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Stability Analysis on Manifolds via Continuous Positive Definite Proper Functions

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***Abstract*— In this paper, we define Lagrange stability and input-to-state stability on manifolds. Furthermore, we present necessary and sufficient conditions for stability by using continuous positive definite proper functions.**

I. INTRODUCTION

Stability is defined by Euclidean norms in Euclidean spaces [1]. Although some definitions are extended to more general topological spaces [2], some are not.

In Euclidean spaces, necessary and sufficient conditions for stability are provided by using Euclidean norms or continuous positive definite proper functions [1]. In metric spaces, the same conditions are obtained by using metrics or continuous positive definite proper functions [2]. However, dependence on metrics causes inconvenience.

In [3], stability on manifolds is analyzed by using sample-and-hold solutions. However, conditions for stability are not rendered by continuous positive definite proper functions.

In this paper, we define Lagrange stability and input-to-state stability on manifolds. Furthermore, we present necessary and sufficient conditions for stability by using continuous positive definite proper functions.

II. PRELIMINARIES

Let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_{\geq 0} := [0, +\infty)$, X and U be connected C^1 manifolds without boundaries, $0 \in X$ and $0 \in U$ the origins, and $\mathcal{U}$ a set of essentially bounded functions $u : \mathbb{R}_{\geq 0} \to U$, i.e. there exist a constant $c \geq 0$ and a continuous positive definite proper function $V_U : U \to \mathbb{R}_{\geq 0}$ satisfying

$$V_U(u(t)) \leq c \quad \text{a.e.} \quad t \geq 0. \tag{1}$$

We consider the following dynamical system:

$$\dot{x} = f(x, u), \tag{2}$$

where $x \in X$ is a state, $u \in \mathcal{U}$ is an input, $f : X \times U \to TX$ is a mapping, and TX is the tangent bundle of X.

We assume that for each $x \in X$ and $u \in \mathcal{U}$, there exist an interval $[0, T)$ with $0 < T \leq +\infty$ and at least one local Carathéodory solution $\varphi : [0, T) \to X; t \mapsto \varphi(t; x, u)$ with $\varphi(0; x, u) = x$. For subsets $X' \subset X$ and $\mathcal{U}' \subset \mathcal{U}$, $\mathcal{S}(X', \mathcal{U}')$ denotes the set of all local Carathéodory solutions with initial states $x \in X'$ and inputs $u \in \mathcal{U}'$.

Substituting $u(t) = 0\ \forall t \geq 0$ into system (2), we obtain

$$\dot{x} = f(x, 0). \tag{3}$$

For autonomous system (3), we simply write $\varphi(t; x)$ or $\mathcal{S}(X)$ instead of $\varphi(t; x, 0)$ or $\mathcal{S}(X, 0)$.

Definition 1 *[1][2] The origin of system (3) is called Lyapunov stable if for each neighborhood W_1 of $0 \in X$ there exists a neighborhood W_2 of $0 \in X$ such that all $\varphi \in \mathcal{S}(W_2)$ are continuable on $\mathbb{R}_{\geq 0}$ and*

$$\varphi(t; x) \in W_1 \quad \forall \varphi \in \mathcal{S}(W_2)\ \forall t \geq 0. \tag{4}$$

□

Definition 2 *[1][2] The origin of system (3) is called globally asymptotically stable if it is Lyapunov stable, all $\varphi \in \mathcal{S}(X)$ are continuable on $\mathbb{R}_{\geq 0}$, and*

$$\lim_{t \to +\infty} \varphi(t; x) = 0 \quad \forall \varphi \in \mathcal{S}(X). \tag{5}$$

□

Remark 1 *If X has a boundary, the existence of a Lyapunov function does not guarantee global asymptotic stability because the solutions may go outside of X.* □

Remark 2 *Note that the choice of V_U causes no change in inputs.* □

Remark 3 *Note that continuity of f is not assumed. If f is continuous and the origin of system (3) is globally asymptotically stable, X is contractible [4].* □

Remark 4 *A local Carathéodory solution $\varphi(t; x, u)$ is continuable on $\mathbb{R}_{\geq 0}$ if all solutions $\bar{\varphi} \in \mathcal{S}(x, u)$ stay in a compact subset of X.* □

Definition 3 *[1] Let $0 < r_1 \leq +\infty$ be a constant. A continuous strictly increasing function $\alpha : [0, r_1) \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ is called class $\mathcal{K}_0$. A class $\mathcal{K}_0$ function defined on entire $\mathbb{R}_{\geq 0}$ is called class $\bar{\mathcal{K}}_0$.*

Let $0 \leq r_2 < +\infty$ be a constant. A continuous strictly increasing function $\alpha : [r_2, +\infty) \to \mathbb{R}_{\geq 0}$ with $\lim_{r \to +\infty} \alpha(r) = +\infty$ is called class $\mathcal{K}^\infty$. A class $\mathcal{K}^\infty$ function defined on entire $\mathbb{R}_{\geq 0}$ is called class $\bar{\mathcal{K}}^\infty$.

A continuous strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ and $\lim_{r \to +\infty} \alpha(r) = +\infty$ is called class $\mathcal{K}_0^\infty$; $\mathcal{K}_0^\infty = \bar{\mathcal{K}}_0 \cap \bar{\mathcal{K}}^\infty$.

