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Lyapunov Stability of Pseudo Euler-Lagrange Systems

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Abstract— **This paper presents a systematic approach to find a Lyapunov function for stability analysis of pseudo Euler-Lagrange systems. There are two main contributions of this paper. First, a systematic procedure is proposed to obtain a Lyapunov function for the system directly from the mathematical structure of the differential equations, without the need to determine any kinetic or potential energy of the system first. Second, energy-based ideas used in Euler-Lagrange systems are extended to the case where generalized velocity variables are not necessarily the derivative of generalized position variables. The method proposed here works for any mathematical model in the class of pseudo Euler-Lagrange systems and is therefore not restricted to models of physical systems, having thus the potential to address economic, biologic and other systems. Several examples illustrate the application of the new approach.**

I. INTRODUCTION

Lyapunov stability has been a very important topic in control theory for over one hundred years. A good introduction to Lyapunov theory can be found in [1] or in [2]. In his famous work [3], Lyapunov suggested a generalized energy-based method to analyze the stability of nonlinear dynamic systems. The only drawback of this technique is that finding a Lyapunov function with properties of a generalized energy for a given nonlinear system is in general a very hard task. An exception is the case of linear systems for which Lyapunov proved that it is necessary and sufficient to find a quadratic Lyapunov function to prove stability. Since then many researchers have dedicated themselves to finding classes of Lyapunov functions for specific classes of systems. This paper gives a step in that direction by proposing a Lyapunov function for systems in a class that will be called pseudo Euler-Lagrange systems. This is a generalization of the well-known class of Euler-Lagrange systems. A comprehensive description of the research on Euler-Lagrange systems up to 1998 using a passivity-based control approach can be found in the book [4] and references therein. Since 1998, other authors addressed the control of Euler-Lagrange systems using techniques such as, for example, nonlinear control [5], optimal control [6], and adaptive control [7]. However, the general procedure on Euler-Lagrange systems research has been to construct Lyapunov functions based on expressions for the system's kinetic and potential energy, using the fact that the generalized velocity variables are the derivatives of the generalized position variables. To the best of the authors' knowledge, there are no results in the literature that extend the Euler-Lagrange ideas to systems for which the variables that usually correspond to generalized velocities are not necessarily the derivatives of the generalized position variables, although there is a nonlinear function relationship between them. Such a generalization is the topic of this paper and the class of systems defined in the paper is called pseudo Euler-Lagrange systems. The paper is organized as follows. A motivating physical example is first introduced in section II and then abstract generalizations are made to the case of single degree of freedom systems in section III, and multiple degrees of freedom in section IV. This is followed by conclusions.

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II. MOTIVATING PHYSICAL EXAMPLE

We start by analyzing a mass-spring-damper system (see Example 3.8 in [2] for a generalized pendulum example using the same ideas) with unit mass as a motivating example The equations of motion of this system are

$$\begin{aligned} \dot{x}_1(t) &= x_2 \\ \dot{x}_2(t) &= -cx_2 - kx_1 \end{aligned} \tag{1}$$

The equations of this system can be derived using the Euler-Lagrange equations [4]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_2}\right) - \frac{\partial L}{\partial x_1} = f \tag{2}$$

where $f = -cx_2$ and $L = 0.5x_2^2 - G_1(x_1)$ with

$$G_1(x_1) = \int g_1(x_1)dx_1 \tag{3}$$

being the anti-derivative of $g_1(x_1) = kx_1$ with $G_1(0) = 0$. Note that the force f for the mass-spring-damper system is dissipative and can also be assumed as the partial derivative of a dissipative potential function with respect to the velocity variable. This dissipative term can then be moved to the left hand side of the equation (2), leaving a null term on the right hand side. However, the format of equation (2) is the one corresponding to the most general force inputs. Since the Lagrangian (L) is the difference between the kinetic (K) and potential (P) energies and the mechanical energy (E) is the sum of these two energies then

$$E = \frac{\partial L}{\partial x_2}x_2 - L = 2K - K + P = K + P \tag{4}$$

and using Lagrange's equations (2) and the chain rule one concludes that (see [4] for an alternative derivation)

