Preprints of the 2012 20th Mediterranean Conference on Control & Automation (MED), Barcelona, Spain, July 3-6, 2012

Adaptive Locally Homogeneous Control of Input Affine Nonlinear Systems and Its Application to Magnetic Levitation System Control

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***Abstract*— We have recently proposed a locally homogeneous stabilizing controller that guarantees a convergence rate for input affine nonlinear control systems. However, adaptive control problem has not been discussed. In this research, we propose an adaptive homogeneous controller for homogeneous control systems. Consecutively, we propose an adaptive locally homogeneous controller with a convergence rate guarantee for input affine nonlinear systems. We apply our proposed controller to a stabilization problem of a magnetic levitation system. We confirm the effectiveness of the proposed method by an experiment.**

I. INTRODUCTION

Convergence speed guarantee of stabilizing controllers is an important problem in nonlinear control theory. Recently, we have proposed a locally homogeneous controller that guarantees a convergence rate owing to local homogeneity [7]. However, we did not discuss parameter adaptive mechanisms in [7], although adaptive control is crucial in practical control problems; i.e., an integral action is necessary such as PID control.

Convergence speed guarantee also attracts attention in adaptive control [1]. Hong et al. proposed a homogeneous adaptive controller that dominates effects of unknown parameters in control systems [3]. However, a stabilizatin problem at a point that is not an equilibrium of the autonomous system is not discussed.

Hence, we propose an adaptive homogeneous controller for stabilization of homogeneous system at an operating point that is not an equilibrium in the paper. Then, we extend the method to an adaptive locally homogeneous controller that guarantees the convergence rate.

Finally, we apply our proposed method to a magnetic levitation system. We confirm the effectiveness of the proposed method by an experiment.

II. PRELIMINARIES

In this section, we introduce definitions and important properties of stability, convergence rate, control Lyapunov functions (CLF), dilated homogeneity and homogeneous approximations.

Throughout the paper, $\|\cdot\|$ denotes a Euclidean norm, $\mathbb{R}_{\geq 0} := [0, +\infty) \subset \mathbb{R}$, $B_\delta^n := \{x \in \mathbb{R}^n | \|x\| \leq \delta\}$, and $S^{n-1} := \{x \in \mathbb{R}^n | \|x\| = 1\}$.

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A. Stability and Convergence Rate

In this subsection, we show definitions of stability and convergence rates. We consider the following time-invariant differential equation:

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $f(0) = 0$. Stability of the origin of (1) and the convergence rates are defined as follows.

Definition 1 (Stability) *The origin of (1) is said to be*

1) *stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq 0; \tag{2}$$

2) *globally asymptotically stable if the origin is stable and all solutions $x(t)$ satisfy the following equation:*

$$\lim_{t\to\infty} \|x(t)\| = 0. \tag{3}$$

Definition 2 (Convergence Rate) *The origin of (1) is said to be*

1) *rationally stable if the origin is stable and there exist positive constants $\delta, b_1, b_2 > 0$ and $0 < \eta \leq 1$ such that*

$$\|x(t)\| \leq b_1(1 + \|x_0\|^{b_2} t)^{1/b_2} \|x(0)\|^\eta, \quad \forall t \geq 0, \forall x(0) \in B_\delta; \tag{4}$$

2) *exponentially stable if the origin is stable and there exist positive constants $\delta, b_1, b_2 > 0$ such that*

$$\|x(t)\| \leq b_1 e^{-b_2 t} \|x(0)\|, \quad \forall t \geq 0, \forall x(0) \in B_\delta; \tag{5}$$

3) *finite-time stable if the origin is stable and there exist a positive constant $\delta > 0$ and a function $T : B_\delta \backslash \{0\} \to \mathbb{R}_{>0}$ such that*

$$\lim_{t\to T(x(0))} x(t) = 0, \quad \forall x(0) \in B_\delta. \tag{6}$$

B. Control Lyapunov Function

This paper discusses problems of adaptive control by using control Lyapunov functions. We consider the following input affine nonlinear system:

$$\dot{x} = f(x) + g(x)u, \tag{7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous, and $f(0) = 0$. Let respective $g_i(x)$ and $g^j(x)$ denote the i-th row vector and the j-th column vector of $g(x)$.

