, \qquad (49)\\
x_{2,k+1} =& x_{2,k} + h(-\gamma x(2,k) - Kpx_{5,k}x_{3,k} + \frac{K}{T}x_{4,k} \\
&+ \frac{1}{\sigma L}u_{2,k}) + E_{2,1}d_k, \qquad (50)\\
x_{3,k+1} =& x_{3,k} + h(\frac{M}{T_r}x(1,k) - \frac{1}{T_r}x_{3,k} - px_{5,k}x_{4,k}) \\
&+ E_{3,1}d_k, \qquad (51)\\
x_{4,k+1} =& x_{4,k} + h(\frac{M}{T_r}x_{2,k} + px_{5,k}X_{3,k} - \frac{1}{T_r}x_{4,k}) \\
&+ E_{4,1}d_k, \qquad (52)\\
x_{5,k+1} =& x_{5,k} + h(\frac{pM}{JL_r}(x_{3,k}x_{2,k} - x_{4,k}x_{1,k}) - \frac{T_L}{J}) \\
&+ E_{5,1}d_k, \qquad (53)\\
y_{1,k+1} =& x_{1,k+1}, \qquad y_{2,k+1} = x_{2,k+1}, \qquad (54)
\end{aligned}$$

where $x_k = [x_{1,k}, \ldots, x_{n,k}]^T = [i_{\text{sak}}, i_{\text{sbk}}, \psi_{\text{rak}}, \psi_{\text{rbk}}, \omega_{\text{k}}]^T$ represents the currents, the rotor fluxes, and the angular speed, respectively, while $u_{\text{k}} = [u_{\text{sak}}, u_{\text{sbk}}]^T$ is the stator voltage control vector, p is the number of pole pairs, and T_L is the load torque. The rotor time constant T_r and the remaining parameters are defined as:

$$T_r = \frac{L_r}{R_r}, \qquad \sigma = 1 - \frac{M^2}{L_S L_r},$$
$$K = \frac{M}{\sigma L_S L_r}, \qquad \gamma = \frac{R_S}{\sigma L_S} + \frac{R_r M^2}{\sigma L_S L_r^2},$$

where R_S, R_r and L_S, L_r are stator and rotor per-phase resistances and inductances, respectively, and J is the rotor moment inertia. The numerical values of the above parameters are as follows:

$$\begin{aligned}
&R_S = 0.18\Omega, \quad R_r = 0.15\Omega, \quad M = 0.068H,\\
&L_S = 0.0699H, \quad L_r = 0.0699H, \quad J = 0.0586kgm^2,\\
&T_L = 10Nm, \quad p = 1, \quad h = 0.0001s.
\end{aligned}$$

The input signals and initial conditions are:

$$u_{1,k} = 350\cos(0.03k), \; u_{2,k} = 300\sin(0.03k), \qquad (55)$$
$$\boldsymbol{x}_0 = [0,0,0,0,0]^T \qquad (56)$$

The unknown input for the system (49)–(53) is defined as:

$$\boldsymbol{E} = [1.2, 0.2, 2.4, 1, 1.6]^T, \qquad (57)$$
$$\boldsymbol{d}_k = 0.3\sin(0.5\pi k)\cos(0.03\pi k), \qquad (58)$$

while the noise covariance matrices are $\boldsymbol{Q} = 10^{-5}\boldsymbol{I}$ and $\boldsymbol{R} = 10^{-5}\boldsymbol{I}$, respectively. Note that, the small values of the process and measurement noise are selected in order to clearly portray the effect of an unknown input.

Figures 1 and 2 show the residual $\boldsymbol{z}_k$ for the conventional UKF without unknown input decoupling. From these results, it is evident that the estimation quality is very low and hence

the residual is significantly different from zero, which may lead to the decrease of the fault detection abilities. After

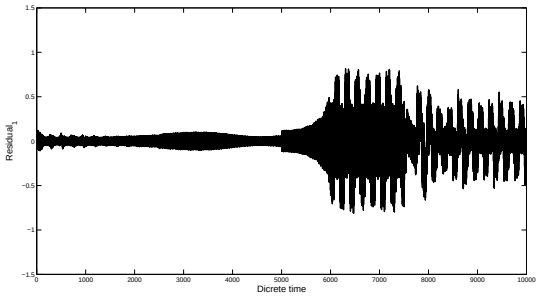

Fig. 1. Residual $z_{1,k}$ for the conventional UKF

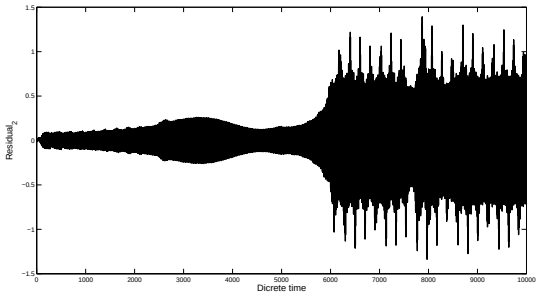

Fig. 2. Residual $z_{2,k}$ for the conventional UKF

the application of the ARS-based algorithm the following estimate was obtained

$$\hat{\boldsymbol{E}} = [0.3651, 0.0609, 0.7303, 0.3043, 0.4869]^T, \qquad (59)$$

Moreover, let us consider an abrupt actuator fault

$$f_k = \begin{cases} 0, & 500 < k < 140, \\ -0.2u_{1,k}, & \text{otherwise.} \end{cases}, \qquad (60)$$

and $\boldsymbol{L} = [1/(\sigma L_s), 0, 0, 0, 0]^T$. It can be easily checked that $\hat{\boldsymbol{E}}$ satisfies (25). Figures 3 and 4 show the residual z_k for the proposed UUIF with unknown input decoupling. From the presented results, it is evident that the estimation quality is very high and hence the residual is almost at the zero level for the fault-free case.

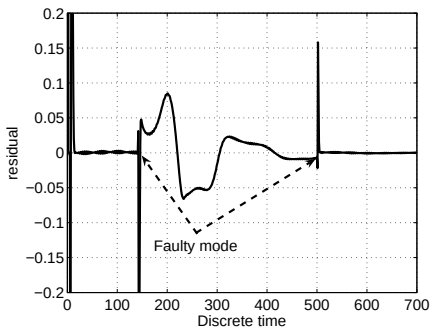

Fig. 3. Residual $z_{1,k}$ for UUIF

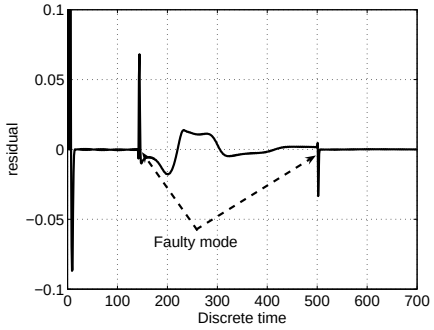

Fig. 4. Residual $z_{2,k}$ for the UUIF

VII. Conclusions

The main objective of this paper was to propose a novel approach that can be used for the estimation of an unknown input distribution matrix for non-linear discrete-time stochastic systems. In particular, based on the UKF an unknown input unscented Kalman filter was proposed for which a fault decoupling condition was developed. Subsequently, an optimisation-based approach for estimating an unknown input distribution matrix without a fault decoupling effect was designed. The proposed algorithm was illustrated with an example regarding an induction motor, which confirms the effectiveness of the proposed approach.

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