Preprints of the 2012 20th Mediterranean Conference on Control & Automation (MED), Barcelona, Spain, July 3-6, 2012

Control, simulation and stability analysis of nonlinear regular proper DAEs

Daniel Labisch, Maximilian Manderla, Ulrich Konigorski

***Abstract*— In this paper we provide an algorithm for nonlinear regular proper DAEs to derive a controller for an associated state space system assuming the algebraic variables and the error of the algebraic equations as fictitious inputs and outputs, respectively. It is shown that the achieved controlled state space representation has the same solution as the original DAE in the case of consistent initial values. Therefore, it can be used for stability analysis, designing control laws, and the numerical integration with standard methods. In contrast to the realization in minimal coordinates this formulation can be derived straightforward and the computation is easy to implement on a computer algebra system.**

I. Introduction

Modeling complex systems naturally and intuitively leads to differential algebraic equations (DAEs) which are also depicted as descriptor systems, irrelevant whether these systems are mechanical, chemical, or economical (see e.g. [7], [9], [13]). DAEs that arise by modeling realizable systems are always regular and proper. A conventional way for the simulation, controller design, and stability analysis for this class of systems is to develop a state space representation in minimal coordinates and use this system for the individual purposes. Unfortunately, this transformation is very involved and, in many cases, increases the complexity of the system vector fields.

In contrast, another possibility is to use the system in the differential algebraic structure for these tasks. Although the simulation of low index DAEs is common, it becomes very challenging and costly for higher index DAEs [2], [5]. Even though in the last decades various design methods for the control of DAEs have been developed, for state space representations there exist many more, of course. Also the stability analysis for state space systems is further developed compared to DAEs [13].

The purpose of this paper is to derive a state space representation which is not in minimal coordinates but is much easier to obtain. The DAE is associated to a fictitious state space system with fictitious in- and output. For this state space system a coupling controller [8] is designed. It is shown that in the case of consistent initial conditions the solutions of a regular and proper DAE and the controlled fictitious state space system are the same. Therefore, a state space representation is derived that can be integrated by standard integration methods. Additionally, the design of this state space system is much less involving than the computation of the representation in minimal coordinates. Although the purpose of the work carried out in [15] is related to ours, their requirements are much more demanding than the conditions in our approach.

The method presented in [10] and [11] for the simulation of DAEs is the basis for our approach. In [11] Manderla et al. derived the state space representation for the DAE of a parallel robotic manipulator. Necessary conditions for the success of the method are derived during the algorithm, but properties of a general DAE necessary for the success of the method are not stated. With a new notion of the regularity for DAEs we can additionally derive necessary and sufficient conditions for the success of a slightly modified method.

This paper is organized as follows. In section II the problem to be solved is introduced and additionally some basic definitions are stated. With the new notion of regularity derived in section III, and the requirements for the state space representation stated in section IV, together with the algorithm for designing the fictitious control law in section V, we will state our main results in section VI. We exemplify our approach with a DAE modeling of a parallel robotic manipulator in section VII and, finally, terminate this paper with the conclusion in section VIII.

II. Problem Statement and Preliminaries

We will consider nonlinear descriptor systems

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, u) \\ 0 &= g(x_1, x_2, u) \\ y &= h(x_1, x_2, u) \end{aligned} \tag{1}$$

in semi-explicit form (see [4]). $x_1 \in \mathcal{M}_{x_1}$ and $x_2 \in \mathbb{R}^m$ denote the differential and the algebraic variables, respectively. Together they form the descriptor variables $x = (x_1, x_2) \in \mathcal{M}_x$. $\mathcal{M}_{x_1}$ and $\mathcal{M}_x$ are smooth r and n-dimensional manifolds with $n = r + m$. Additionally, $u \in \mathbb{R}^p$ is the input and $y \in \mathbb{R}^q$ the output of the system. The vector field $f : (\mathcal{M}_{x_1} \times \mathbb{R}^m \times \mathbb{R}^p) \to T_{x_1}\mathcal{M}_{x_1}$

The authors gratefully thank the Deutsche Forschungsgemeinschaft (DFG) for funding this work by the project with the reference number Ko 1876/11-2.