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The control Lyapunov function (CLF) is defined as follows:

Definition 3 (Control Lyapunov Function) *A C^1 propser positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function (CLF) for system (7) if*

$$\inf_{u\in\mathbb{R}^m}[L_f V + L_g V \cdot u] < 0 \quad (\forall x \in \mathbb{R}^n\backslash\{0\}), \tag{8}$$

where $L_f V := \partial V/\partial x \cdot f$ and $L_g V := \partial V/\partial x \cdot g$.

C. Dilated Homogeneity

Definition 4 (Dilation) *The mapping $\Delta_\varepsilon^r : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows is said to be a dilation:*

$$\Delta_\varepsilon^r x := (\varepsilon^{r_1}x_1, \cdots, \varepsilon^{r_n}x_n), \tag{9}$$

where $\varepsilon > 0$ and $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n$ ($r_i > 0,\ 1 \leq i \leq n$).

Note that we often refer r in the dilation mappings as "dilation exponent."

Definition 5 (Homogeneous Function) *A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a homogeneous function of degree $k \in \mathbb{R}$ with respect to a dilation exponent r if the following equality holds for all $\varepsilon > 0$:*

$$V(\Delta_\varepsilon^r x) = \varepsilon^k V(x). \tag{10}$$

Definition 6 (Homogeneous System) *Consider an input affine nonlinear system (7). (7) is said to be homogeneous of degree $\tau \in \mathbb{R}$ with respect to a dilation exponent (r, s) if the following equality holds for all $\varepsilon > 0$:*

$$f(\Delta_\varepsilon^r x) + g(\Delta_\varepsilon^r x)\Delta_\varepsilon^s u = \varepsilon^\tau \Delta_\varepsilon^r[f(x) + g(x)u]. \tag{11}$$

Definition 7 (Homogeneous Feedback) *Consider a homogeneous system (7) of degree τ with respect to dilation exponent (r, s). Then, the feedback $u(x)$ such that the following equality holds is said to be a homogeneous feedback:*

$$u(\Delta_\varepsilon^r x) = \Delta_\varepsilon^s u(x). \tag{12}$$

Definition 8 (Homogeneous Approximation) *Consider a function $V : \mathbb{R}^n \to \mathbb{R}$. A homogeneous function $V_h : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous approximation of V with respect to a dilation exponent r if there exists a function $V_o : \mathbb{R}^n \to \mathbb{R}$ such that the following two conditions are satisfied uniformly on S^{n-1}.*

$$V(x) = V_h(x) + V_o(x), \tag{13}$$

$$\lim_{\varepsilon\to 0}\frac{V_o(\Delta_\varepsilon^r x)}{\varepsilon^k} = 0. \tag{14}$$

In the same manner, the homogeneous system

$$\dot{x} = f_h(x) + g_h(x)u \tag{15}$$

of degree k with respect to a dilation exponent (r, s) is said to be a homogeneous approximation of the system

$$\dot{x} = f(x) + g(x)u \tag{16}$$

with respect to a dilation exponent (r, s) if there exist vector fields f_o and $g_o : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that the following two conditions are satisfied uniformly on $S^{n+m-1} := \{(x, u) \in \mathbb{R}^{n+m} | \|(x, s)\| = 1\}$

$$f(x) + g(x)u = f_h(x) + g_h(x)u + f_o(x) + g_o(x)u, \tag{17}$$

$$\lim_{\varepsilon\to 0}\frac{f_{o,i}(\Delta_\varepsilon^r x) + g_{o,i}(\Delta_\varepsilon^r x)\delta_\varepsilon^s u}{\varepsilon^{\tau+r_i}} = 0, \quad \forall i = 1, \ldots, n. \tag{18}$$

The following lemma (Corollary 5.5 in [2] with taking into consideration of Lemma 2 in [7]) plays a central role in the homgeneity-based control systems design.

Lemma 1 *Consider a differential equation $\dot{x} = f(x)$, where f has a homogeneous approximation $\dot{x} = f_h(x)$ of degree τ with respect to dilation exponent r. Assume that the origin of the homogeneous differential equation $\dot{x} = f_h(x)$ is asymptotically stable. Then,*

1) *if $k > 0$ the origin is rationally stable;*
2) *if $k = 0$ the origin is exponentially stable;*
3) *if $k < 0$ the origin is finite-time stable.*

Note that Lemma 1 was implicitly applied to the proof of Theorem 2 in [7].

