DIRECTIONAL CHANGE IN LQ CONTROL WITH ACTUATOR FAILURE

Dariusz Horla *

* *Poznan University of Technology, Institute of Control and Information Engineering, ul. Piotrowo 3a, 60-965 Poznan, POLAND,*
Dariusz.Horla@put.poznan.pl

Abstract: A discrete-time LQ control with actuator failure is considered with directional change in controls phenomenon taken into account, as a result of actuator saturation case included. The control problem is analyzed in terms of algebraic Ricatti equations. Computer simulations of two-input two-output system are given to illustrate the performance of the reliable LQ controller with respect to directional change phenomenon.

Keywords: LQ control. Multivariable control. Directional change.

1. INTRODUCTION

Reliable LQ control is an area of the recent research, as it has been described in [2]. The LQ control problem with failure of actuator has been described in [3]. General ideas on applications of fault-tolerant control laws have been discussed in [4,3,2].

In this paper, a discrete-time LQ control problem with actuator failure modeled by a scaling factor is considered. The aim of the paper is to check how does the control law behave in case of no directional controls requirement for a plant with equal number of inputs and outputs.

The directional change phenomenon, as it has been described in [1] takes place when saturation nonlinearity is taken into account, changing the proportions in between control vector components. Such a change of direction usually relates to decoupling, but there are applications (e.g. in robotics of chemical processes, ratio control), where control vector must be of unchanged direction, despite active constraints.

It is therefore of value to check in what way the requirement of no directional change can be related to control performance in the case of modeled actuator failure, describing in an indirect way effects of nonlinearity. In this case, amplitude and simultaneous rate constraints imposed on the control vector.

The novelty of the paper is the introduction of directional change notion to static controllers, as LQ, absent in the literature.

2. PLANT AND FAILURE MODEL

The following plant model is taken into consideration

$$\underline{x}_{t+1} = \boldsymbol{A}\underline{x}_t + \boldsymbol{B}\underline{u}_{t-d}\,, \tag{1}$$

$$\underline{y}_t = \boldsymbol{C}\underline{x}_t\,, \tag{2}$$

where the state-space representation

$$\boldsymbol{A} = \begin{bmatrix} -\boldsymbol{A}_1 & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \boldsymbol{0} \\ -\boldsymbol{A}_{n-1} & \cdots & \cdots & \boldsymbol{I} \\ -\boldsymbol{A}_n & \cdots & \cdots & \boldsymbol{0} \end{bmatrix}, \tag{3}$$

$$\boldsymbol{B} = [\boldsymbol{B}_1^T, \ldots, \boldsymbol{B}_n^T]^T\,, \tag{4}$$

$$\boldsymbol{C} = [\boldsymbol{I}, \boldsymbol{0}, \ldots, \boldsymbol{0}]^T \tag{5}$$

can be derived from left coprime representation of matrix of transfer functions with $d \geq 1$ as a dead-time, and submatrices in $\boldsymbol{A}$, $\boldsymbol{B}$ result from input-output representation using polynomial matrices.

The LQ control performance index

$$J = \sum_{t=0}^{\infty} \left(\underline{x}_t^T \boldsymbol{Q} \underline{x}_t + \underline{u}_t^T \boldsymbol{R} \underline{u}_t \right), \tag{6}$$

and $\boldsymbol{Q} \geq 0$, $\boldsymbol{R} > 0$ are weighting matrices. In the most general case, the actuator failure can be transformed into a single actuator failure [3,4,5]

$$u_{t,i}^F = \varrho_i v_{t,i} \quad (i = 1, 2, \ldots, m)\,, \tag{7}$$

where $v_{t,i}$ is the i–th computed control signal and

$$0 \leq \varrho_{-,i} \leq \varrho_i \leq \varrho_{+,i} \quad (i = 1, 2, \ldots, m) \tag{8}$$

and $\varrho_{-,i} \leq 1$, $\varrho_{+,i} \geq 1$ are given constants.

If $\rho_{-,t,i}^k = \rho_{+,t,i}^k = 0$, then there is no failure, whereas for $\rho_{-,t,i}^k = \rho_{+,t,i}^k = 1$ there is an outage case. For $0 < \rho_{-,t,i}^k \leq \rho_{+,t,i}^k < 1$, where is a partial failure in the actuator and information loss carried by $\mathrm{sat}(v_{t,i})$.

For $\varrho_{-,i} = \varrho_{+,i}$ there are no constraints imposed on the control system and $u^F_{t,i} = v_{t,i}$, the case $\varrho_{-,i} > 0$ corresponds to partial failure, and $\varrho_{-,i} = 0$ corresponds to outage case.