D. Labisch is with the Department of Control Engineering and Mechatronics, Technische Universität Darmstadt, 64283 Darmstadt, Germany dlabisch@iat.tu-darmstadt.de

M. Manderla is with the Voith Hydro Holding GmbH & Co. KG, 89522 Heidenheim, Germany Maximilian.Manderla@voith.com

U. Konigorski is head of the Department of Control Engineering and Mechatronics, Technische Universität Darmstadt, 64283 Darmstadt, Germany ukonigorski@iat.tu-darmstadt.de

978-1-4673-2529-5/12/$31.00 ©2012 IEEE

and the maps $g : (\mathcal{M}_{\mathrm{x}_1} \times \mathbb{R}^m \times \mathbb{R}^p) \to \mathbb{R}^m$ and $h : (\mathcal{M}_{\mathrm{x}_1} \times \mathbb{R}^m \times \mathbb{R}^p) \to \mathbb{R}^q$ are assumed to be sufficiently smooth. $T_{x_1}\mathcal{M}_{\mathrm{x}_1}$ denotes the tangent space of $\mathcal{M}_{\mathrm{x}_1}$ at the point x_1.

The modeling procedure for many systems leads to descriptor systems in semi-explicit form. Moreover, a general implicit DAE

$$0 = F(\dot{x}, x, u) \tag{2}$$

with $F : (T\mathcal{M}_{\mathrm{x}} \times \mathbb{R}^p) \to \mathbb{R}^n$, where $T\mathcal{M}_{\mathrm{x}}$ denotes the tangent bundle of $\mathcal{M}_{\mathrm{x}}$, can always be transformed in semi-explicit form via

$$\begin{aligned} \dot{x} &= w \\ 0 &= F(w, x, u) \end{aligned} \tag{3}$$

with $w \in \mathbb{R}^n$ (see [4]).

For nonlinear DAEs different definitions for regularity exist in the literature. Actually, some are not consistent with the well-known definition in the linear case (see [3]). In this paper we will consider regularity definitions which are consistent with the linear case, such as [12], [16]. In the sense of [12] the following definitions are used.

Definition 2.1: [12] A function $x : \mathbb{I} \to \mathcal{M}_{\mathrm{x}}$ is called a *solution* of system (1) if $x \in \mathcal{C}^1(\mathbb{I}, \mathcal{M}_{\mathrm{x}})$ and x satisfies (1) pointwise for a given input function u. It is called a solution of the initial value problem consisting of (1) and

$$x(t_0) = \hat{x} \tag{4}$$

if x is a solution of (1) and satisfies (4). An initial condition (4) is called consistent if the corresponding initial value problem has at least one solution.

$\mathcal{C}^\kappa(\mathbb{I}, \mathcal{M})$ denotes the set of all κ-times continuously differential maps from $\mathbb{I} \subset \mathbb{R}$ to $\mathcal{M}$.

Definition 2.2: [12] A system (1) is called *regular* in a neighborhood of $(\hat{x}, \hat{u})$ satisfying (1) if it has a unique solution for every sufficiently smooth input function u and every initial value in a neighborhood of $(\hat{x}, \hat{u})$ that is consistent.

Definition 2.3: A regular system (1) is called *proper* if it has a unique solution for every input function $u \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^p)$ and every initial value in a neighborhood of $(\hat{x}, \hat{u})$ that is consistent for the system. Otherwise it is called *improper*. If the unique solution already exists for every input function $u \in \mathcal{C}^0(\mathbb{I}, \mathbb{R}^p)$, it is called *strictly proper*.

Additionally to the definitions for DAEs, some definitions for state space systems are required.

Definition 2.4: [1] A state space system

$$\begin{aligned} \dot{\xi} &= a(\xi, \omega) \\ \phi &= c(\xi, \omega) \end{aligned} \tag{5}$$

with the state $\xi \in \mathcal{M}_\xi$ on a smooth manifold $\mathcal{M}_\xi$, the input $\omega \in \mathbb{R}^p$, and the output $\phi \in \mathbb{R}^p$, where $a : (\mathcal{M}_\xi \times \mathbb{R}^p) \to T_\xi\mathcal{M}_\xi$ and $c : (\mathcal{M}_\xi \times \mathbb{R}^p) \to \mathbb{R}^p$ are sufficiently smooth, is called *invertible* in a neighborhood of $(\hat{\xi}, \hat{\omega})$ if the system has rank p in this neighborhood.

For the definition of the rank of a state space system see [1], [14].

It is possible to locally determine the unique input function $\omega(t)$ from the knowledge of the initial value $\hat{\xi}$ at time t_0 and the output function $\phi(t)$ if and only if a systems is invertible [17, condition (NL4)].

Definition 2.5: A *regular static feedback* for a state space system (5) with rank p in a neighborhood of $(\hat{\xi}, \hat{\omega})$ is of the form

$$\omega = r(\xi, v),$$

$v \in \mathbb{R}^p$, such that the rank of the controlled system is again p in a neighborhood of $(\hat{\xi}, \hat{v})$ with $\hat{\omega} = r(\hat{\xi}, \hat{v})$.