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A continuous decreasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lim_{r \to +\infty} \alpha(r) = 0$ is called class $\mathcal{L}$.

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called class $\mathcal{LK}$ if it is class $\mathcal{L}$ with respect to the first variable and class $\bar{\mathcal{K}}_0$ with respect to the second one. □

III. LAGRANGE STABILITY

In this section, we define Lagrange stability for autonomous system (3).

In Euclidean spaces, Lagrange stability is defined [1]. In subspaces of Euclidean spaces, Lagrange stability for bounded sets is defined [5]. Although these definitions are equivalent in Euclidean spaces, Lagrange stability for bounded sets is not appropriate as the extension of Lagrange stability by the following reasons:

i) Lyapunov stability, Lagrange stability and asymptotic stability are invariant under diffeomorphic coordinate transformations. However, Lagrange stability does not hold under diffeomorphic ones.
ii) Global asymptotic stability implies Lagrange stability on Euclidean spaces [5]. However, it does not guarantee Lagrange stability for bounded sets on subspaces of Euclidean spaces [5].

We extend the definition of Lagrange stability to manifolds as follows:

Definition 4 *System (3) is called Lagrange stable if for each compact neighborhood $\bar{W}_1$ of $0 \in X$ there exists a compact neighborhood $\bar{W}_2$ of $0 \in X$ such that all $\varphi \in \mathcal{S}(\bar{W}_1)$ are continuable on $\mathbb{R}_{\geq 0}$ and*

$$\varphi(t;x) \in \bar{W}_2 \quad \forall \varphi \in \mathcal{S}(\bar{W}_1)\ \forall t \geq 0. \tag{6}$$

□

Remark 5 *In Euclidean spaces, Lagrange stability in Definition 4 coincides with the traditional Lagrange stability. Moreover, it is invariant under diffeomorphic coordinate transformations. Furthermore, global asymptotic stability implies the extended Lagrange stability as shown in the following section.* □

IV. NECESSARY AND SUFFICIENT CONDITIONS

In this section, we show necessary and sufficient conditions for stability of autonomous system (3) by using continuous positive definite proper functions.

Let $\operatorname{dom}\varphi$ denote the domain of φ. Then, we obtain the following propositions.

Proposition 1 *For system (3), the following are equivalent:*

i) the origin is Lyapunov stable;
ii) for a given continuous positive definite proper function $V_X : X \to \mathbb{R}_{\geq 0}$, there exist a positive constant $0 < r_1 \leq +\infty$ and a class $\mathcal{K}_0$ function $\alpha_1 : [0, r_1) \to \mathbb{R}_{\geq 0}$ satisfying

$$\begin{aligned} &V_X(\varphi(t;x)) \leq \alpha_1(V_X(x)) \\ &\qquad \forall \varphi \in \mathcal{S}(\{x \in X \mid V_X(x) < r_1\})\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{7}$$

□

Proposition 2 *For system (3), the following are equivalent:*

i) system (3) is Lagrange stable;
ii) for a given continuous positive definite proper function $V_X : X \to \mathbb{R}_{\geq 0}$, there exists a function $\alpha_2 \in \bar{\mathcal{K}}^\infty$ satisfying

$$V_X(\varphi(t;x)) \leq \alpha_2(V_X(x)) \quad \forall \varphi \in \mathcal{S}(X)\ \forall t \in \operatorname{dom}\varphi. \tag{8}$$

□

Proposition 3 *For system (3), the following are equivalent:*

i) the origin is Lyapunov stable and Lagrange stable;
ii) for a given continuous positive definite proper function $V_X : X \to \mathbb{R}_{\geq 0}$, there exists a function $\alpha_3 \in \mathcal{K}_0^\infty$ satisfying

$$V_X(\varphi(t;x)) \leq \alpha_3(V_X(x)) \quad \forall \varphi \in \mathcal{S}(X)\ \forall t \in \operatorname{dom}\varphi. \tag{9}$$

□

Proposition 4 *For system (3), the following are equivalent:*

i) the origin is globally asymptotically stable;
ii) the origin is globally asymptotically stable and Lagrange stable;
iii) for a given continuous positive definite proper function $V_X : X \to \mathbb{R}_{\geq 0}$, there exists a function $\beta \in \mathcal{LK}$ satisfying

$$V_X(\varphi(t;x)) \leq \beta(t, V_X(x)) \quad \forall \varphi \in \mathcal{S}(X)\ \forall t \in \operatorname{dom}\varphi. \tag{10}$$

□

All of the proofs are given in Section VI.