For homogeneous systems design, Young's inequality [3] is commonly used. We show another useful lemma for the purpose:

Lemma 2 (Generalized Discriminant) *Consider the following C^1 homogeneous function $V(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ of degree k with respect to dilation exponent $(r_1, 1)$:*

$$V(x_1, x_2) := a|x_1|^{\frac{k}{r_1}} + b|x_1|^{\frac{k-1}{r_1}} \operatorname{sgn} x_1 \cdot x_2 + c|x_2|^k, \tag{19}$$

where $a, c > 0$ are positive constants and $b \in \mathbb{R}$.

Then, $V(x)$ is positive definite if and only if the following condition holds:

$$k^k a^{k-1}c > (k-1)^{k-1}|b|^k. \tag{20}$$

Proof: Let x_1 be a constant x_{10}. Then, the partial derivative of V with respect to x_2 is obtained as follows:

$$\frac{\partial V}{\partial x_2}(x_{10}, x_2) = b|x_{10}|^{\frac{k-1}{r_1}} \operatorname{sgn} x_{10} + ck|x_2|^{k-1}\operatorname{sgn} x_2. \tag{21}$$

Note that $\partial V/\partial x_2 = 0$ at $x_2 = x_{20} := -(|b|/ck)^{1/(k-1)}|x_{10}|^{1/r_1}\operatorname{sgn}(bx_{10})$, $\partial V/\partial x_2 > 0$ for all $x_2 < x_{20}$, and $\partial V/\partial x_2 < 0$ for all $x_2 > x_{20}$. Then, for every x_{10}, x_{20} takes the global minimum of V. $V(x_{10}, x_{20})$ is calculated as follows:

$$V(x_{10}, x_{20}) = \left[a + \left(\frac{1-k}{k^{k/(k-1)}}\right)\frac{|b|^{k/(k-1)}}{c^{1/(k-1)}}\right]|x_{10}|^{k/r_1} \tag{22}$$

Therefore, if $k^k a^{k-1}c > (k-1)^{k-1}|b|^k$, $V(x_{10}, x_{20}) > 0$ for all x_{10}. Otherwise, there exists x_{10} and x_{20} such that $V(x_{10}, x_{20}) < 0$. ■

Remark 1 *For $r_1 = 1$ and $k = 2$, (20) becomes $4ac > b^2$. This is clearly the discriminant of the quadratic function.*

III. Adaptive Homogeneous Control of Homogeneous Systems

A. Problem Statement

This section aims to develop an adaptive homogeneous controller for the following control system:

$$\dot{x} = f(x) + g(x)\theta + g(x)u, \tag{23}$$

where $x \in \mathbb{R}^n$ is a state, $\theta \in \mathbb{R}^m$ is an unknown parameter vector, $u \in \mathbb{R}^m$ is an input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous, $f(0) = 0$, $g^j(0) \neq 0$ for all $j = 1, \ldots, m$ and constant vectors $g^j(0)$ are linear independent.

Remark 2 *Note that we assume $g(0) \neq 0$. This does not mean that g is always a constant matrix. For example,*

$$\dot{x}_1 = x_2^3 + x_2^2 u \tag{24}$$

$$\dot{x}_2 = u, \tag{25}$$

is homogeneous of degree 0 with respect to dilation exponent $r = (3, 1)$ and $s = 3$, $g(0) \neq 0$, and $g(x)$ is not a constant matrix.

In this section, we suppose the following assumption for (23)

Assumption 1 *1) The following control system is homogeneous of degree τ with respect to a dilation exponent (r, s):*

$$\dot{x} = f(x) + g(x)u, \tag{26}$$

where $s = (s_1, \ldots, s_1)$ and $\tau + s_1 > 0$; that is, the dilation exponent regarding the input is supposed to consist of the same elements.

2) A C^1 homogeneous control Lyapunov function (CLF) V of degree k with respect to r for stabilization of the origin of (26) is known.

The objective of the section is to design an adaptive controller that asymptotically stabilizes the origin $x = 0$ of (23) with a certain convergence rate. Note that the case that $g(0) \neq 0$ is included in the paper; the adaptive control discussed in [3] cannot be applied to our problem.