This definition directly leads to

Lemma 2.6: The invertibility of a state space system (5) is invariant under regular static feedback.

At this point we want to extend some definitions for state space systems stated above to state space representations with time-variant parameters.

Definition 2.7: A system

$$\begin{aligned} \dot{\xi} &= a(\xi, \omega, \zeta) \\ \phi &= c(\xi, \omega, \zeta) \end{aligned} \tag{6}$$

with state $\xi \in \mathcal{M}_\xi$, input $\omega \in \mathbb{R}^p$, output $\phi \in \mathbb{R}^p$, and the time-variant parameter vector $\zeta \in \mathbb{R}^m$, where the vector field $a : (\mathcal{M}_\xi \times \mathbb{R}^p \times \mathbb{R}^m) \to T_\xi\mathcal{M}_\xi$ and the map $c : (\mathcal{M}_\xi \times \mathbb{R}^p \times \mathbb{R}^m) \to \mathbb{R}^p$ are sufficiently smooth, is called *invertible* in a neighborhood of $(\hat{\xi}, \hat{\omega}, \hat{\zeta})$ with respect to $\langle \phi, \omega \rangle$ if it has rank p in this neighborhood for all sufficiently smooth $\zeta \in \mathcal{C}^\kappa(\mathbb{I}, U \subset \mathbb{R}^m)$ with appropriate κ.

Finally, we need the notion of an attractive submanifold.

Definition 2.8: [8] A submanifold $\mathcal{N}$ of $\mathcal{M}_\xi$ defined by $\mathcal{N} = \{\xi \in \mathcal{M}_\xi \mid \gamma(\xi) = 0\}$ is called *locally attractive* for system (5) in a neighborhood U of $(\hat{\xi}, \hat{\omega})$ with $\gamma(\hat{\xi}) = 0$ if $\lim_{t\to\infty} \gamma(\xi(t)) = 0$ holds for for every $(\xi(t_0), \omega(t)) \in U$, $t \geq t_0$.

The problem to be solved in this paper is finding a state space representation of a regular and proper DAE (1) which features the same solutions for consistent initial values. As the derivation of the state space representation in minimal coordinates is very involving, we are not interested in reducing the dynamical order of the system, i.e. we are seeking for a state space representation of order r. If we are successful, as the solutions coincide, controller design tasks and stability analysis can be transferred to the state space system.

Using the achieved system also for simulation tasks additional requirements must hold. Obviously, the coinciding solutions will evolve on a submanifold of the state manifold of the state space system. Because of numerical errors in every integration method the invariance of the submanifold will not be sufficient for this task. Additionally, the attractiveness of the submanifold for the state space system is required.

For solving this problem, a new notion of regularity of DAEs will be presented in the next section.

III. Regularity

We now consider the fictitious state space system

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, u) \\ z &= g(x_1, x_2, u) \end{aligned} \tag{7}$$

with the vector field f and the map g of (1), the r-dimensional state $x_1 \in \mathcal{M}_{\mathrm{x}_1}$, the fictitious input $x_2 \in \mathbb{R}^m$, the time-variant parameter vector $u \in \mathbb{R}^p$, and the fictitious output $z \in \mathbb{R}^m$. Note that the fictitious in- and output of (7) are different from the in- and output of (1).

Together with the definition of invertibility the regularity of (1) can be checked with

Proposition 3.1: Suppose $(\hat{x}_1, \hat{x}_2)$ is a consistent initial value for the DAE (1). The DAE (1) is regular in a neighborhood of $(\hat{x}_1, \hat{x}_2, \hat{u})$ if and only if (7) is invertible with respect to $\langle z, x_2 \rangle$ in a neighborhood of $(\hat{x}_1, \hat{x}_2, \hat{u})$.

Proof: Sufficiency: If $\varphi(t) = (\varphi_{x_1}(t), \varphi_{x_2}(t), \varphi_u(t))$ is a function with $x_1 = \varphi_{x_1}(t)$, $x_2 = \varphi_{x_2}(t)$, $u = \varphi_u(t)$, and $\hat{x} = (\varphi_{x_1}(t_0), \varphi_{x_2}(t_0))$ pointwise fulfilling (1), then $\varphi(t)$ also pointwise fulfills (7) with $z = 0$. Hence, the regularity of the DAE (1) is equivalent to the possibility of finding a unique function $\varphi_{x_2}(t)$ for a given initial value $\hat{x}$ and an arbitrary sufficiently smooth input function $\varphi_u(t)$. Then $\varphi_{x_1}(t)$ is uniquely specified with (7). Because (7) is invertible with respect to $\langle z, x_2 \rangle$, $\varphi_{x_2}(t)$ is uniquely specified by $z(t) = 0$ and $\hat{x}$. Therefore, the DAE (1) is regular.

