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Almost Sure Asymptotic Stabilizability for Deterministic Systems with Wiener Processes

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Abstract— **In this paper, we discuss almost sure asymptotic stability for deterministic systems, which are randomized by adding one-dimensional standard Wiener processes. First, we clarify the difference between the deterministic Lyapunov stability theory and the stochastic stability theory proposed by Hasminskii, and show that local asymptotic stability in probability causes trouble in randomization problems. Second, we describe the Lyapunov stability theory based on almost sure stability, as proposed by Bardi and Cesaroni, and show that the stability is "almost the same" as the deterministic Lyapunov stability. Third, we survey randomization problems briefly, and show that Stratonovich integrals are valid for such problems based on one-dimensional Wiener processes. Finally, we derive the necessary conditions for stochastic Lyapunov functions to ensure almost sure stabilities and discuss the difference between Hasminskii's stabilities and those of Bardi and Cesaroni via linear stochastic systems with numerical simulations.**

I. INTRODUCTION

In this paper, we derive the necessary conditions for the origins of systems, which are randomized by one-dimensional Wiener processes, to become almost surely asymptotically stable.

Stabilization problems for deterministic systems due to the addition of white noises have been widely discussed from 1960s to the present. For example, Appleby, Mao, and Rodkina [1], Mao [13], and Nishimura et al. [15] have clarified that suitably designed diffusion terms make the unstable origins of deterministic systems asymptotically stable with probability one.[1] However, two important issues remain with these problems: what stochastic stability is almost equivalent to the deterministic Lyapunov stability, and what stochastic system is generated when white noises are added to a deterministic system.

The stochastic Lyapunov stability theory was formalized by Hasminskii [8] and Kushner [11] in the 1960s, was extended to control problems by Florchinger [6] in the 1990s, and was developed into stochastic input-to-state stability by Deng et al. [5] and Liu et al. [12] during the first decade of the 21st century. Hasminskii and Kushner's Stochastic Lyapunov functions (SLFs) are sometimes considered compatible with deterministic Lyapunov functions (DLFs) [1], [13]. Unfortunately, the assertion is not always true, because SLFs do not ensure their sublevel sets to be invariance sets. The fact implies that even if global SLFs (GSLFs) exist for stochastic systems, then the origins are not *locally asymptotically stable with probability one* while they are *globally asymptotically stable with probability one*. In [1], [13], [15], local stability was not well investigated. In contrast, almost all sublevel sets of almost Lyapunov functions (ALFs), which are proposed by Bardi and Cesaroni [3], are invariance sets. Therefore, in this paper, we claim that the almost sure asymptotic stability based on ALFs is "almost the same" as the asymptotic stability based on DLFs.

To compare deterministic and stochastic systems, we have to consider randomization problems, i.e., if one-dimensional Wiener processes are added into deterministic systems, then we obtain Stratonovich-type stochastic systems [17], [20]. In [1], [2], [16], randomized systems are considered as Ito type; therefore, these results are not solutions to the stabilization problems for deterministic systems due to the addition of white noises. Although Mao [13] and Nishimura et al. [15] consider the Stratonovich-type systems, their stabilities are not equivalent to those for deterministic systems, as shown in the preceding paragraph.

In this paper we compare the conditions for DLFs to exist for deterministic systems with those for ALFs to exist for the corresponding Stratonovich-type stochastic systems. Then, we prove that any one-dimensional standard Wiener process cannot make the origin of the randomized systems almost surely asymptotically stable if the original deterministic systems are not asymptotically stable.

This paper is organized as follows. Sections II to IV provide preliminary discussions, and Section V shows the main results of the paper. In Section II, the difference between deterministic Lyapunov stability theory and Hasminskii's stochastic theory [8] is clarified by using Kushner's result [11]. In Section III, Bardi and Cesaroni's almost sure stability theory [3] is described, and the sublevel sets of ALFs are shown to be invariance sets with probability one. In Section IV, we survey the randomization problems and illustrate that Stratonovich integrals are valid for the problems with one-dimensional Wiener processes. In Section V, we derive the necessary conditions for ALFs to exist for randomized systems, discuss the difference between SLFs and ALFs via linear systems with linear multiplicative white noises, and show the results of some numerical simulations.