We consider the following adaptive control law for (23).

$$\begin{aligned} u &= u_s - \hat{\theta}, \\ \dot{\hat{\theta}} &= \beta(x) = [\beta_1(x), \ldots, \beta_m(x)]^T, \end{aligned} \tag{27}$$

where $\hat{\theta}$ denotes an estimate of θ.

Let $e = \theta - \hat{\theta}$, and the closed-loop system of (23) with the adaptive control law (27) is obtained as follows:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u_s + g(x)e, \\ \dot{e} &= \beta(x). \end{aligned} \tag{28}$$

If the system (28) is homogeneous and the origin $(x, e) = (0, 0)$ is globally asymptotically stable, we can guarantee the convergence rate. The condition for the homogeneity of the system (28) is summarized as follows:

Lemma 3 *The system (28) is homogeneous of degree τ with respect to dilation exponent (r_a, s) if and only if function $\beta_i(x)$ is a homogeneous function of degree $\tau + s_1$ with respect to the dilation exponent $(r_a := (r, s), s)$ for every $i = 1, \ldots, m$.*

The proof is obvious and omitted.

Remark 3 *Continuous β is preferred for practical implementations and mathematical analyses. $\tau + s_1 > 0$ in Assumption 1 is supposed for the realizability of continuous homogeneous β. If the assumption is not satisfied, there does not exist a continuous β when $\tau + s_1 < 0$ and $\beta \equiv 0$ is the only continuous homogeneous β when $\tau + s = 0$. Note that the condition $\tau + s_1 > 0$ may become a problem when $\tau < 0$.*

B. Adaptive Controller Design

This section proposes a new adaptive controller as follows:

Theorem 1 *Consider a control system (23) satisfying Assumption 1.*

Then, the following controller is continuous with respect to x and asymptotically stabilizes the origin of (28):

$$u = u_s(x) - \hat{\theta}, \tag{29}$$

$$\dot{\hat{\theta}} = \beta(x) = V^{\frac{2s_1}{k}-1}\Gamma(L_g V)^T, \tag{30}$$

where $u_s(x)$ is a continuous homogeneous feedback (of degree s) such that $L_f V + L_g V \cdot u_s(x) < 0$ for all $x \neq 0$, and Γ is a positive definite symmetric matrix.

Moreover, the following properties hold.

1) *If $\tau > 0$, the origin is rationally stable.*
2) *If $\tau = 0$, the origin is exponentially stable.*
3) *If $\tau < 0$, the origin is finite-time stable.*

Proof: Since V is C^1, $\partial V/\partial x$ is continuous. Moreover, $(\partial V/\partial x)g^i(x)$ is a homogeneous function of degree $k + \tau - s_1$ with respect to dilation exponent r for all $i = 1, \ldots, m$, and $V^{(2s_1/k)-1}$ is a homogeneous function of degree $2s_1 - k$. Hence, (30) is homogeneous of degree $\tau + s_1$.

Then, if we show (30) is continuous and the origin of the closed-loop system (28) is asymptotically stable, we can guarantee the convergence rate by Lemma 1.

We show continuity of (30). Due to the fact that $\partial V/\partial x$ is continuous, β is continuous except at the origin. Then, we show the continuity of β at the origin. Recall that the fact (30) is homogeneous of degree $\tau + s_1$. This implies that $\beta \to 0$ uniformly as $x \to 0$ due to the assumption that $\tau + s_1 > 0$. Therefore, β is continuous on $\mathbb{R}^n$.

We show asymptotic stability of the origin of (28). Consider the following continuous positive definite proper function V_a:

$$V_a(x, e) = V^{\frac{2s_1}{k}}(x) + \frac{s_1}{k} e^T \Gamma^{-1} e. \tag{31}$$

Note that the following properties of V_a.

1) V_a is homogeneous of degree $2s$ with respect to dilation exponent (r, s).
2) V_a is differentiable on $x \neq 0$.
3) V_a may not differentiable on $x = 0$.