Necessity: If (1) is regular, there exists a unique $\varphi(t)$ pointwise fulfilling (1) with $u = \varphi_u(t)$ and $\hat{x} = (\varphi_{x_1}(t_0), \varphi_{x_2}(t_0))$. Suppose (7) is not invertible with respect to $\langle z, x_2 \rangle$. Hence, a $\tilde{\varphi}(t)$ exists, such that $\tilde{\varphi}_{x_2}(t) \neq \varphi_{x_2}(t)$, $\tilde{\varphi}_{x_2}(t_0) = \varphi_{x_2}(t_0)$, $\tilde{\varphi}_{x_1}(t_0) = \varphi_{x_1}(t_0)$, and $\tilde{\varphi}_u(t) = \varphi_u(t)$ pointwise fulfill (7). Therefore, also $\tilde{\varphi}(t)$ pointwise fulfills (1) and therefore (1) is not regular. ■

This proposition will be used later to prove our main results. Prior to this, in the following section we will derive requirements for a state space system to have the same solution as the DAE (1). These requirements will directly lead to the algorithm stated in section V.

IV. Requirements

Let $\varphi_u(t)$ be the input and $(\hat{x}_1, \hat{x}_2)$ a consistent initial condition for the regular DAE (1). If we can find a fictitious control law

$$x_2 = \alpha(x_1, u), \tag{8}$$

such that $z(t) = 0$, $t \geq t_0$ for (7) with the same initial value $\hat{x}_1$ and $u = \varphi_u$, the solutions of (1) and (7) will be identical.

In the next section an algorithm deriving (8) is provided. Together with the necessary and sufficient conditions for the success of the algorithm, also the existence of (8) will be shown. However, at this point we only assume that a fictitious control law (8) exists.

All solutions of (7) controlled by (8) with arbitrary consistent initial values for (1) and arbitrary inputs will evolve locally on a submanifold $\mathcal{N}$ of the state manifold $\mathcal{M}_{\mathrm{x}_1}$. This submanifold has the following properties

- output-nulling:

$$0 = g\big(x_1, \alpha(x_1, u), u\big)$$

- controlled invariance:

$$f(x_1, \alpha(x_1, u), u) \in T_{x_1}\mathcal{N}$$

for all $x_1 \in U_1 \subset \mathcal{N}, u \in U_2 \subset \mathbb{R}^p$ with some neighborhoods U_1 and U_2.

Comparing these properties with the constrained dynamics algorithm of Nijmeijer and van der Schaft [14], $\mathcal{N}$ is the maximal locally controlled invariant output-nulling submanifold for (7).

If the controlled system is numerically integrated, even with high accuracy, $f(x_1, \alpha(x_1, u), u)$ will not exactly lie in $T_{x_1}\mathcal{N}$ because of numerical errors. As $\mathcal{N}$ will not be a locally attractive submanifold in general, the solution of the state space system with any standard integration method will leave $\mathcal{N}$ noticeably and is therefore unfeasible. But if we can design (8) such that $\mathcal{N}$ is locally attractive for the controlled system, solutions will always lie on $\mathcal{N}$ except for small numerical errors.

In [8] the authors present a method for deriving an attractive maximal locally controlled invariant output-nulling submanifold and the feedback required for input-affine nonlinear state space systems. In the next section this algorithm is applied to (7) considering the non-affine structure of (7) and the time-variant parameter vector u.

V. Algorithm

We will use the abbreviation

$$\partial_{x_1} = \frac{\partial}{\partial x_1}$$

for the partial derivative in the coordinates x_1 and

$$L_f z = (\partial_{x_1} z)\, f$$

for the Lie derivative of a map z in the direction of a vector field f.

Step 0

Set $\mathcal{M}_0 = \mathcal{M}_{\mathrm{x}_1}$, $m_0 = m$, $m_1 = m - 1$, …, $m_m = 0$, and $i = 0$. Denote the system (7) by Σ_0 and set $z_0 = g(x_1, x_2, u)$. The elements of z_0 are denoted by $z_0^1, \ldots, z_0^m$.