$$\dot{E} = -\frac{\partial L}{\partial t} + fx_2$$

This expression is important because if the partial derivative of L with respect to time is non-negative then the system is dissipative because the time rate of increase of energy ($\dot{E}$) is smaller than the rate of supplied energy (fx_2). Sometimes in the literature the energy function (E) is referred to as the Hamiltonian (H) or the Jacobi energy function (J). An interesting aspect of equation (4) is that one can find an energy function for a system from knowledge of its Lagrangian. For the case of the mass-spring-damper system (1), looking at the dynamics of x_2 as being in the Euler-Lagrange form (2) we note that making

$$-g_1 = -kx_1 = \frac{\partial L}{\partial x_1}$$

one gets by integration

$$L = -G_1(x_1) + l(x_2) \tag{5}$$

where $G_1(x_1)$ is defined in (3) and $l(x_2)$ is an integration function of x_2. Introducing (5) in expression (4) yields

$$\begin{aligned} E &= l'(x_2)x_2 - l(x_2) + G_1(x_1) = \\ &= \int l''(x_2)x_2dx_2 + \int g_1(x_1)dx_1 \end{aligned}$$

If this energy function corresponds to the mechanic energy of the system with unit mass we must have $l''(x_2) = 1$. This shows that one can form an energy function by equating $g_1(x_1)$ to the partial derivative of the Lagrangian with respect to x_1, equating the expression $l''(x_2)x_2$ to x_2 and then integrating these expressions.

The next section extends this physical idea to a class of mathematical models and proposes a Lyapunov function that proves stability for such models, called pseudo Euler-Lagrange systems.

III. SINGLE DEGREE OF FREEDOM

In this section we consider the following system

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_2) \\ \dot{x}_2(t) &= -g_1(x_1) + g_2(x_1)f_2(x_2) \end{aligned} \tag{6}$$

where $x(t) = [x_1 \ x_2]^T \in I\!R^2$, $f_1,\ g_1$ are class $\mathcal{C}^1$ functions not identically zero and $f_2,\ g_2$ are continuous functions not identically zero. When $f_1(x_2) = x_2$ the system dynamics are cast in the general form of the Euler-Lagrange equations (2). In this work however, we do not assume $f_1(x_2) = x_2$ and therefore we call the more general system (6) a pseudo Euler-Lagrange system. We now present the main stability result of this section.

Theorem 1: For system (6) define a class $\mathcal{C}^1$ function

$$V(x_1, x_2) = F_1(x_2) + G_1(x_1) \tag{7}$$

where $G_1(x_1)$ is defined in (3) and

$$F_2(x_2) = \int f_1(x_2)dx_2 \tag{8}$$

Then if either

$$g_2(x_1) \geq 0,\ f_1(x_2)f_2(x_2) \leq 0, \tag{9}$$

or

$$g_2(x_1) \leq 0,\ f_1(x_2)f_2(x_2) \geq 0, \tag{10}$$

and

$$\begin{aligned} f_1'(x_2) &> 0,\ x_2 \neq 0 \quad (11)\\ g_1'(x_1) &> 0,\ x_1 \neq 0 \quad (12)\end{aligned}$$

the system (6) is stable and V is a Lyapunov function provided it is radially unbounded. If furthermore

- $g_i(x_1) = 0$ implies $x_1 = 0$ for $i = 1, 2$,
- $f_1(x_2) = 0$ implies $x_2 = 0$,
- $f_2(x_2) = 0$ is equivalent to $x_2 = 0$,

then the system (6) is asymptotically stable to the origin.