Remark 6 *If $X = \mathbb{R}^n$ and $V_X(x) = \|x\| := (\sum_{i=1}^n x_i^2)^{1/2}$, Propositions 1–4 coincide with results in [1].* □

Remark 7 *Lyapunov stability means stability in a neighborhood of the origin. This locality appears in the domain of α_1.* □

Remark 8 *Lagrange stability means stability far from the origin. This locality appears in the image of α_2; Im$(\alpha_2) = [r_2, +\infty)$ $(0 \leq r_2 < +\infty)$.* □

Remark 9 *Although every smooth σ-compact manifold can be smoothly embedded in $\mathbb{R}^n$ [6], Euclidean norm is not appropriate for evaluation of stability. For example, $X_1 := \{(x,y) \in \mathbb{R}^2 \mid (x+2)^2 + y^2 > 1\}$ can be embedded in $\mathbb{R}^2$. However, Euclidean norm is not proper on X_1 as shown in Figs. 1 and 2. On the contrary, Figs. 3 and 4 illustrate a continuous positive definite proper function on X_1.* □

Remark 10 *In [3], smooth σ-compact manifolds are embedded in Euclidean spaces as closed subsets. However, the embedded manifolds may not be smooth [6].* □

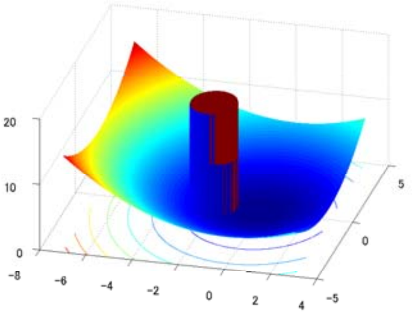

Fig. 1. Euclidean norm on X_1

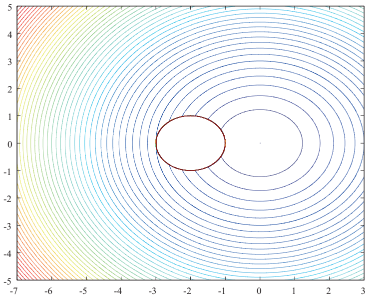

Fig. 2. Level sets of Euclidean norm on X_1

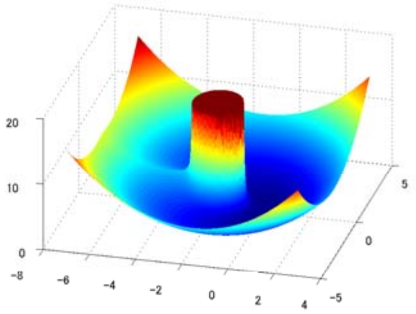

Fig. 3. Continuous positive definite proper function on X_1

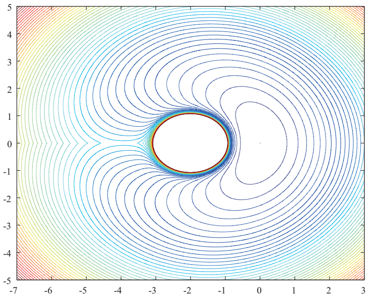

Fig. 4. Level sets of continuous positive definite proper function on X_1

Continuous positive definite proper functions on manifold X have the following relation:

Proposition 5 *For continuous positive definite proper functions $V_1, V_2 : X \to \mathbb{R}_{\geq 0}$, there exist functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_0^\infty$ satisfying*

$$\underline{\alpha}(V_1(x)) \leq V_2(x) \leq \overline{\alpha}(V_1(x)) \quad \forall x \in X. \tag{11}$$

□

The proof is also given in Section VI. Proposition 5 is the extension of a result in Euclidean spaces [1].

V. INPUT-TO-STATE STABILITY

In this section, we define input-to-state stability (ISS) for system (2) and show that the existence of an ISS-Lyapunov function guarantees input-to-stability.

For $u \in \mathcal{U}$, set

$$V_U^\infty(u) := \inf\{c \geq 0 |\ V_U(u(t)) \leq c \text{ a.e. } t \geq 0\}. \tag{12}$$

Input-to-state stability for system (2) is defined by adding an input term to (10).

Definition 5 *Let $V_X : X \to \mathbb{R}_{\geq 0}$ be a continuous positive definite proper function. System (2) is called input-to-state stable if all $\varphi \in \mathcal{S}(X, \mathcal{U})$ are continuable on $\mathbb{R}_{\geq 0}$ and there exist functions $\beta \in \mathcal{LK}$ and $\gamma \in \bar{\mathcal{K}}_0$ satisfying*

$$\begin{aligned} V_X(\varphi(t; x, u)) \leq \beta(t, V_X(x)) + \gamma(V_U^\infty(u)) \\ \forall \varphi \in \mathcal{S}(X, \mathcal{U})\ \forall t \geq 0. \end{aligned} \tag{13}$$

□

Remark 11 *If $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $V_X(x) = \|x\|$ and $V_U(u) = \|u\|$, input-to-state stability in Definition 5 coincides with the traditional input-to-state stability [1]. Our definition is also compatible with the definition for sample-and-hold solutions on Riemannian manifolds [3]. The level sets of V_X correspond to precompact sets in [3].* □

Remark 12 *Input-to-state stability in Definition 5 does not depend on the choice of V_X and V_U. Assume that (13) is satisfied, and let $\bar{V}_X : X \to \mathbb{R}_{\geq 0}$ and $\bar{V}_U : U \to \mathbb{R}_{\geq 0}$ be continuous positive definite proper functions. By Proposition 5, there are functions $\underline{\alpha}_X, \overline{\alpha}_X, \underline{\alpha}_U, \overline{\alpha}_U \in \mathcal{K}_0^\infty$ satisfying*

$$\underline{\alpha}_X(V_X(x)) \leq \bar{V}_X(x) \leq \overline{\alpha}_X(V_X(x)) \quad \forall x \in X \tag{14}$$

$$\underline{\alpha}_U(V_U(u)) \leq \bar{V}_U(u) \leq \overline{\alpha}_U(V_U(u)) \quad \forall u \in \mathcal{U}. \tag{15}$$