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[1] There are previous works on stability in the mean, namely, Ariaratnam and Graefe [2] and Rabotnikov [16], who have proved that if a linear system has an unstable pole, then no diffusion term exists for the origin of the corresponding Ito-type stochastic system to be asymptotically stable in the mean. In general, however, the stability in the mean is not the property for sample paths. Therefore, one does not consider the stability.

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In this paper, $\mathbb{R}^n$ denotes an n-dimensional Euclidean space; in particular, $\mathbb{R}$ denotes $\mathbb{R}^1$. If $\gamma : [0,\infty) \to [0,\infty)$ satisfies $\gamma \in K_\infty$, then $\gamma(t)$ is monotone increasing, $\gamma(0) \equiv 0$, and $\gamma(r) \to \infty$ as $r \to \infty$. The conditional probability and the conditional expectation of some event A, under some condition B, are written as $\mathbb{P}\{A|B\}$ and $\mathbb{E}\{A|B\}$, respectively. Further, $W(t) := (w_1, w_2, \dots, w_d)^T \in \mathbb{R}^d$ is a d-dimensional standard Wiener process, and especially $w(t) \in \mathbb{R}$ denotes a one-dimensional standard Wiener process. The differential forms of Ito and Stratonovich integrals of $\sigma : \mathbb{R}^n \to \mathbb{R}^d$ in $W(t) \in \mathbb{R}^d$ are denoted by $\sigma(x)dW$ and $\sigma(x) \circ dW$, respectively.

II. Conventional Lyapunov Stability Theories

This section provides a preliminary discussion of conventional Lyapunov stability theories. Here we investigate the difference between the Lyapunov stability theory for deterministic systems and that for stochastic systems.

A. Deterministic Lyapunov Theory

In this subsection, we describe the most basic Lyapunov theory, the details of which are shown in [7], [10]. Let the deterministic system

$$\dot{x} = f(x),\ x \in \mathbb{R}^n \tag{1}$$

be considered, where $x \in \mathbb{R}^n$ is a state vector, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. The stability properties for the origin of (1) are defined as follows:

Definition 1 (Lyapunov Stability [7], [10]): The origin of (1) is *Lyapunov stable* if, for all $\varepsilon > 0$ and $t \geq 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x(0)| < \delta$, then $|x(t)| < \varepsilon$. □

Definition 2 (Local Asymptotic Stability [7], [10]): The origin of (1) is *locally asymptotically stable (LAS)* if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x(0)| < \delta$, then $\lim_{t\to\infty} x(t) = 0$. □

Definition 3 (Global Asymptotic Stability [7], [10]): The origin of (1) is *globally asymptotically stable (GAS)* if it is Lyapunov stable and $\lim_{t\to\infty} x(t) = 0$ for all $x(0) \in \mathbb{R}^n$. □

Using the above stability properties, the Lyapunov stability theory is described as follows:

Definition 4 (Lyapunov Function [7], [10]): The function $V : \mathbb{R}^n \to \mathbb{R}$ is a local deterministic Lyapunov function (LDLF) if $V(x)$ is C^1, positive definite, and proper in D, where D is an open set with $0 \in D \subset \mathbb{R}^n$, and if $(dV/dt)(x)$ is negative definite in D. Further, if $V(x)$ is an LDLF in $D \equiv \mathbb{R}^n$, then $V(x)$ is said to be a global deterministic Lyapunov function (GDLF). □

Theorem 1 (Lyapunov Stability Theory [7], [10]): The origin of (1) is LAS (GAS) if and only if there exists an LDLF (GDLF) for (1). ♦

Remark 1: Note that LDLFs (GDLFs) are commonly said to be local (global) Lyapunov functions [7], [10]. In this paper, the word "deterministic" is added to avoid misunderstanding. ♦

B. Stochastic Lyapunov Theory

In this subsection, we describe the most popular stochastic Lyapunov theory, which is proposed by Hasminskii [8]. Let the stochastic system

$$dx = f(x)dt + \sigma(x)dW \tag{2}$$

be considered, where $x \in \mathbb{R}^n$ is a state vector, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$ are smooth functions. The stability properties *in probability* for the origin of (2) are defined as follows:

Definition 5: (Stability in Probability [8]): The origin of (2) is *stable in probability (SIP)* if

$$\lim_{x_s \to 0} \mathbb{P}\left\{ \sup_{s \leq t} |x(t)| > \varepsilon \middle| x(s) = x_s \right\} = 0 \tag{3}$$

for any $s \geq 0$ and $\varepsilon > 0$. □

Definition 6: (Local Asymptotic Stability in Probability [8]): The origin of (2) is *locally asymptotically stable in probability (LASIP)* if it is SIP and

$$\lim_{x_s \to 0} \mathbb{P}\left\{ \lim_{t\uparrow\infty} |x(t)| = 0 \middle| x(s) = x_s \right\} = 1 \tag{4}$$

for any $s \geq 0$. □

Definition 7: (Global Asymptotic Stability in Probability [8]): The origin of (2) is *globally asymptotically stable in probability (GASIP)* if it is SIP and

$$\mathbb{P}\left\{ \lim_{t\uparrow\infty} |x(t)| = 0 \middle| x(s) = x_s \right\} = 1 \tag{5}$$

for any $s \geq 0$ and for all $x_s \in \mathbb{R}^n$. □

Roughly speaking, SIP is the condition for almost all sample paths starting from the origin to stay on the origin; LASIP is the condition for almost all sample paths to converge to the origin as the initial states tend to the origin; and GASIP is the condition for almost all sample paths to converge to the origin. In other words, although GASIP might be considered similar to GAS, LASIP are obviously different from LAS. The reason is that the operation "$\lim_{x_s \to 0}$" is included in (4). The difference can be investigated in more detail via the stochastic Lyapunov stability theory. The following notion is important for the discussion:

Definition 8 (Infinitesimal Operator): Let a Markov process $x(t) \in \mathbb{R}^n$, $t \geq 0$ and $x(t) = x_t \in \mathbb{R}^n$ be considered. If

$$(\mathscr{L}v)(x(t)) = \lim_{h\downarrow 0} \frac{\mathbb{E}\{v(x(t+h))|x(t) = x_t\} - v(x(t))}{h} \tag{6}$$

can be defined for a function $v : \mathbb{R}^n \to \mathbb{R}$, then $\mathscr{L}$ is said to be *an infinitesimal operator* for $x(t)$. □

For (2), the inifinitesimal operator is calculated as follows:

Theorem 2 (Hasminskii [8]): The infinitesimal operator for (2) is represented as

$$\mathscr{L}(\cdot) = \left[\frac{\partial(\cdot)}{\partial x}\right] f(x) + \frac{1}{2}\mathrm{tr}\left[\sigma(x)^T \sigma(x) \left[\frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^T\right]\right]. \tag{7}$$

Now we define the stochastic Lyapunov functions:

Definition 9 (Stocahstic Lyapunov Function [8]): The function $V:\mathbb{R}^n \to \mathbb{R}$ is a local stochastic Lyapunov function (LSLF) if $V(x)$ is C^2, positive definite, and proper in D, where D is an open set with $0 \in D \subset \mathbb{R}^n$, and if $(\mathscr{L}V)(x)$ is negative definite in D. Further, if $V(x)$ is an LSLF in $D \equiv \mathbb{R}^n$, then $V(x)$ is said to be a global stochastic Lyapunov function (GSLF). □

Using the above definitions, we derive the following theorem:

Theorem 3 (Hasminskii's Stochastic Stability Theory [8]): The origin of (2) is LASIP (GASIP) if there exists an LSLF (GSLF) for (2). ♦

As previously mentioned, LASIP should be considered carefully. Hence, LSLFs should also be treated attentively. This sensitivity clearly appears in the other stochastic stability theory, as introduced by Kushner:

Theorem 4 (Kushner's Stochastic Stability Theory [11]): Suppose that an LSLF $V(x)$ exists for (2) in $Q_m := \{x|V(x) < m, m > 0\}$. If an initial state x_0 is in Q_m, then the sample paths of (2) converge to the origin at least with probability $1 - V(x_0)/m$. ♦