Proof. Using V as a candidate Lyapunov function and computing its derivative over time using (6) yields

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2\\ &= g_1 f_1 + f_1\left[g_2 f_2 - g_1\right]\\ &= g_2 f_1 f_2 \leq 0\end{aligned}$$

which proves that the system is globally stable because V is positive definite under the constraints (11)–(12) and it is assumed to be radially unbounded. Let the following three sets be defined

$$\begin{aligned} M_1 &= \{(x_1, x_2) \mid f_1(x_2) = 0\}\\ M_2 &= \{(x_1, x_2) \mid f_2(x_2) = 0\}\\ M_3 &= \{(x_1, x_2) \mid g_2(x_1) = 0\}\end{aligned}$$

Using LaSalle's Invariance Principle (see Theorem 3.4 in [2]), we define the set

$$M = M_1 \cup M_2 \cup M_3$$

If $f_1(x_2) = 0$ then $x_2 = 0$ (constant) and $\dot{x}_1,\ \dot{x}_2,\ f_2(0)$ are zero. This implies that $g_1(x_1) = 0$, which in turn implies that $x_1 = 0$. If $f_2(x_2) = 0$ then $x_2 = 0$, and $\dot{x}_2$ is zero, which implies that $g_1(x_1) = 0$, and consequently $x_1 = 0$. Finally, if $g_2(x_1) = 0$ then $x_1 = 0$, $\dot{x}_1 = 0$ and consequently $f_1(x_2) = 0$, which implies $x_2 = 0$. Therefore, the largest invariant set of system (6) contained inside M is the origin and system (6) is asymptotically stable to the origin. This finishes the proof. □

Remark 1: For the case where $l''(x_2) = 1$, one recovers the Euler-Lagrange structure such as, for example, for the mass-spring-damper system (1), for which $f_1(x_2) = x_2,\ g_2(x_1) = 1,\ f_2(x_2) = -cx_2,\ g_1(x_1) = kx_1$.

Example 1: For the following system

$$\begin{aligned} \dot{x}_1 &= x_2^3\\ \dot{x}_2 &= -x_1^5 - x_1^2 x_2\end{aligned} \tag{13}$$

we have

$$\begin{aligned} &f_1(x_2) = x_2^3, \quad f_2(x_2) = -x_2,\\ &g_1(x_1) = x_1^5, \quad g_2(x_1) = x_1^2\end{aligned}$$

Therefore,

$$V(x_1, x_2) = \frac{x_1^6}{6} + \frac{x_2^4}{4} + \beta$$

Since one must have $V(0, 0) = 0$, this yields $\beta = 0$. The function V is obviously positive definite. The derivative over time of V is

$$\dot{V} = x_2^3(-x_1^2 x_2 - x_1^5) + x_1^5 x_2^3 = -x_1^2 x_2^4 \leq 0$$

LaSalle's Invariance Principle enables us to conclude that this system is asymptotically stable to the origin.

Remark 2: Note that if the Lyapunov function V were to be interpreted as the sum of kinetic and potential energies, the kinetic energy would not be a quadratic function of x_2 as in the standard literature on Euler-Lagrange systems. In Euler-Lagrange systems, the kinetic energy is defined as $K = 0.5\dot{q}^T M(q)\dot{q}$, where q is the vector of generalized positions, $\dot{q}$ is the vector of generalized velocities, and $M(q)$ is the generalized mass matrix of the system and is a positive definite matrix.

IV. MULTIPLE DEGREES OF FREEDOM

The objective of this section is to generalize the construction of a Lyapunov function to higher order systems of the form

$$\begin{aligned}\dot{x}_{i-1}(t) &= f_{i-1}(x_i)\\ \dot{x}_i(t) &= -g_{i-1}(x_{i-1}) - \textstyle\sum_{j\neq i} g_{ij}(\tau_{ij}) + \\ & \quad g_i(x_1,\dots,x_{2n-1}) f_i(x_2,\dots,x_{2n})\end{aligned} \tag{14}$$

where $i = 2m,\ m = 1,\dots,n$ is an even index, $\tau_{ij} = x_{i-1} - x_{j-1},\ i = 2p, j = 2m,\ p, m = 1,\dots,n$. The functions $f_i,\ i = 1,\dots,n$ are continuous and not identically zero. It is further assumed that the functions $f_{i-1},\ i = 2m,\ m = 1,\dots,n$ are class $\mathcal{C}^1$ and the functions g_i for i an even number are continuous and not identically zero. Both $g_{i-1},\ g_{ij},\ i = 2m, j = 2p,\ m, p = 1,\dots,n,\ j \neq i$ are assumed to be of class $\mathcal{C}^1$, with g_{i-1} not identically zero. Note that when there is only one degree of freedom, $i = 2$ and the system (14) transforms to the system (6). The main result of this section is presented next.