By (12)–(15),

$$\begin{aligned} &\bar{V}_X(\varphi(t; x, u)) \\ &\leq \overline{\alpha}_X(\beta(t, V_X(x)) + \gamma(V_U^\infty(u))) \\ &\leq \overline{\alpha}_X(2\beta(t, V_X(x))) + \overline{\alpha}_X(2\gamma(V_U^\infty(u))) \\ &\leq \overline{\alpha}_X(2\beta(t, \underline{\alpha}_X^{-1}(\bar{V}_X(x)))) + \overline{\alpha}_X(2\gamma(\underline{\alpha}_U^{-1}(\bar{V}_U^\infty(u)))). \end{aligned} \tag{16}$$

By $\beta \in \mathcal{LK}$, $\gamma \in \bar{\mathcal{K}}_0$ and $\underline{\alpha}_X, \overline{\alpha}_X, \underline{\alpha}_U \in \mathcal{K}_0^\infty$,

$$\bar{\beta}(t, s) := \overline{\alpha}_X(2\beta(t, \underline{\alpha}_X^{-1}(s))) \in \mathcal{LK} \tag{17}$$

$$\bar{\gamma}(s) := \overline{\alpha}_X(2\gamma(\underline{\alpha}_U^{-1}(s))) \in \bar{\mathcal{K}}_0. \tag{18}$$

□

For a locally Lipschitz continuous positive definite proper function $V : X \to \mathbb{R}_{\geq 0}$, $DV(x,v)$ denotes the Dini derivative defined by

$$DV(x,v) := \liminf_{\varepsilon\to 0+} \frac{V(x+\varepsilon v)-V(x)}{\varepsilon}. \tag{19}$$

Definition 6 *Let $V_X : X \to \mathbb{R}_{\geq 0}$ be a continuous positive definite proper function. A locally Lipschitz continuous positive definite proper function $V : X \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov function if there exist a function $\alpha_1 \in \bar{\mathcal{K}}_0$ and a continuous positive definite function $\alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying*

$$\begin{aligned} &x\in X,\ u\in\mathcal{U} \text{ and } V_X(x)\geq \alpha_1(V_U(u)) \\ &\qquad\Rightarrow\ DV(x,f(x,u))\leq -\alpha_2(V_X(x)). \end{aligned} \tag{20}$$

□

Theorem 1 *System (2) is input-to-state stable if an ISS-Lyapunov function exists.* □

Proof: By Proposition 5, there exists functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_0^\infty$ satisfying

$$\underline{\alpha}(V_X(x)) \leq V(x) \leq \overline{\alpha}(V_X(x)) \quad \forall x\in X. \tag{21}$$

Let $\Omega := \{x\in X|\ V(x)\leq \overline{\alpha}\circ\alpha_1(V_U^\infty(u))\}$. For simplicity, we write $\varphi(t)$ instead of $\varphi(t;x,u)$. By (20), (21), and Lemma 6.1 in [1], we obtain

$$\begin{aligned} \varphi(t)\notin\Omega\ &\Rightarrow\ V_X(\varphi(t))\geq \overline{\alpha}^{-1}\circ V(\varphi(t))\geq \alpha_1(V_U^\infty(u)) \\ &\Rightarrow\ V_X(\varphi(t))\geq \alpha_1(V_U(u)) \text{ a.e. } t \\ &\Rightarrow\ \dot{V}(\varphi(t))\leq -\alpha_2(V_X(\varphi(t))) \text{ a.e. } t \\ &\Rightarrow\ V_X(\varphi(t))\leq \beta(t,V_X(\varphi(0))), \end{aligned} \tag{22}$$

where $\beta\in\mathcal{LK}$. By (21),

$$\begin{aligned} \varphi(t)\in\Omega\ &\Rightarrow\ V(\varphi(t))\leq \overline{\alpha}\circ\alpha_1(V_U^\infty(u)) \\ &\Rightarrow\ V_X(\varphi(t))\leq \underline{\alpha}^{-1}\circ\overline{\alpha}\circ\alpha_1(V_U^\infty(u)). \end{aligned} \tag{23}$$

By $\underline{\alpha}, \overline{\alpha}\in\mathcal{K}_0^\infty$ and $\alpha_1\in\bar{\mathcal{K}}_0$, $\gamma := \underline{\alpha}^{-1}\circ\overline{\alpha}\circ\alpha_1\in\bar{\mathcal{K}}_0$. Hence by (22) and (23), system (2) is input-to-state stable. ■

Remark 13 *If $\mathcal{U} = \{u : \mathbb{R}_{\geq 0}\to 0\}$, an ISS-Lyapunov function becomes a Lyapunov function. Namely, the origin of autonomous system (3) is globally asymptotically stable if*

$$DV(x,f(x,0))\leq -\alpha_2(V_X(x)) \quad \forall x\in X \tag{24}$$

for a locally Lipschitz continuous positive definite proper function $V : X\to\mathbb{R}_{\geq 0}$, a continuous positive definite proper function $V_X : X\to\mathbb{R}_{\geq 0}$, and a continuous positive definite function $\alpha_2 : \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$. □

VI. PROOFS OF PROPOSITIONS 1–5

Let $\mathbb{R}_{>0} := (0,+\infty)$, $\Lambda := \{\ldots,1/3,1/2,1,2,3,\ldots\}$, and $\lceil\cdot\rceil : \mathbb{R}_{>0}\to\Lambda$ be the ceiling function defined by

$$\lceil s\rceil := \min\{\lambda\in\Lambda|\ s\leq\lambda\}. \tag{25}$$

For a subset $X'\subset X$, $\operatorname{Int}X'$ denotes the interior of X'. For a subset $\mathcal{I}\subset\mathbb{R}_{>0}$, we define $\sup\mathcal{I} := +\infty$ if $\mathcal{I}$ is not bounded above.