This theorem implies that the following results: if $D \neq \mathbb{R}^n$, LSLFs do not ensure that almost all sample paths starting from any neighborhood D converge to the origin. The reason is that Dynkin's formula

$$\mathbb{E}\{V(x(t))|x(s)=x_s\} - V(x(t)) = \mathbb{E}\left\{\int_s^t (\mathscr{L}V)(x(\tau))d\tau \middle| x(s)=x_s\right\} \quad (8)$$

and supermartingale property

$$\mathbb{E}\{V(x(s+h))|x(s)=x_s\} \le V(x(s)), \quad h>0, \quad (9)$$

are used in the proof of Theorem 4. These formulas do not prohibit the sample paths from falling out the neighborhood $D \neq \mathbb{R}^n$. Consequently, LSLFs provide stability properties distinctly weaker than those for deterministic systems.

Remark 2: Note that, almost all sample paths converge to the origin if $V(x)$ is a GSLF because $D \to \mathbb{R}^n$ as $m \uparrow \infty$ in Theorem 4. Nevertheless, GASIP is different from GAS; the reason is shown in Section V. ♦

III. Lyapunov Stability Theory Based on Almost Sure Stability

In this section, we show the stochastic Lyapunov stability theory based on almost sure stability, which was introduced by Bardi and Cesaroni [3]. Further, we clarify that the theory is "almost the same" as the deterministic Lyapunov stability theory. We define the stability notions as follows:

Definition 10 (Almost Sure Stable [3]): The origin of (2) is *almost surely stable (ASS)* if, for any initial state $x(0) = x_0 \in \mathbb{R}^n$ and any $\eta > 0$ satisfying $|x_0| \le \delta$, there exists $\delta > 0$ almost surely such that $|x(t)| \le \eta$ for all $t > 0$. □

Definition 11: (Local Almost Sure Asymptotic Stability [3]): The origin of (2) is *locally almost surely asymptotically stable (LASAS)* if it is ASS and

$$\lim_{t\uparrow\infty} |x(t)| = 0 \quad (10)$$

almost surely for $x \in D$, where D is an open set with $0 \in D \subset \mathbb{R}^n$. □

Definition 12: (Global Almost Sure Asymptotic Stability [3]): The origin of (2) is *globally almost surely asymptotically stable (GASAS)* if there exists $\gamma \in \mathscr{K}_\infty$, and if $|x(t)| \le \gamma(|x|)$ and (10) hold almost surely for all $x \in \mathbb{R}^n$ and $t \ge 0$. □

The difference between LASIP and LASAS is the existence or the nonexistence of the operation $\lim_{x_s \to 0}$. Furthermore, the difference between LAS and LASAS is either the space such that the probability measure is zero is included in the discussion or it is not. Hence, if the origin of (2) is LASAS, then almost all sample paths with initial states $x(0) \in D$ converge to the origin. That explains why LASAS is "almost the same" as LAS.

Now, we consider Lyapunov functions based on LASAS and GASAS.

Definition 13 (Almost Lyapunov Function [3]): Let $V:\mathbb{R}^n \to \mathbb{R}$ be C^2, positive definite, and proper in D, where D is an open set with $0 \in D \subset \mathbb{R}^n$. The function $V(x)$ is a local almost Lyapunov function (LALF) if there exists a positive definite function $l:\mathbb{R}^n \to \mathbb{R}$ such that

$$-(\mathscr{L}V)(x) \le l(x), \ \forall x \in D\backslash\{0\}, \quad (11)$$

and if

$$\frac{\partial V}{\partial x}(x)\sigma(x) = 0, \ \forall x \in D\backslash\{0\}. \quad (12)$$

Further, if $V(x)$ is an LALF in $D \equiv \mathbb{R}^n$, then $V(x)$ is said to be a global almost Lyapunov function (GALF). □

Using LALFs and GALFs, we derive the following stability theory:

Theorem 5 (Almost Sure Stability Theory [3]): The origin of (2) is LASAS (GASAS) if there exists an LALF (GALF) for (2). ♦