For each $x \neq 0$, $\dot{V}_a$ is calculated as follows:

$$\begin{aligned}\dot{V}_a(x,e) &= \frac{2s_1}{k} V^{\frac{2s}{k}-1} \left(\frac{\partial V}{\partial x}(f(x) + g(x)u_s(x)) \right) \\ &\quad + \frac{2s_1}{k} V^{\frac{2s_1}{k}-1} \frac{\partial V}{\partial x} g(x)e \\ &\quad - \frac{2s_1}{k} V^{\frac{2s_1}{k}-1} \frac{\partial V}{\partial x} g(x)e \quad (32) \\ &= \frac{2s_1}{k} V^{\frac{2s}{k}-1} \left(\frac{\partial V}{\partial x}(f(x) + g(x)u_s(x)) \right). \quad (33)\end{aligned}$$

$(\partial V/\partial x \cdot (f(x)+g(x)u_s(x)))$ is a homogeneous function of degree $k+\tau$ according to [6], and $V^{2s/k-1}$ is homogeneous of dgree $2s-k$. Thus, $V^{2s/k-1}(\partial V/\partial x \cdot (f(x)+g(x)u_s(x)))$ is homogeneous of degree $2s + \tau$. Due to the assumption that $s + \tau > 0$, $V^{2s/k-1}(\partial V/\partial x \cdot (f(x) + g(x)u_s(x))) \to 0$ uniformly as $x \to 0$.

Therefore, V_a itself is not differentiable; however, $\dot{V}_a(x, e)$ is continuous entire on $\mathbb{R}^{n+m}$ due to the above discussion and the continuity of β. Then, we can obtain the following inequality:

$$\dot{V}_a = \frac{2s_1}{k} V^{\frac{2s}{k}-1} \left(\frac{\partial V}{\partial x}(f(x) + g(x)u_s(x)) \right) \leq 0. \quad (34)$$

Then, $x(t) \to 0$ as $t \to +\infty$ for every solutions $x(t)$ by Barbalat's lemma [4].

Note that if $e \neq 0$, $x = 0$ is not an equilibrium by the assumptions that $g^j(0) \neq 0$ for all $j = 1, \ldots, m$. Due to the fact that constant vectors $g^j(0)$ are linear independent and $x(t) \to 0$ as $t \to +\infty$, $e \to 0$ as $t \to +\infty$.

Consequently, the origin of (28) is stable and all solutions $(x(t), e(t)) \to 0$. Hence, the origin of (28) is asymptotically stable.

The system (28) is continuous and homogeneous of degree τ with respect to the dilation exponent (r_a, s), and the origin is asymptotically stable. Therefore, the theorem is proved by Lemma 1. ■

Remark 4 *We can find the necessity of the assumption that* $s = (s_1, \ldots, s_1)$ *in (31). If we apply the usual discussion of CLF based adaptive control, quadratic* e *in (31) is necessary. This is equivalent to the statement that* s *consists of the same element* s_1.

According to Theorem 1, the following corollary holds:

Corollary 1 *Consider the following input affine control system satisfying Assumption 1 and adaptive control law:*

$$\dot{x} = f(x) + g(x)H\theta + g(x)u, \quad (35)$$

$$u = u_s(x) - \hat{\theta}, \quad (36)$$

$$\dot{\hat{\theta}} = \beta(x) = V^{\frac{2s_1}{k}-1} \Gamma H^T g^T(x) \left(\frac{\partial V}{\partial x} \right)^T, \quad (37)$$

where, $x \in \mathbb{R}^n$*,* u *and* $\theta \in \mathbb{R}^m$*,* f *and* g *are continuous,* $f(0) = 0$*,* H *is a nonsingular matrix,* $u_s(x)$ *is a continuous homogeneous feedback (of degree* s*) such that* $L_f V + L_g V \cdot u_s(x) < 0$ *for all* $x \neq 0$*, and* Γ *is a positive definite symmetric matrix.*

Then, the origin of the following system is globally asymptotically stable:

$$\dot{x} = f(x) + g(x)u_s(x) + g(x)He, \quad (38)$$

$$\dot{e} = \beta(x). \quad (39)$$

Moreover, the following holds.

1) If $\tau < 0$*, the origin of (45) is finite-time stable.*
2) If $\tau = 0$*, the origin of (45) is exponentially stable.*
3) If $\tau > 0$*, the origin of (45) is rationally stable.*

IV. Adaptive Locally Homogeneous Control

In the previous section, we proposed an adaptive homogeneous control law for homogeneous control systems having $s = (s_1, \ldots, s_1)$. This section consecutively proposes an adaptive locally homogeneous control law for non-homogeneous input-affine nonlinear control systems.