Theorem 2: For system (14) define a class $\mathcal{C}^1$ function

$$V = \sum_i \Big(\int g_{i-1} dx_{i-1} + \int f_{i-1} dx_i + \sum_{j>i} \int g_{ij} d\tau_{ij} \Big) \tag{15}$$

Then if

$$g_{ij}(\tau_{ij}) = -g_{ji}(\tau_{ij}), \tag{16}$$

for $i = 2m, j = 2p,\ m, p = 1,\dots,n,\ j \neq i$ and

$$\sum_i g_i f_{i-1} f_i \leq 0 \tag{17}$$

the system (14) is stable and V is a Lyapunov function, provided V is positive definite and radially unbounded. In particular, the condition (17) can be met if either

$$g_i \geq 0,\ f_{i-1} f_i \leq 0, \tag{18}$$

or

$$g_i \leq 0,\ f_{i-1} f_i \geq 0, \tag{19}$$

for $i = 2m,\ m = 1,\dots,n$. If furthermore

- $g_i = g$ for all $i = 2m,\ m = 1,\dots,n$,
- $g = 0$ implies $(x_1, x_3, \dots, x_{2n-1}) = 0$,
- $f_{i-1}(x_i) = 0$ implies $x_i = 0$ for each $i = 2m,\ m = 1,\dots,n$,
- $f_{i-1} f_i \geq 0$ or $f_{i-1} f_i \leq 0$, for all $i = 2m,\ m = 1,\dots,n$,
- $x_i = 0$ for a given $i = 2m,\ m = 1,\dots n$, implies $f_i = 0$ for the same index i,
- $f_i = 0$ for all $i = 2m,\ m = 1,\dots n$ implies $(x_2, x_4, \dots, x_{2n}) = 0$,
- $g_{i-1}(x_{i-1}) = -\sum_{j\neq i} g_{ij}(\tau_{ij})$ for $i = 2m, j = 2p, m, p = 1,\dots n,\ j \neq i$, implies $(x_1, x_3, \dots, x_{2n-1}) = 0$

then the system (14) is asymptotically stable to the origin.

Proof.

$$\begin{aligned}\dot{V} &= \sum_i [g_{i-1} f_{i-1} + f_{i-1}(-g_{i-1} - \sum_{j\neq i} g_{ij} \\ &\quad + g_i f_i)] + \sum_i \sum_{j>i} g_{ij}\left(\dot{x}_{i-1} - \dot{x}_{j-1}\right)\end{aligned} \tag{20}$$

where, as before, i, j are indices that can only take even values. Using now the skew-symmetry property (16) one can see that

$$\sum_i \left[-f_{i-1} \sum_{j\neq i} g_{ij} + \sum_{j>i} g_{ij}\left(f_{i-1} - f_{j-1}\right) \right] = 0 \tag{21}$$

and therefore from (21) expression (20) yields

$$\dot{V} = \sum_i g_i f_{i-1} f_i \leq 0,$$

which proves that the system is globally stable under the assumption that V is positive definite and radially unbounded. If $g_i = g$ the derivative of V over time becomes

$$\dot{V} = g \sum_i f_{i-1} f_i \leq 0, \tag{22}$$

Using LaSalle's Invariance Principle, the set to analyze is $M = M_1 \cup M_2$, where

$$\begin{aligned} M_1 &= \{(x_1, x_3, \dots, x_{2n-1}) \mid g = 0\} \\ M_2 &= \{(x_2, x_4, \dots, x_{2n}) \mid \sum_i f_{i-1} f_i = 0\}\end{aligned}$$