Obviously, LSLFs (GSLFs) satisfying the condition (12) are LALFs (GALFs). What is important is that all sublevel sets of LALFs and GALFs are invariance sets [3]. To be specific, Ito's formula implies that the stochastic differentiation of $V(x)$ is represented as

$$dV(x(t),t) = (\mathscr{L}V)(x(t))dt + \frac{\partial V}{\partial x}(x(t))\sigma(x(t))dW(t). \quad (13)$$

That is, if $V(x)$ satisfies the condition (12), then

$$dV(x(t),t) = (\mathscr{L}V)(x(t))dt \quad (14)$$

holds with probability one. Therefore, if $V(x)$ is an LALF (a GALF), then dV is negative definite with probability one. In contrast, if $V(x)$ is an LSLF (a GSLF) not satisfying (12), then (14) does not hold, and only

$$\mathbb{E}\{dV(x(t),t)|x(0)=x_0\} = (\mathscr{L}V)(x(t))dt \quad (15)$$

holds.

Furthermore, if $n = 1$, then there is no LALF or GALF for (2) except the situation $\sigma(x) \equiv 0$; the fact is obtained from the condition (12). However, if $n \ge 2$, LALFs or GALFs may exist, because "escape routes" may exist for $V(x)$ to decrease.

Remark 3: Note that LALFs (GALFs) are commonly said to be local (global) strict Lyapunov functions [3]. In this paper, the word "strict" is deleted to simplify the names, and "almost" is added to avoid misunderstanding. ♦

Remark 4: In [3], LALFs (GALFs) are defined as lower semicontinuous functions; in this paper, however, they are defined as C^2 functions to simplify the discussion. Note that the aim of this paper is not to analyze discontinuity, but to show the relationship between LDLFs (GDLFs) and LALFs (GALFs) via randomization problems. ♦

IV. Randomization Problems

In this section, we summarize and discuss the randomization problems of deterministic systems.

In the previous section, we claimed that ALFs are compatible with DLFs. However, to compare deterministic and stochastic systems, we should take randomization problems into account. Such problems were initially discussed by Wong and Zakai [20] and Stratonovich [17], and these authors concluded that if a one-dimensional or a specific multi-dimensional Wiener process is added into a deterministic system, the system becomes a Stratonovich-type stochastic system. However, McShane [14] has shown that if a multi-dimensional Wiener process that does not satisfy a specific condition is added into a deterministic system, the randomized system is not always of Stratonovich type. Then, Ikeda and Watanabe [9] and Sussmann [19] proved that general randomized systems are represented as Stratonovich type with additional terms. However, the additional terms are not unique. Moreover, if $d \geq 1$, Ito mappings $(W,t) \mapsto x$ are discontinuous with probability one [18]. To avoid the difficulties of nonuniqueness and discontinuity, we only consider one-dimensional Wiener processes. The following theorem is the one-dimensional version of Ikeda and Watanabe's theorem.

Theorem 6 (Ikeda and Watanabe [9]): For $t \in [0,T]$ with $T > 0$, let $0 = t_0 < t_1 < \cdots < t_N = T$ with $N = 1,2,\ldots$,

$$\delta := \max_{j=1,2,\ldots,N} \{|t_{j+1} - t_j|\}, \tag{16}$$

and $w^D(t) \in \mathbb{R}$ be a piecewise continuous function satisfying $w^D(t) \equiv w(t)$ for all $t = t_0, t_1, \ldots, t_N$. Let a system

$$dx^D(t) = f(x^D(t))dt + \sigma(x^D(t))dw^D(t) \tag{17}$$

and a Stratonovich-type stochastic system

$$dx = f(x)dt + \sigma(x) \circ dw \tag{18}$$

be considered, where x and f are same as (2), and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is smooth enough, and $x^D(t) \in \mathbb{R}^n$. Then, for every $T > 0$,

$$\lim_{\delta \downarrow 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|x(t) - x^D(t)\right|^2\right] = 0 \tag{19}$$

holds. ♦

Consequently, the randomization problems with one-dimensional Wiener processes are described as follows. Let a control problem of the deterministic system (1) by adding a controller be considered, i.e.,

$$\dot{x} = f(x) + \sigma(x)u, \tag{20}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function, and $u \in \mathbb{R}$ is a control input. If we design u as a Gaussian white noise, i.e., $udt = dw$, then we obtain the Stratonovich-type stochastic system (18).