We consider the following input affine control system

$$\dot{x} = f(x) + g(x)h(x)\theta + g(x)u, \quad (40)$$

where $x \in \mathbb{R}^n$, $u, \theta \in \mathbb{R}^m$, f, g, h are continuous, $f(0) = 0$, $g^j(0) \neq 0$ for all $j = 1, \ldots, m$, constant vectors $g^j(0)$ are linear independent and $h(x)$ is nonsingular for each x.

In this paper, we suppose the following assumption for system (43):

Assumption 2 *1) The following system is supposed to have a homogeneous approximation of degree* τ *with respect to dilation exponent* (r, s)*:*

$$\dot{x} = f(x) + g(x)u, \quad (41)$$

where $s = (s_1, \ldots, s_1)$*.*

2) A C^1 *function* V *is a CLF for (41).*
3) V *has a homogeneous approximation* V_h *with respect to dilation exponent* r*.*
4) V_h *is a* C^1 *homogeneous CLF for the following homogeneous system:*

$$\dot{x} = f_h(x) + g_h(x)u. \quad (42)$$

Note that the first condition of Assumption 2 implies that (40) can be rewritten as

$$\begin{aligned}\dot{x} = &f_h(x) + g_h(x)h(x)\theta + g_h(x)u \\ &+ f_o(x) + g_o(x)u + g_o(x)h(x)\theta. \quad (43)\end{aligned}$$

By the same discussion as Corollary 1 and [7], we can obtain the following main theorem of the paper:

Theorem 2 *Consider system (40) satisfying Assumption 2.*

Then, the following adaptive controller is continuous with respect to x:

$$\begin{aligned}&u = u_s(x) - \hat{\theta}, \\ &\dot{\hat{\theta}} = \beta(x) = V^{\frac{2s_1}{k}-1} \Gamma h^T(x)(LgV)^T,\end{aligned} \quad (44)$$

where $u_s(x)$ globally asymptotically stabilizes the origin of (41) and has a locally homogeneous of degree s, and Γ is an arbitrary positive definite symmetric matrix.

Moreover, (44) asymptotically stabilizes the origin of the following closed-loop system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u_s(x) + g(x)h(x)e \\ \dot{e} &= \beta(x),\end{aligned} \tag{45}$$

where $e = \theta - \hat{\theta}$.

Further, the following holds.

1) *If $\tau < 0$, the origin of (45) is finite-time stable.*
2) *If $\tau = 0$, the origin of (45) is exponentially stable.*
3) *If $\tau > 0$, the origin of (45) is rationally stable.*

Proof: Note that the homogeneous approximation β_h of β is obtained as follows:

$$\beta_h(x) = V_h^{\frac{2s_1}{k}-1}\Gamma h^T(0) g_h^T(x)\left(\frac{\partial V_h}{\partial x}\right)^T. \tag{46}$$

By the same discussion of the proof of Theorem 1, β_h is homogeneous of degree $\tau + s_1$. Then, the closed-loop system (45) has a homogeneous approximation of degree τ with respect to dilation exponent$((r, s), s)$. Hence, if the closed-loop system (45) is globally asymptotically stable, the convergence rate of the closed-loop system is guaranteed by Lemma 1.

We consider the following function V_a:

$$V_a = V^{\frac{2s_1}{k}} + \frac{s_1}{k}e^T\Gamma^{-1}e. \tag{47}$$

Note that $\dot{V}_a$ is continuous by the same discussion as the proof of Theorem 1. Then,

$$\begin{aligned}\dot{V}_a &= \frac{2s_1}{k}V^{\frac{2s}{k}-1}\left(\frac{\partial V}{\partial x}(f(x) + g(x)u_s(x))\right) \\ &\quad + \frac{2s_1}{k}V^{\frac{2s}{k}-1}\frac{\partial V}{\partial x}g(x)h(x)e \\ &\quad - \frac{2s_1}{k}V^{\frac{2s}{k}-1}\frac{\partial V}{\partial x}g(x)h(x)e \qquad (48) \\ &= \frac{2s_1}{k}V^{\frac{2s}{k}-1}\left(\frac{\partial V}{\partial x}(f(x) + g(x)u_s(x))\right) < 0. \qquad (49)\end{aligned}$$

Taking into consideration of Barbalat's lemma [4] and the assumption that $g^j(0) \neq 0$ for all $j = 1, \ldots, m$, the fact that the origin of (45) is asymptotically stable is proved. ■

Remark 5 *Note that the locally homogeneous controller proposed in [7] can be used as u_s.*

V. Magnetic Levitation System

In this section, we apply our proposed adaptive locally homogeneous control law to a magnetic levitation system.