In order to prove Proposition 1, we provide the following four lemmas.

Lemma 1 *Let $V_X : X\to\mathbb{R}_{\geq 0}$ be a continuous positive definite proper function. Then for each neighborhood X' of $0\in X$, there exists a positive constant $0<r<+\infty$ satisfying $\{x\in X|\ V(x)\leq r\}\subset X'$.* □

Proof: Since X is a locally compact Hausdorff space, X has the one point compactification X^* by Theorem 11.3, I [6]. Let $X^\infty := X^*\backslash X$, $\mathbb{R}_{\geq 0}^* := \mathbb{R}_{\geq 0}\cup\{+\infty\}$, and $V_X^* : X^*\to\mathbb{R}_{\geq 0}^*$ be the extension of V_X with $V_X^*(X^\infty) = +\infty$.

For each neighborhood X' of $0\in X$ there is a compact neighborhood $\bar{W}\subset X'$ of $0\in X$ by Theorem 29.2 [7]. Note that $\min\{V_X^*(x)|\ x\in X^*\backslash\operatorname{Int}\bar{W}\}>0$. Then for any positive constant $r<\min\{V_X^*(x)|\ x\in X^*\backslash\operatorname{Int}\bar{W}\}$, $\{x\in X|\ V_X(x)\leq r\}\subset\{x\in X^*|\ V_X^*(x)\leq r\}\subset\operatorname{Int}\bar{W}\subset\bar{W}\subset X'$. ■

Lemma 2 *Let $V_1, V_2 : X\to\mathbb{R}_{\geq 0}$ be continuous positive definite proper functions and assume that $\delta : \Lambda\to\mathbb{R}_{>0}$ is a positive valued function satisfying*

$$\begin{aligned} &V_2(\varphi(t;x))\leq\lambda \\ &\quad\forall\varphi\in\mathcal{S}(\{x\in X|\ V_1(x)\leq\delta(\lambda)\})\ \forall t\in\operatorname{dom}\varphi. \end{aligned} \tag{26}$$

Then, there exists a positive valued increasing function $\bar{\delta} : \Lambda\to\mathbb{R}_{>0}$ such that

i) for each $\lambda\in\Lambda$ there exists $\xi\in\Lambda$ satisfying $\xi<\lambda$ and $\delta(\xi)\geq\bar{\delta}(\lambda)$;
ii) $\lim_{\lambda\to+\infty}\bar{\delta}(\lambda) = \sup\{\delta(\lambda)|\ \lambda\in\Lambda\}$;
iii) $\lim_{\lambda\to 0}\bar{\delta}(\lambda) = 0$.

□

Proof: Define $\bar{\delta} : \Lambda\to\mathbb{R}_{>0}$ by

$$\bar{\delta}(\lambda) := \begin{cases} \min\left\{\delta(\xi)\,\middle|\ \xi\in\Lambda \text{ and } \frac{\lambda}{\lambda+1}\leq\xi\leq 1\right\} \\ \qquad\qquad\qquad\qquad\qquad (\lambda\leq 1) \\ \max\left\{\delta(\xi)|\ \xi\in\Lambda \text{ and } \frac{1}{\lambda}\leq\xi\leq\lambda-1\right\} \\ \qquad\qquad\qquad\qquad\qquad (\lambda>1). \end{cases} \tag{27}$$

Note that $\lim_{\lambda\to 0}\delta(\lambda) = 0$ by (26). Then by (27), $\bar{\delta}$ is an increasing function satisfying ii), iii) and

$$\begin{aligned} &\lambda\in\Lambda,\ \lambda\leq 1 \\ &\quad\Rightarrow\ \xi := \frac{\lambda}{\lambda+1}\in\Lambda,\ \xi<\lambda,\ \delta(\xi)\geq\bar{\delta}(\lambda) \end{aligned} \tag{28}$$

$$\begin{aligned} &\lambda\in\Lambda,\ \lambda>1 \\ &\quad\Rightarrow\ \text{there is } \xi\in\Lambda \text{ with } \xi\leq\lambda-1<\lambda,\ \delta(\xi) = \bar{\delta}(\lambda). \end{aligned} \tag{29}$$

Therefore, i) is satisfied. ■

Lemma 3 *Let $V_1, V_2 : X \to \mathbb{R}_{\geq 0}$ be continuous positive definite proper functions and assume that $\delta : \Lambda \to \mathbb{R}_{>0}$ is a positive valued function satisfying (26). Then, there exists a function $\hat{\delta} \in \bar{\mathcal{K}}_0$ satisfying*

$$\begin{aligned} & V_2(\varphi(t;x)) \leq s \\ & \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_1(x) \leq \hat{\delta}(s)\})\ \forall t \in \operatorname{dom}\varphi \end{aligned} \tag{30}$$

$$\lim_{s \to +\infty} \hat{\delta}(s) = \sup\{\delta(\lambda) |\ \lambda \in \Lambda\}. \tag{31}$$