V. Almost Sure Asymptotic Stabilizability on Randomization Problems

In this section, the almost sure asymptotic stabilizability problem on randomization problems is solved. Subsection V-A shows the main results of this paper; the necessary conditions for LALFs and GALFs on randomization problems are described. In Subsection V-B, the main theorem and corollary are proven. In Subsection V-C, ALFs are compared with SLFs via linear systems. In Subsection V-D, some numerical simulations are shown.

A. Conditions for Almost Sure Asymptotic Stability

Let the Stratonovich-type stochastic system (18) be considered as the randomized system of the deterministic system (1). Considering the conditions of LALFs and GALFs, the following theorem is derived:

Theorem 7: Let $V : \mathbb{R}^n \to \mathbb{R}$ be considered as a C^∞, positive definite, and proper function in a neighborhood of the origin $D \subset \mathbb{R}^n$. If $V(x)$ is an LALF for (18), then

$$\frac{\partial F_j}{\partial x} f(x) = 0, \tag{21}$$

$$F_j(x) = 0, \tag{22}$$

are necessary to hold for all $j = 1,2,3,\ldots$, where

$$F_1(x) := \frac{\partial V}{\partial x}(x)\sigma(x), \tag{23}$$

$$F_{j+1}(x) := \frac{\partial F_j}{\partial x}(x)\sigma(x). \tag{24}$$

♦

Further, this theorem provides the following important corollary:

Corollary 1: Let $V : \mathbb{R}^n \to \mathbb{R}$ be considered as a C^∞, positive definite, and proper function in a neighborhood of the origin $D \subset \mathbb{R}^n$. One of necessary conditions for $V(x)$ to be an LALF for (18) is that $V(x)$ is an LDLF for the deterministic system (1). ♦

This corollary implies that, if the origin of a deterministic system (1) is not LAS (GAS), then there is no one-dimensional Wiener process such that the origin becomes LASAS (GASAS). The fact is coincident with the conclusion of Section III, i.e., LASAS and GASAS are "almost the same" as LAS and GAS, respectively. Further, Theorem 7 also implies that even if a one-dimensional Wiener process is added into a deterministic system having the LAS (GAS) origin as disturbance, the resulting stochastic system has the potential to be LASAS (GASAS).

Note that, in our previous work [15], a non-LAS deterministic system was changed to a GASIP stochastic system by randomization. Hence, *stability with probability one* could not be considered as the same as *stability in the deterministic sense*.

B. Proofs of Theorem 7 and Corollary 1

Theorem 7 and Corollary 1 are proven as follows. In the proof, the formula

$$\frac{\partial K_1^T(x)K_2(x)}{\partial x} = K_1^T(x)\frac{\partial K_2}{\partial x}(x) + K_2^T(x)\frac{\partial K_1}{\partial x}(x) \qquad (25)$$

is used on several occasions, where $K_1 : \mathbb{R}^n \to \mathbb{R}$ and $K_2 : \mathbb{R}^n \to \mathbb{R}$ are C^1. The Stratonovich-type stochastic system (18) is transformed into an Ito-type stochastic system

$$dx = \left\{ f(x) + \frac{1}{2}\frac{\partial \sigma}{\partial x}(x)\sigma(x) \right\} dt + \sigma(x)dw. \qquad (26)$$

Therefore, $(\mathscr{L}V)(x)$ is calculated as

$$\begin{aligned}(\mathscr{L}V)(x) = &\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\left\{ \frac{\partial V}{\partial x}(x)\frac{\partial \sigma}{\partial x}(x)\sigma(x) \right.\\ &\left. + \sigma^T(x)\left[\frac{\partial}{\partial x}\left[\frac{\partial V}{\partial x}\right]^T\right](x)\sigma(x)\right\}\\ = &\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}F_2(x) \end{aligned} \qquad (27)$$

by using (25) with $K_1 = (\partial V/\partial x)^T$ and $K_2 = \sigma$. The stochastic differentiation of $V(x)$ is obtained as

$$dV = \left\{ \frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}F_2(x) \right\} dt + F_1(x)dw. \qquad (28)$$