TABLE I

Identified Parameters

m [kg]	μ [-]	g_0 $[m/s^2]$	a [V/N · m ^{4}]	b [m]
0.12	4.5	9.80665	40118.9	0.056464

A. Problem Statement

The magnetic levitation system considered in the paper is illustrated in Fig. 1. We consider the stabilization problem of the disc to the desired operating point ξ^*[m] using attractive force generated by the upper magnetic coil. In this case, the magnetic levitation system can be modeled as Fig. 2, where ξ[m] is the position of the magnet from the upper coil, F_u[N] denotes an attractive force for the disc generated by the upper drive magnetic coil, m[kg] is the mass of the disc, μ[-] represents a friction constant and g_0[m/s^2] is the gravitational acceleration.

The dynamical equation for the disc illustrated in Fig. 2 is obtained as

$$m\ddot{\xi} = F_u - m\mu\dot{\xi} - mg_0, \tag{50}$$

where the magnetic force F_u can be modeled by

$$F_u = \frac{u}{a(-\xi + b)^4}, \tag{51}$$

where a and b are constants.

Let ξ^*[m] be a desired operating point of the disc and introduce a new variable $x = [x_1, x_2]^T$, where $x_1 = \xi - \xi^*$[m] and $x_2 = \dot{x}_1$[m/s]. According to (50) and (51), we can obtain the following state equation:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 - g_0 + \frac{u}{ma(-x_1 - \xi^* + b)^4}.\end{aligned} \tag{52}$$

We show parameters in system (56) in Table I.

Throughout the section, we use the following notations.

$$f(x) = \begin{bmatrix} x_2 \\ -\mu x_2 \end{bmatrix}, \tag{53}$$

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{1}{ma(-x_1 - \xi^* + b)^4} \end{bmatrix}, \tag{54}$$

$$h(x) = -ma(-x_1 - \xi^* + b)^4 g_0. \tag{55}$$

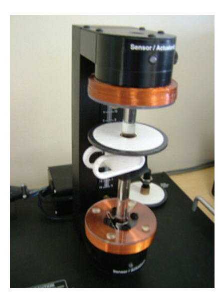

Fig. 1. Magnetic Levitation System

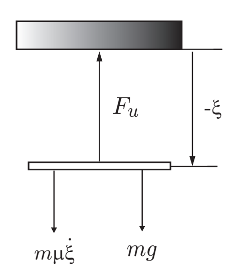

Fig. 2. System Setting

Then, (52) is equivalent to

$$\dot{x} = f(x) + g(x)h(x)\theta + g(x)u, \tag{56}$$

where $\theta = 1$. Therefore, the problem of the paper is to design an asymptotic stabilizing controller for (52).

Note that $f(0) + g(0)h(0)\theta \neq 0$ in (56). This means that the locally homogeneous control cannot be applied directly to the problem of the paper. Then, we apply our proposed adaptive control to the system.

B. Loaclly Homogeneous Control Lyapunov Function Design

Consider the following homogeneous function V of degree $k = 4$ with respect to dilation exponent $r = [4, 3]$.

$$V = 10|x_1|^3 + |x_1|^{\frac{9}{4}} \operatorname{sgn} x_1 \cdot x_2 + \frac{1}{4000} x_2^4. \tag{57}$$

According to Lemma 2, V is positive definite and $L_f V < 0$ whenever $L_g V = 0$.

Note that every positive definite homogeneous function is proper, and V is a homogeneous CLF.