Step 1

If $\partial_{x_2} z_i^1 = 0$ holds, increase i by 1, define the new constraint $z_i^1 = L_f z_{i-1}^1$, set $\gamma_1 = \big(z_0^1, \ldots, z_{i-1}^1\big)$, and assume that the rank of γ_1 with respect to x_1 is constant around $(\hat{x}_1, \hat{u})$ and

$$\operatorname{rank}(\gamma_1) = i. \tag{9}$$

978-1-4673-2529-5/12/$31.00 ©2012 IEEE

Step 2

Repeat step 1 until $\partial_{x_2} z_i^1 \neq 0$. Set $\delta_1 = i$ and $\mathcal{M}_1 = \{x_1 \in \mathcal{M}_0 \mid \gamma_1 = 0\}$. Choose parameters ν_j^1, such that the roots of the polynomial

$$P^1(s) = \sum_{j=0}^{\delta_1} \nu_j^1 \cdot (s)^j \tag{10}$$

are all in $\mathbb{C}^-$. It is possible to find a control law $x_2 = \alpha_1(x_1, u, x_{21})$ with the m_0-dimensional vector field α_1 and $x_{21} \in \mathbb{R}^{m_1}$, in order to achieve

$$\sum_{j=0}^{\delta_1} \nu_j^1 z_j^1(x_1, u, x_2)\Big|_{x_2 = \alpha_1(x_1, u, x_{21})} = 0$$

locally on $\mathcal{M}_1$. Denote the system Σ_0 controlled by $x_2 = \alpha_1(x_1, u, x_{21})$ by Σ_1.

In the following step we proceed with the second output which is denoted by $k = 2$.

Step 3

Let $\dot{x}_1 = \tilde{f}(x_1, x_{2(k-1)}, u)$ with $x_{2(k-1)} \in \mathbb{R}^{m_{k-1}}$ be the description of the system Σ_{k-1} and set $i = 0$.

Step 4

If $\partial_{x_{2(k-1)}} z_i^k = 0$ holds, increase i by 1, define the new constraint $z_i^k = L_{\tilde{f}} z_{i-1}^k$, set $\gamma_k = \left(z_0^k, \ldots, z_{i-1}^k\right)$ and assume that the rank of γ_k is constant with respect to x_1 around $(\hat{x}_1, \hat{u})$ and

$$\operatorname{rank}(\gamma_1, \ldots, \gamma_k) = i + \sum_{j=1}^{k-1} \delta_j. \tag{11}$$

Step 5

Repeat step 4 until $\partial_{x_2} z_i^k \neq 0$. Set $\delta_k = i$ and $\mathcal{M}_k = \{x_1 \in \mathcal{M}_{k-1} \mid \gamma_k = 0\}$. Choose parameters ν_j^k to ensure that the roots of the polynomial

$$P^k(s) = \sum_{j=0}^{\delta_k} \nu_j^k \cdot (s)^j \tag{12}$$

are all in $\mathbb{C}^-$. It is possible to find a control law $x_{2(k-1)} = \alpha_k(x_1, u, x_{2k})$ with the m_{k-1}-dimensional vector field α_k and $x_{2k} \in \mathbb{R}^{m_k}$ in order to achieve

$$\sum_{j=0}^{\delta_k} \nu_j^k z_j^k(x_1, u, x_{2(k-1)})\Big|_{x_{2(k-1)} = \alpha_k(x_1, u, x_{2k})} = 0$$

locally on $\mathcal{M}_k$. Denote the system Σ_{k-1} controlled by $x_{2(k-1)} = \alpha_k(x_1, u, x_{2k})$ by Σ_k.

Step 6

Increase k by 1 and repeat steps 3 – 6 until $k = m$.

VI. Main Results

If the constant rank assumptions at step 1 and 4 are satisfied, $(\hat{x}_1, \hat{u})$ is a regular point for this algorithm.

If (9) or (11) are not fulfilled in the k-th step for any i, the k-th output z_0^k is not influenced by any of the remaining inputs $x_{2(k-1)}$. In this case the algorithm is not successful.

If the algorithm is successful, it leads to the manifold $\mathcal{M}_m$, the smooth control law

$$x_2 = \alpha_1(x_1, u, \alpha_2(x_1, u, \ldots)) = \alpha(x_1, u), \tag{13}$$

and the system Σ_m

$$\begin{aligned} \dot{x}_1 &= f(x_1, \alpha(x_1, u), u) = \tilde{f}(x_1, u) \\ y &= h(x_1, \alpha(x_1, u), u) = \tilde{h}(x_1, u). \end{aligned} \tag{14}$$

Note that the choice of the feedback in this algorithm is always suitable for the constrained dynamics algorithm in [14]. As shown there, the particular choice of the feedback has no influence on the success of the algorithm and the computed manifold $\mathcal{M}_m$.

The following theorems state necessary and sufficient conditions for the success of the presented algorithm and the connection to the DAE (1) via their solutions.