□

Proof: By linear interpolation of $\bar{\delta}$ in Lemma 2, we obtain the following continuous positive definite increasing function $\tilde{\delta} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$;

$$\tilde{\delta}(s) := \begin{cases} 0 & (s = 0) \\ \frac{1}{\lceil s \rceil^2} \left\{ (\lceil s \rceil + 1)s - \lceil s \rceil \right\} \bar{\delta}(\lceil s \rceil) & \\ \quad + \frac{\lceil s \rceil + 1}{\lceil s \rceil^2} (\lceil s \rceil - s) \bar{\delta}\left(\frac{\lceil s \rceil}{\lceil s \rceil + 1} \right) & (0 < s \leq 1) \\ (s + 1 - \lceil s \rceil)\bar{\delta}(\lceil s \rceil) + (\lceil s \rceil - s)\bar{\delta}(\lceil s \rceil - 1) & \\ & (s > 1). \end{cases} \tag{32}$$

By (32) and ii) in Lemma 2,

$$\lim_{s \to +\infty} \tilde{\delta}(s) = \lim_{\lambda \to +\infty} \bar{\delta}(\lambda) = \sup\{\delta(\lambda) |\ \lambda \in \Lambda\}. \tag{33}$$

Since $\tilde{\delta}$ is a continuous positive definite increasing function, there is a function $\hat{\delta} \in \bar{\mathcal{K}}_0$ such that

$$\hat{\delta}(s) \leq \tilde{\delta}(s) \quad \forall s \geq 0 \tag{34}$$

$$\lim_{s \to +\infty} \hat{\delta}(s) = \lim_{s \to +\infty} \tilde{\delta}(s) = \sup\{\delta(\lambda) |\ \lambda \in \Lambda\}. \tag{35}$$

Therefore, (31) is satisfied. By (32), (34) and i) in Lemma 2, for each $s > 0$ there exists $\xi \in \Lambda$ such that

$$\xi < \lceil s \rceil \tag{36}$$

$$\delta(\xi) \geq \bar{\delta}(\lceil s \rceil) \geq \tilde{\delta}(s) \geq \hat{\delta}(s). \tag{37}$$

By $\xi \in \Lambda$ and (36),

$$0 < s \leq 1 \ \Rightarrow\ \xi \leq \frac{\lceil s \rceil}{\lceil s \rceil + 1} < s \leq \lceil s \rceil \tag{38}$$

$$s > 1 \ \Rightarrow\ \xi \leq \lceil s \rceil - 1 < s \leq \lceil s \rceil. \tag{39}$$

By (26) and (37)–(39),

$$\begin{aligned} & 0 < V_1(x) \leq \hat{\delta}(s) \\ & \Rightarrow\ V_2(\varphi(t;x)) < s \\ & \qquad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_1(x) \leq \hat{\delta}(s)\})\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{40}$$

$$\begin{aligned} & V_1(x) = 0 \\ & \Rightarrow\ V_2(\varphi(t;x)) = 0 \quad \forall \varphi \in \mathcal{S}(x)\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{41}$$

Therefore, (30) is satisfied. ■

Lemma 4 *Let $V_1, V_2 : X \to \mathbb{R}_{\geq 0}$ be continuous positive definite proper functions and assume that $\delta : \Lambda \to \mathbb{R}_{>0}$ is a positive valued function satisfying (26). Then, there exist a positive constant $0 < r \leq +\infty$ and a class $\mathcal{K}_0$ function $\alpha : [0, r) \to \mathbb{R}_{\geq 0}$ such that*

$$\begin{aligned} & V_2(\varphi(t;x)) \leq \alpha(V_1(x)) \\ & \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_1(x) < r\})\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{42}$$

Moreover, α can be chosen to be class $\mathcal{K}_0^\infty$ if $\sup\{\delta(\lambda) |\ \lambda \in \Lambda\} = +\infty$. □

Proof: Let $\hat{\delta}$ be a function in Lemma 3, $r := \sup\{\delta(\lambda) |\ \lambda \in \Lambda\}$ and $\alpha := \hat{\delta}^{-1}$. Then α is a class $\mathcal{K}_0$ function defined on $[0, r)$ and satisfies (42). Moreover, α becomes class $\mathcal{K}_0^\infty$ if $\sup\{\delta(\lambda) |\ \lambda \in \Lambda\} = +\infty$. ■

By Lemmas 1 and 4, Proposition 1 is proved as follows:

Proof: [Proof of Proposition 1] Prove i)⇒ii). Let $V_X : X \to \mathbb{R}_{\geq 0}$ be a continuous positive definite proper function. Then by continuity of V_X, $\{x \in X |\ V_X(x) \leq \lambda\}$ is a neighborhood of $0 \in X$ for each $\lambda \in \Lambda$. By i), for each $\lambda \in \Lambda$ there is a neighborhood W of $0 \in X$ such that

$$V_X(\varphi(t;x)) \leq \lambda \quad \forall \varphi \in \mathcal{S}(W)\ \forall t \in \operatorname{dom}\varphi. \tag{43}$$

By Lemma 1 and (43), there is a positive valued function $\delta : \Lambda \to \mathbb{R}_{>0}$ satisfying

$$\begin{aligned} & V_X(\varphi(t;x)) \leq \lambda \\ & \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_X(x) \leq \delta(\lambda)\})\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{44}$$

Hence by Lemma 4, there are a positive constant $0 < r_1 \leq +\infty$ and a class $\mathcal{K}_0$ function $\alpha_1 : [0, r_1) \to \mathbb{R}_{\geq 0}$ satisfying (7).