In M_1 one has $x_1 = x_3 = \ldots = x_{2n-1} = 0$ and therefore $\dot{x}_1 = \dot{x}_3 = \ldots = \dot{x}_{2n-1} = 0$, which implies $f_{i-1}(x_i) = 0,\ i = 2m,\ m = 1,\ldots,n.$ This in turn implies $x_2 = x_4 = \ldots = x_{2n} = 0$. In M_2, since $f_{i-1}f_i \geq 0$ or $f_{i-1}f_i \leq 0$ for all $i = 1,\ldots,M$ and $\sum_i f_{i-1}f_i = 0$, one must have $f_{i-1}f_i = 0,\ i = 2m,\ m = 1,\ldots,n$. We now show that this implies $f_i = 0,\ i = 2m,\ m = 1,\ldots,n$. In fact, if for some index $i = 2m,\ m = 1,\ldots,n$ one has $f_{i-1} = 0$, then $x_i = 0$, which implies $f_i = 0$ for the same index i. Therefore, one has $f_i = 0,\ i = 2m,\ m = 1,\ldots,n,$ and consequently $(x_2,\ldots,x_{2n}) = 0$, which implies $(\dot{x}_2,\ldots,\dot{x}_{2n}) = 0$. Under these conditions, $g_{i-1}(x_{i-1}) = -\sum_{j\neq i} g_{ij}(\tau_{ij})$ for $i = 2m, j = 2p, m, p = 1,\ldots n,\ j \neq i$, which implies $(x_1, x_3, \ldots, x_{2n-1}) = 0$. Therefore, the largest invariant set of the trajectories of system (14) contained in M is the origin. This finishes the proof. $\square$

Remark 3: Note that the Lyapunov function will clearly be positive definite if the antiderivatives of $f_{i-1},\ g_{i-1},\ g_{ij},\ i = 2m, j = 2p,\ m, p = 1,\ldots n,\ j \neq i$ are even functions of its arguments.

Example 2: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= -x_1^5 - (x_1 - x_3)^5 - (x_1^2 + x_3^2)x_2 \\ \dot{x}_3 &= x_4^3 \\ \dot{x}_4 &= -x_3^5 - (x_3 - x_1)^5 - (x_1^2 + x_3^2)x_4 \end{aligned} \tag{23}$$

Using the results of Theorem 2 we consider the candidate Lyapunov function

$$V = \frac{x_2^4}{4} + \frac{x_4^4}{4} + \frac{x_1^6}{6} + \frac{x_3^6}{6} + \frac{(x_1 - x_3)^6}{6}$$

where the constants have been found such that $V(0) = 0$. We note that V is clearly positive definite. Taking its derivative over time yields

$$\begin{aligned} \dot{V} &= x_2^3\left[-x_1^5 - (x_1 - x_3)^5 - (x_1^2 + x_3^2)x_2\right] \\ &+ x_4^3\left[-x_3^5 - (x_3 - x_1)^5 - (x_1^2 + x_3^2)x_4\right] \\ &+ x_1^5x_2^3 + x_3^5x_4^3 + (x_1 - x_3)^5x_2^3 - (x_1 - x_3)^5x_4^3 \end{aligned}$$

After simplification one gets

$$\dot{V} = -(x_1^2 + x_3^2)\left(x_2^4 + x_4^4\right) \leq 0$$

and therefore the system is stable. LaSalle's Invariance Principle allows one to conclude that the system is asymptotically stable to the origin.

V. CONCLUSIONS

The main conclusion of the work presented here is that knowledge of physics, in particular the relationship between the Lagrangian and the energy of a physical system, can be used to generalize the construction of a physical energy function to an abstact Lyapunov function for pseudo Euler-Lagrange systems. In pseudo Euler-Lagrange systems, the generalized velocities are not necessarily the derivative of generalized positions but there exists a nonlinear functional relationship between them. Additionaly, an important outcome of this work is that the Lyapunov function proposed in this paper is obtained directly from the mathematical structure of the differential equations describing the system, without the need to determine any kinetic or potential energy of the system first. The method proposed here therefore works for any mathematical model in the class of pseudo Euler-Lagrange systems and is not restricted to models of physical systems, having thus the potential to address economic, biologic and other systems.

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