Theorem 6.1: The algorithm terminates successfully if and only if the DAE (1) is regular and proper.

Proof: Sufficiency: As the DAE is proper, $\partial_u z_i^k = 0, \forall i < \delta_k$ holds and therefore $L_{\tilde{f}} z_i^k = \dot{z}_i^k$. Hence, z_i^k denotes the i-th derivative of the k-th output of (7). Because the DAE is regular, (7) is invertible with respect to $\langle z, x_2 \rangle$ and therefore all z_i^k, $i < \delta_k$ are independent (see [17]). Hence, the rank assumptions during the algorithm are fulfilled.

Necessity: Suppose the algorithm is successful. Then, a function α in (13) is derived which is unique on $\mathcal{M}_\mathrm{m}$. Therefore, with the knowledge of $\hat{x}_1$, $\hat{u}$, and $z = 0$ it is possible to determine x_2 uniquely. Hence, (7) is invertible with respect to $\langle z, x_2 \rangle$ which is equivalent to the regularity of (1) because of proposition 3.1. ■

Theorem 6.2: Let $u(t)$ be the input and $(\hat{x}_1, \hat{x}_2)$ a consistent initial condition for the system (1). With the initial value $\hat{x}_1$ and the input $u(t)$ the solution of (14) is equal to the solution of (1) if and only if the system (1) is regular and proper.

Proof: Sufficiency: Let (1) be regular and proper. Because of theorem 6.1 the state space system (14) can be derived. Clearly, $\hat{x}_1$ lies on $\mathcal{M}_m$. As $\mathcal{M}_m$ is invariant for system (14) (see [8]), $z(t) = 0 \; \forall t \geq 0$ holds. As (7) is invertible (see proposition 3.1) (13) computes the unique x_2 satisfying $z = 0$. Hence, the solutions of (1) and (14) coincide.

Necessity: Suppose the solutions coincide. If (1) is not regular, (1) has more than one or no solution. As $\tilde{f}$ is smooth, (14) has a unique solution and, therefore, not all solutions coincide. If (1) is regular but not proper, $L_{\tilde{f}} z_{i-1}^k = \dot{z}_{i-1}^k$ does not hold for all $1 \leq i < \delta_k$, $1 \leq k \leq m$. Furthermore, as $\exists i < \delta_k$: $\partial_u z_{i-1}^k \neq 0$ it follows that in

(13) α has to depend on derivatives of u if $\mathcal{M}_m$ should be controlled invariant for (7). Therefore, no state space representation in the form of (14) with the property $z = 0$ for all u could be found. ■

As seen in the last proof, $\mathcal{M}_m$ fulfills the requirements described in section IV and, therefore,

$$\mathcal{N} = \mathcal{M}_m \tag{15}$$

is set.

At this point, we stated necessary and sufficient conditions for the successful termination of the algorithm and the possibility of deriving a state space representation (14) of (1) with equal solutions. Comparing this result with the literature, there are two main differences.

First, in contrast to [10] and [11], our necessary and sufficient conditions are stated as properties of the original DAE, i.e. the DAE is regular and proper. Therefore, these basic properties are actually known before applying the algorithm. Second, the pole placement procedure inducing the attractiveness of $\mathcal{N}$ described in [15] is only valid if (7) has relative degree (see [6]) for all $u \in U \subset \mathbb{R}^p$. Obviously, requiring invertibility via proposition 3.1 in our new approach is less restrictive.

Just like the approach given in [11] the presented algorithm consists of simple derivations and rank condition tests which are easy to implement on a computer algebra system.

As the solutions of both systems, the DAE (1) and the state space system (14), are equal under the stated conditions, a controller designed for (14) satisfying a specified behavior can directly be used for (1) satisfying the same requirements. Therefore, both controlled systems provide the same input-output behavior.

Furthermore, (1) and (14) locally have the same equilibria on $\mathcal{N}$ and also the stability properties of these equilibria coincide. This directly leads to

Proposition 6.3: Let (1) be regular and proper. $(\hat{x}_1, \hat{x}_2)$ is an asymptotic stable equilibrium for (1) if and only if $(\hat{x}_1, \hat{x}_2)$ is consistent and $\hat{x}_1$ is an asymptotic stable equilibrium for (14).

Proof: The proposition is a direct consequence of theorem 6.2. ■

Moreover, (14) can be used for a numerical simulation of the DAE (1) with common integration methods for ordinary differential equations. As stated in section IV the solution will always remain in $\mathcal{N}$ because of the attractiveness of $\mathcal{N}$ for (14).