C. Adaptive Locally Homogeneous Control

In this subsection, we design an adaptive locally feedback for system (56). Let an estimate of θ be $\hat{\theta}$, $e_\theta = \theta - \hat{\theta}$, $\dot{\hat{\theta}} = \beta(x)$ and $u = u_s + ma(-x_1 - \xi^* + b)^4 g_0 \hat{\theta}$. Then, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu x_2 + \frac{1}{ma(-x_1 - \xi^* + b)^4} \\ &\qquad [u_s - mag(-x_1 - \xi^* + b)^4 e_\theta]] \\ \dot{e}_\theta &= \beta(x), \end{aligned} \tag{58}$$

According to Theorem 2 and Theorem 2 in [7], we can obtain the following control law.

$$u_s = \begin{cases} -\frac{11}{18}\left[\frac{L_f V}{|L_g V|^{\frac{11}{9}}} + \left|\frac{L_f V}{|L_g V|^{\frac{11}{9}}}\right| + c\right] \\ |L_g V|^{\frac{2}{9}} \operatorname{sgn}(L_g V) \\ \qquad\qquad\qquad\qquad (L_g V \neq 0) \\ 0 \qquad\qquad\qquad\quad (L_g V = 0), \end{cases} \tag{59}$$

$$\dot{\hat{\theta}} = \gamma \frac{g_0}{V^{\frac{2}{3}}} L_g V. \tag{60}$$

D. Experimental Result

Figs. 3 and 4 illustrate the experimental result of the proposed method. We confirm that the proposed controller asymptotically stabilizes the origin.

VI. Conclusion

In this paper, we propose an adaptive homogeneous controller for homogeneous systems and an adaptive locally homogeneous controller for general nonlinear systems.

We confirm the effectiveness of the proposed method by an experiment of a magnetic levitation system. Experimental results demonstrate the effectiveness of the proposed method.

We suppose some conditions in the paper; e.g., $s = (s_1, \ldots, s_1)$. Relaxation of the assumptions remains a challenge for future works.

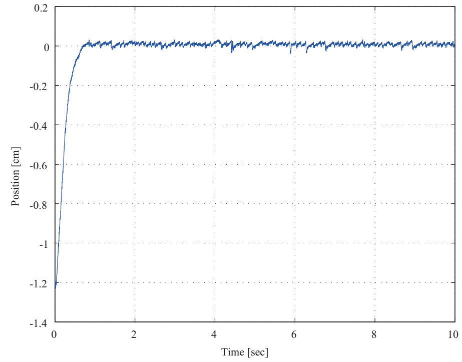

Fig. 3. Experimental Result: Adaptive Finite-time State

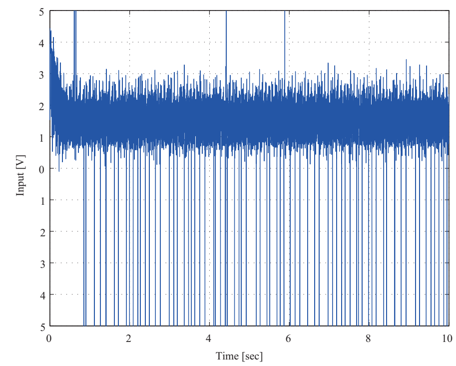

Fig. 4. Experimental Result: Adaptive Finite-time Input

References

[1] V. Adetola and M. Guay, “Performance improvement in adaptive control of linearly parametrized nonlinear systems,” IEEE Transactions on Automatic Control, vol. 55, no. 9, pp. 2182–2186, 2010.

[2] A. Bacciotti and L. Rosier, “Liapunov Functions and Stability in Control Theory,” Sprinver-Verlagm, Berlin, 2005.

[3] Y. Hong, Z.-P. Jiang and G. Feng, “Finite-time input-to-state stability and applications to finite-time control design,” SIAM Journal on Control and Optimization, vol. 48, no. 7, pp. 4395–4418, 2010.

[4] H. K. Khalil, “Nonlinear Systems,” 3rd ed., Prentice Hall, 2002.

[5] M. Kristić and H. Deng, “Stabilization of Nonlinear Uncertain Systems,” Springer-Verlag, London, 1998.

[6] N. Nakamura, H. Nakamura, Y. Yamashita and H. Nishitani, “Homogeneous stabilization of input affine homogeneuos systems,” IEEE Transactions on Automatic Control, vol. 54, no. 9, pp. 2271–2275, 2009.

[7] N. Nakamura, H. Nakamura and H. Nishitani, “Global inverse optimal control with guaranteed convergence rate of input affine nonlinear systems,” IEEE Transactions on Automatic Control, vol. 56, no. 2, pp. 358–369, 2011.