One of conditions for LALFs (GALFs) is (12); i.e., $F_1 \equiv 0$ with probability one. Therefore, $dF_1 \equiv 0$ should also be satisfied. Thus, $(\mathscr{L}F_1)(x) = F_2(x) \equiv 0$ is derived by

$$\begin{aligned} dF_1 &= (\mathscr{L}F_1)(x)dt + F_2(x)dw \\ &= \left\{ \frac{\partial F_1}{\partial x}(x)f(x) + \frac{1}{2}F_3(x) \right\} dt + F_2(x)dw \end{aligned} \qquad (29)$$

using (25) with $K_1 = (\partial F_1/\partial x)^T$ and $K_2 = \sigma$. By (27) and condition $F_1 = F_2 \equiv 0$,

$$dV = (\mathscr{L}V)(x) = \frac{\partial V}{\partial x}(x)f(x) \qquad (30)$$

are derived. This equation and (11) prove that $(\partial V/\partial x)(x)f(x)$ is necessary to be negative definite, which is same as the condition for LDLFs (GDLFs). Thus, Corollary 1 is derived. Then, by using (29),

$$dF_j = \left\{ \frac{\partial F_j}{\partial x}(x)f(x) + \frac{1}{2}F_{j+2}(x) \right\} dt + F_{j+1}(x)dw \qquad (31)$$

is derived for $j = 1, 2, \ldots$ a posteriori. From this equation and condition (12), Theorem 7 is derived.

C. Linear Systems with Linear Multiplicative White Noises

In this subsection, the difference between conditions for LALFs (GALFs) and LSLFs (GSLFs) is discussed by using Theorem 7 via a linear system with a linear multiplicative white noise:

$$dx(t) = Ax(t)dt + Bx(t) \circ dw(t), \qquad (32)$$

where $x \in \mathbb{R}^n$ is a state vector, $w \in \mathbb{R}$ is a standard Wiener process, and A and B are $n \times n$ matrices.

The sufficient condition for the origin of (32) to be GASIP is represented as the following theorem:

Theorem 8: If there exist $n \times n$ positive definite matrices P and Q such that

$$2PA + PB^2 + B^T PB = -Q, \qquad (33)$$

then the origin of (32) is GASIP. ♦

Proof: For $V(x) = x^T Px$,

$$\begin{aligned}(\mathscr{L}V)(x) = &2x^T P\left(Ax + \frac{1}{2}B^2\right)x + \frac{1}{2}\mathrm{tr}[Bxx^T B^T(2P)]\\ = &x^T(2PA + PB^2 + B^T PB)x \end{aligned} \qquad (34)$$

is calculated. Therefore, if (33) holds, then $V(x)$ is a GSLF for (32) because $(\mathscr{L}V)(x) = -x^T Qx$ is derived. ■

This theorem implies that, even if a deterministic linear system

$$\dot{x}(t) = Ax(t) \qquad (35)$$

is not GAS, it is possible for the randomized system (32) to become GASIP. In contrast, Theorem 7 shows that no diffusion term exists such that non-GAS deterministic systems become GASAS stochastic systems. The following corollary is obviously obtained from Corollary 1.

Corollary 2: If the origin of (35) is not LAS (GAS), then there exists no matrix B such that the origin of (32) is LASAS (GASAS). ♦

Instead of proving this corollary, let $V(x) = x^T Px$ be considered a candidate of an LALF (GALF) for (32). By condition (12),

$$\frac{\partial V}{\partial x}(x)Bx = 2x^T PBx = 0 \qquad (36)$$

is derived. By substituting $x^T BPx = 0$ into (33),

$$(\mathscr{L}V)(x) = 2x^T PAx = -x^T Qx \qquad (37)$$

is derived. This equation yields that a necessary condition for $(\mathscr{L}V)(x)$ of (32) to be negative definite is the same as that for $(dV/dt)(x)$ of (35) to be negative definite.