Summing up the last results we have that the state space representation (14) can be used for

- controller design,
- stability analysis, and
- numerical integration

for the DAE (1).

Remark 6.4: All results stated for the DAE (1) can also be applied to the fully implicit DAE (2). The requirement of a regular proper DAE (1) has to be changed to a regular, strictly proper DAE (2). In this manner, the requirements are not necessary any more, but they are still sufficient.

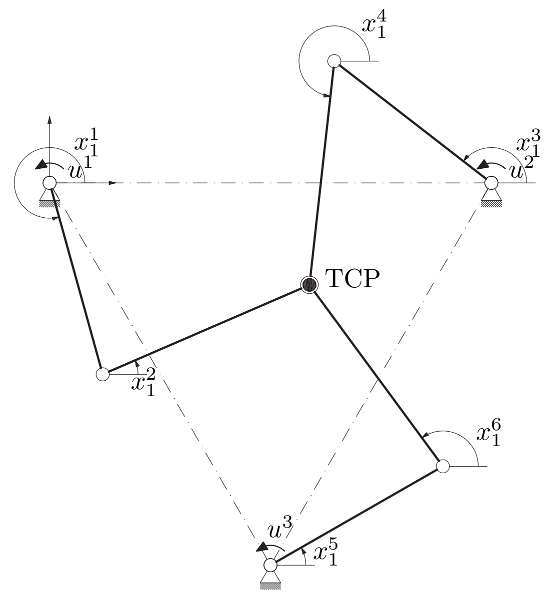

Fig. 1. Parallel robot

VII. Example

To show the feasibility of the presented approach the same parallel robot as described in [11] is considered, which is presented in Fig. 1. The robot consists of three arms with two links and legs each. The first link is actuated in every arm, the second is a passive joint. The ends of the second legs are connected with a further passive joint corresponding to the tool center point (TCP). Every link turns around a vertical axis, hence, the tool center point moves in the horizontal plane.

As in [11] the equations of motion are derived via decomposing the system into three subsystems consisting of one arm each. The internal forces at the end of each arm are regarded as external forces in the subsequent Lagrangian modeling processes. Appending the algebraic constraints, i.e. the positions of the ends of the second legs have to be the same, a differential algebraic system description (1) can be derived. Therein, x_1 denotes the angles and angular speeds of the six links, x_2 contains the internal forces, and u consists of the torques of the actuated joints.

This modeling procedure is very descriptive and easy to implement on a computer algebra system. For the simulation using standard integration methods the proposed coupling controller (13) is designed. This method is also easy to realize on a computer algebra system. As stated in the previous section, the system has to be initialized with consistent initial values for deriving the solution of the differential algebraic system.

As the submanifold $\mathcal{N}$ is attractive for (14), trajectories with non-consistent initial values will asymptotically

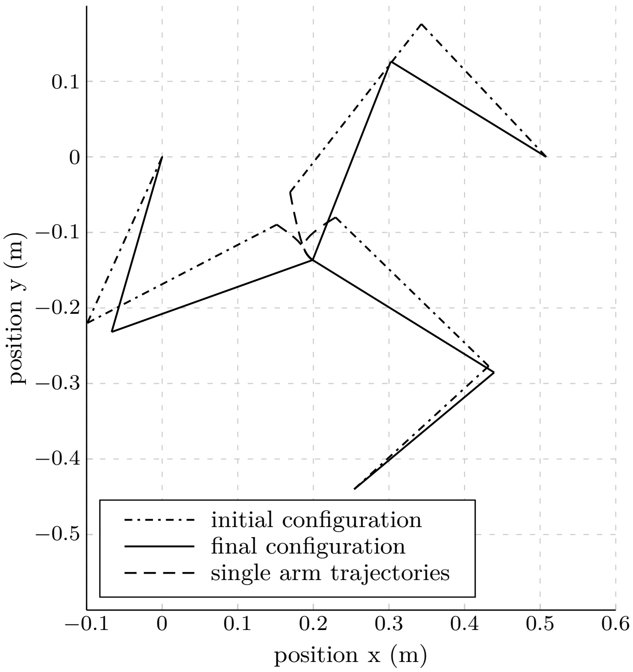

Fig. 2. Simulation of the non-consistent coupled system

tend to $\mathcal{N}$. Hence, if no consistent initial values are known in advance, non-consistent initial values could be used to derive consistent initial values, as shown in Fig. 2. The arms will meet asymptotically and a consistent initial value could be found with little effort.

For further control purposes the position of the TCP and the sum of the input torques are assumed as the output vector of the system. Therefore, a decoupling control law for the input-output behavior of (14) is designed. Because of theorem 6.1 this control law also decouples the behavior of (1). Using the same feedback and feedforward design as described in [11], Fig. 3 shows the measurement of a trajectory driven on a real test stand of the parallel robot.