Prove ii)⇒i). By (7) and Lemma 1, for each neighborhood W of $0 \in X$ there is a positive constant $0 < s < r_1$ such that

$$\begin{aligned} & \varphi(t;x) \in \{x \in X |\ V_X(x) \leq \alpha_1(s)\} \subset W \\ & \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_X(x) \leq s\})\ \forall t \in \operatorname{dom}\varphi. \end{aligned} \tag{45}$$

Since $\{x \in X |\ V_X(x) \leq \alpha_1(s)\}$ is compact, all $\varphi \in \mathcal{S}(\{x \in X |\ V_X(x) \leq s\})$ are continuable on $\mathbb{R}_{\geq 0}$. Therefore i) is satisfied. ■

Proposition 2 is also proved in the same manner of the proof of Proposition 1. Proposition 3 is easily proved by Propositions 1 and 2. In order to prove Proposition 4, we give the following lemma:

Lemma 5 *Let $V_X : X \to \mathbb{R}_{\geq 0}$ be a continuous positive definite proper function and assume that the origin of system (3) is globally asymptotically stable and Lagrange stable. Then, there exists a function $\gamma : \mathbb{R}_{\geq 0} \times \Lambda \to \mathbb{R}_{\geq 0}$ such that*

i) for all $\varphi \in \mathcal{S}(X)$ and for all $t \geq 0$,

$$V_X(\varphi(t;x)) \leq \begin{cases} \gamma\left(t, \frac{\lceil V_X(x) \rceil}{\lceil V_X(x) \rceil + 1}\right) & (0 < V_X(x) \leq 1) \\ \gamma(t, \lceil V(x) \rceil - 1) & (V_X(x) > 1); \end{cases} \tag{46}$$

ii) $\gamma(t, \lambda)$ is class $\mathcal{L}$ with respect to t;

iii) $\gamma(t, \lambda)$ is increasing with respect to λ;

iv) $\lim_{\lambda \to 0} \gamma(t, \lambda) = 0 \quad \forall t \geq 0.$

□

Proof: By Proposition 3 and global convergence, there are functions $\bar{\gamma} : \mathbb{R}_{\geq 0} \times \Lambda \to \mathbb{R}_{\geq 0}$ and $\alpha_3 \in \mathcal{K}_0^\infty$ such that

- i') $V_X(\varphi(t;x)) \leq \bar{\gamma}(t,\lambda) \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_X(x) \leq \lambda\})\ \forall t \geq 0$;
- ii') $\bar{\gamma}(t,\lambda)$ is class $\mathcal{L}$ with respect to t;
- iii') $\bar{\gamma}(0,\lambda) = \alpha_3(\lambda) \quad \forall \lambda \in \Lambda$.

Define $\gamma : \mathbb{R}_{\geq 0} \times \Lambda \to \mathbb{R}_{\geq 0}$ by

$$\gamma(t,\lambda) := \begin{cases} \min\left\{\bar{\gamma}(t,\xi) \,\middle|\, \xi \in \Lambda \text{ and } \frac{\lambda}{1-\lambda} \leq \xi \leq 1\right\} \\ \hfill (0 < \lambda < 1) \\ \max\{\bar{\gamma}(t,\xi) |\ \xi \in \Lambda \text{ and } 1 \leq \xi \leq \lambda + 1\} \\ \hfill (\lambda \geq 1). \end{cases} \tag{47}$$

Then by i') and (47),

$$\begin{aligned} & 0 < V_X(x) \leq 1 \\ & \quad \Rightarrow\ V_X(\varphi(t;x)) \leq \gamma\left(t, \frac{\lceil V_X(x) \rceil}{\lceil V_X(x) \rceil + 1}\right) \\ & \qquad \forall \varphi \in \mathcal{S}(\{x \in X |\ 0 < V_X(x) \leq 1\})\ \forall t \geq 0 \end{aligned} \tag{48}$$

$$\begin{aligned} & V_X(x) > 1 \\ & \quad \Rightarrow\ V_X(\varphi(t;x)) \leq \bar{\gamma}(t, \lceil V_X(x) \rceil) \leq \gamma(t, \lceil V_X(x) \rceil - 1) \\ & \qquad \forall \varphi \in \mathcal{S}(\{x \in X |\ V_X(x) > 1\})\ \forall t \geq 0. \end{aligned} \tag{49}$$

Therefore i) is satisfied. By ii') and (47), ii) and iii) are satisfied. By ii), iii') and (47),

$$0 < \lambda < 1\ \Rightarrow\ \gamma(t,\lambda) \leq \gamma(0,\lambda) = \alpha_3\left(\frac{\lambda}{1-\lambda}\right)\ \forall t \geq 0 \tag{50}$$

$$\lim_{\lambda \to 0} \gamma(t,\lambda) \leq \lim_{\lambda \to 0} \alpha_3\left(\frac{\lambda}{1-\lambda}\right) = 0 \quad \forall t \geq 0. \tag{51}$$