D. Numerical Simulations

In this subsection, two numerical examples are shown to compare SLFs and ALFs. The following example is the case in which no ALF exists although an SLF exists:

Example 1: Consider (32) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}. \qquad (38)$$

Then, $V_1(x) = x_1^2 + (1/2)x_2^2$ is a GSLF of (32), because $(\mathscr{L}V_1)(x) = -6x_1^2 - 2x_2^2$. However, there is no LALF nor GALF, because A has a positive eigenvalue. ♦

Fig. 1 and the blue dushed line of Fig. 3 show a simulation result of Example 1. The sample path converges to the origin with probability one, because the system is GASIP. However, the path of $V_1(x)$ does not decrease monotonically while $\mathscr{L}V_1(x)$ keeps negative definiteness. Moreover, this system does not have any ALF, because (35) with (38) is neither LAS nor GAS.

The following example is the case in which an ALF exists:

Example 2: Consider (32) with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}. \tag{39}$$

Then, $V_2(x) = (1/2)(x_1^2 + x_2^2)$ is a GALF of (32), because $dV_2(x) = (\mathscr{L}V_2)(x)dt = -(x_1^2 + 10x_2^2)dt$. ♦

Fig. 2 and the red straight line of Fig. 3 show a simulation result of Example 2. The sample path converges to the origin with probability one, because the system is GASAS. Moreover, the path of $V_2(x)$ decreases monotonically due to the negative definitteness of $dV_2(x) = \mathscr{L}V_2(x)dt$. This example implies that even if a state constraint $2x_1^2 + x_2^2 \le a$ for $a > 0$ is added, the sample paths converge to the origin with probability one because the origin of the system keeps the LASAS property.

VI. Conclusion

In this paper, we clarified the necessary conditions for almost Lyapunov functions of randomized systems to exist, compared the difference between stochastic Lyapunov functions and almost Lyapunov functions via linear systems with linear multiplicative white noises, and showed the validity of the results by illustrating numerical examples.

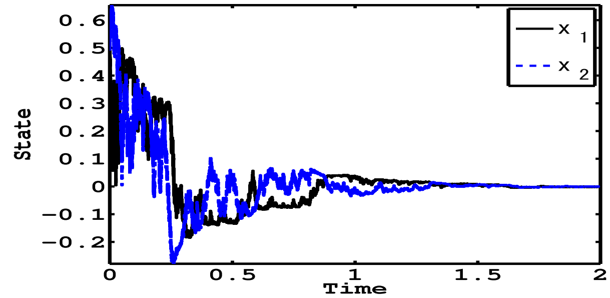

Fig. 1. Sample path of Ex.1.

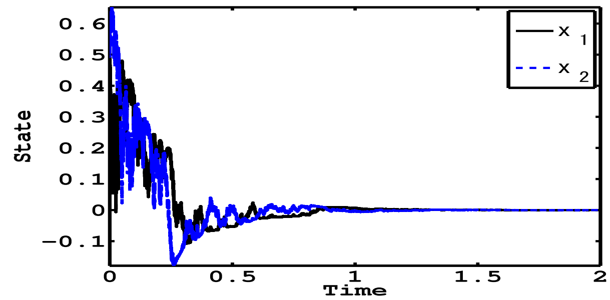

Fig. 2. Sample path of Ex.2.

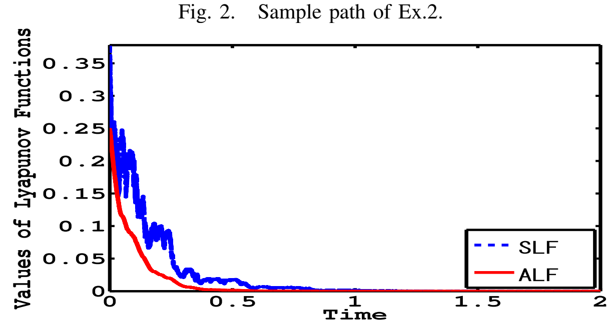

Fig. 3. SLF $V_1(x)$ and ALF $V_2(x)$.

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