The deviation from the reference trajectory is mainly caused by the high friction and backlash in the gearbox of the actuated joints.

VIII. Conclusion

We presented a constructive algorithm for deriving a coupling control law for a fictitious state space representation of a regular and proper DAE. It was shown, that this controlled system locally has the same solutions as the original DAE in case of consistent initial values. Consequently, the controlled system can be used for controller design tasks, numerical integration of the DAE, and, in addition, also for the investigation of the stability of equilibria. It was proven that the DAE being regular and proper is necessary and sufficient for the success of our approach. Finally, we showed the feasibility of our approach via simulation and measurement of a parallel robot test stand.

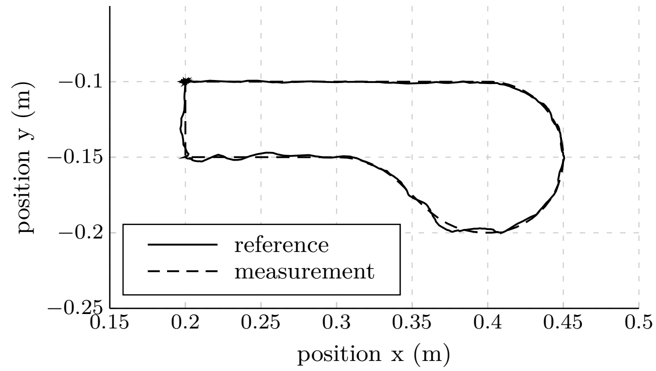

Fig. 3. Measurement of trajectory

References

[1] M. D. Di Benedetto, J. W. Grizzle, and C. H. Moog. Rank invariants of nonlinear systems. 27(3):658–672, 1989.

[2] K.E. Brenan, S.L. Campbell, and L.R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Classics in applied mathematics. Society for Industrial and Applied Mathematics, 1996.

[3] L. Dai. *Singular Control Systems*. Springer, Berlin, Heidelberg, 1989.

[4] C.W. Gear. Differential-algebraic equation index transformations. *Siam Journal on Scientific and Statistical Computing*, 9:39–47, 1988.

[5] E. Hairer, C. Lubich, and M. Roche. *The numerical solution of differential-algebraic systems by Runge-Kutta methods*. Lecture notes in mathematics. Springer-Verlag, 1989.

[6] A. Isidori. *Nonlinear Control Systems*. Springer, London, 1995.

[7] A. Kumar and P. Daoutidis. *Control of nonlinear differential algebraic equation systems: with application to chemical processes*. Chapman & Hall/CRC research notes in mathematics series. Chapman & Hall/CRC, 1999.

[8] D. Labisch and U. Konigorski. Attractive invariant submanifold-based coupling controller design. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, December 12-15, 2011*, pages 7747–7752, Orlando, Florida, 2011.

[9] D. Luenberger. Non-linear descriptor systems. *Journal of Economic Dynamics and Control*, 1:219–242, 1979.

[10] M. Manderla. *Eine Methodik zum Regler- und Beobachterentwurf für Mehrgrößensysteme in Deskriptorform*. VDI-Fortschrittsberichte, Reihe 8, Nr. 1196, VDI-Verlag, Düsseldorf, Dissertation, Technische Universität Darmstadt, 2011.

[11] M. Manderla, D. Schmitt, and U. Konigorski. Modelling, simulation and control of a redundant parallel robotic manipulator based on invariant manifolds. *Mathematical and Computer Modelling of Dynamical Systems*, 16(2):95–113, 2010.

[12] V. Mehrmann and T. Stykel. Descriptor systems: A general mathematical framework for modelling, simulation and control. *at - Automatisierungstechnik*, 54:405–415, 2006.

[13] P.C. Müller. Stability and optimal control of nonlinear descriptor systems: A survey. *International Journal of Applied Mathematics and Computer Science*, 8:269–286, 1998.

[14] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, New York, 1990.

[15] P. S. Pereira da Silva, E. Delaleau, and I.S. de Souza. On geometric control and numeric integration of DAE's. In *Proceedings of the IFAC World Congress Praha*, 2005.

[16] S. Reich. On an existence and uniqueness theory for nonlinear differential-algebraic equations. *Circuits Systems Signal Process*, 10:343–359, 1991.

[17] W. Respondek. Right and left invertibility of nonlinear control systems. In H. J. Sussmann, editor, *Nonlinear Control and Optimal Control*, pages 133–176. Marcel Dekker, 1990.