Therefore iv) is satisfied. ■

Then Proposition 4 is proved as follows:

Proof: [Proof of Proposition 4] Prove i)⇒ii). By global convergence, for any compact neighborhoods $\bar{W}_1, \bar{W}_2$ of $0 \in X$ there is $T > 0$ such that

$$\varphi(t;x) \in \bar{W}_1 \quad \forall \varphi \in \mathcal{S}(\{x \in X |\ x \in \bar{W}_2\})\ \forall t \geq T. \tag{52}$$

Hence,

$$\begin{aligned} & \varphi(t;x) \in \bar{W}_1 \cup \{\varphi(t;x) |\ t \in [0,T] \text{ and } x \in \bar{W}_2\} \\ & \qquad \forall \varphi \in \mathcal{S}(\{x \in X |\ x \in \bar{W}_2\})\ \forall t \geq 0. \end{aligned} \tag{53}$$

Since $\bar{W}_1, \bar{W}_2$ are compact and φ is continuous with respect to t, $\bar{W}_1 \cup \{\varphi(t;x) |\ t \in [0,T] \text{ and } x \in \bar{W}_2\}$ is also compact. Therefore, the system is Lagrange stable.

Prove ii)⇒iii). By linear interpolation of γ in Lemma 5, we obtain the following continuous function $\tilde{\gamma} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$;

$$\begin{aligned} & \tilde{\gamma}(t,s) \\ & := \begin{cases} 0 & (s = 0) \\ \frac{1}{\lceil s \rceil^2} \{((\lceil s \rceil + 1)s - \lceil s \rceil\} \gamma(t, \lceil s \rceil) & \\ \quad + \frac{\lceil s \rceil + 1}{\lceil s \rceil^2}(\lceil s \rceil - s)\gamma\left(t, \frac{\lceil s \rceil}{\lceil s \rceil + 1}\right) & (0 < s \leq 1) \\ (s + 1 - \lceil s \rceil)\gamma(t, \lceil s \rceil) & \\ \quad + (\lceil s \rceil - s)\gamma(t, \lceil s \rceil - 1) & (s > 1). \end{cases} \end{aligned} \tag{54}$$

By (54), ii) and iii) in Lemma 5, $\tilde{\gamma}(t,s)$ is class $\mathcal{L}$ with respect to t and increasing with respect to s. Hence, there is a function $\beta \in \mathcal{LK}$ such that

$$\beta(t,s) \geq \tilde{\gamma}(t,s) \quad \forall t \geq 0\ \forall s \geq 0. \tag{55}$$

By (54), (55) and i) in Lemma 5,

$$\begin{aligned} V_X(\varphi(t;x)) \leq \tilde{\gamma}(t, V_X(x)) \leq \beta(t, V_X(x)) \\ \forall \varphi \in \mathcal{S}(X)\ \forall t \geq 0. \end{aligned} \tag{56}$$

Therefore iii) is satisfied.

Prove iii)⇒i). By (10),

$$\begin{aligned} \varphi(t;x) \in \{y \in X |\ V_X(y) \leq \beta(0, V_X(x))\} \\ \forall \varphi \in \mathcal{S}(X)\ \forall t \in \operatorname{dom} \varphi. \end{aligned} \tag{57}$$

By $\beta \in \mathcal{LK}$, $\alpha(\cdot) := \beta(0,\cdot) \in \bar{\mathcal{K}}_0$. Hence by Proposition 1, the origin is Lyapunov stable. Moreover, all solutions $\varphi \in \mathcal{S}(X)$ are continuable on $\mathbb{R}_{\geq 0}$ because $\{y \in X |\ V_X(y) \leq \beta(0, V_X(x))\}$ is compact for all $x \in X$. By (10) and Lemma 1, for each point $x \in X$ and each neighborhood W of $0 \in X$ there is $T \geq 0$ such that

$$\varphi(t;x) \in W \quad \forall \varphi \in \mathcal{S}(x)\ \forall t \geq T. \tag{58}$$

Therefore (5) is satisfied. ■

Proof: [Proof of Proposition 5] By Lemma 1, there is a positive valued function $\delta : \Lambda \to \mathbb{R}_{>0}$ satisfying

$$V_1(x) \leq \delta(\lambda)\ \Rightarrow\ V_2(x) \leq \lambda \tag{59}$$

$$\lim_{\lambda \to +\infty} \delta(\lambda) = +\infty. \tag{60}$$

Substituting $\varphi(t;x) := x\ \forall t \geq 0$ into Lemma 4, there is a function $\overline{\alpha} \in \mathcal{K}_0^\infty$ with $V_2(x) \leq \overline{\alpha}(V_1(x))\ \forall x \in X$. Similarly, there is a function $\underline{\alpha} \in \mathcal{K}_0^\infty$ with $\underline{\alpha}(V_1(x)) \leq V_2(x)\ \forall x \in X$. ■

VII. CONCLUSION

We have extended Lagrange stability and input-to-stability to manifolds. Moreover, we have obtained necessary and sufficient conditions for stability by using continuous positive definite proper functions. Our results contains the traditional ones on Euclidean spaces.

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