Actuator failure may give rise to changes in direction between computed $\underline{v}_t$ and applied $\underline{u}_t$ control vectors.

3. DIRECTIONAL CHANGE PHENOMENON

Depending on the method of imposing constraints on the control vector one can observe directional change, illustrated in Fig. 1a in the case of cut-off saturation (for future reference), which is not present when saturation is performed according to imposed constraints (dashed lines) with constant direction, Fig. 1b (for future reference: DP).

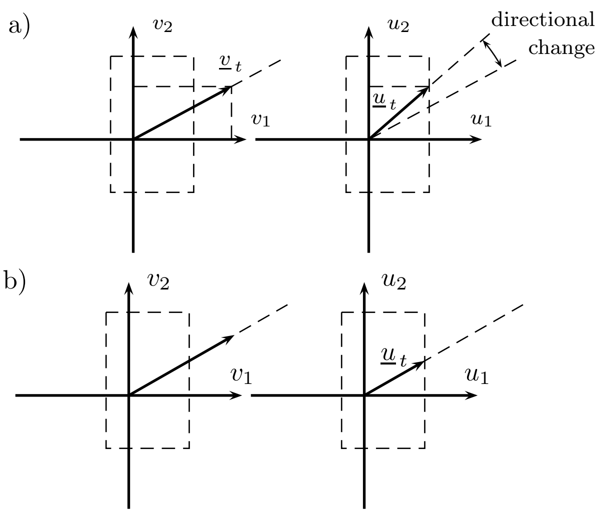

Fig. 1. a) direction-changing, b) direction-preserving saturation (left: control vector before saturation, right: after saturation)

Let two-input two-output system be not coupled and both loops be driven by separate I controllers (with no cross-coupling in the linear case). The output vector $\underline{y}_t$ is to track reference vector of two sinusoid waves, what corresponds to drawing a circular shape in the (y_1, y_2) plane [1].

As it can be seen in the Fig. 2a, the system with no constraints performs best, whereas in the case of cut-off saturation of both the elements of control vector (Fig. 2b) the tracking performance is poor. This is because of directional change in controls, changing proportions between its components. In the application for, e.g., shape-cutting, performance of the system from Fig. 2c (direction-preserving) is superior. Nevertheless, it is to be borne in mind that the system is always perfectly decoupled.

4. CONTROL LAW

The considered controller is of the form

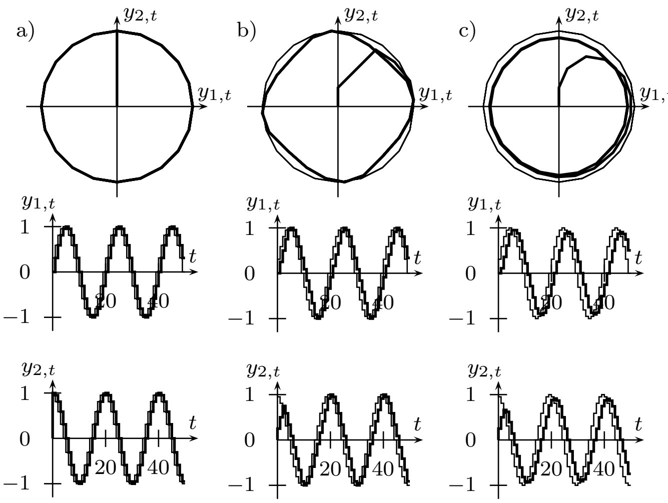

Fig. 2. a) unconstrained system, b) cut-off saturation, c) direction-preserving saturation

$$\underline{v}_t = \boldsymbol{F}\underline{x}_t\,, \tag{9}$$

where $\underline{x}_t$ is the plant state vector (or its estimate), is called reliable if the control performance indexes does not exceed some prescribed value. It is connected to $\boldsymbol{P}$ matrix, equations (1), (2) and performance index (6), if $\boldsymbol{P}$ satisfies [3,4,2]

$$(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{\varrho}\boldsymbol{F})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{\varrho}\boldsymbol{F})-\boldsymbol{P}+\boldsymbol{F}^T\boldsymbol{\varrho}\boldsymbol{R}\boldsymbol{\varrho}\boldsymbol{F}+\boldsymbol{Q} \leq 0 \tag{10}$$

Then, the closed-loop system

$$\underline{x}_{t+1} = (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{\varrho}\boldsymbol{F})\,\underline{x}_t \tag{11}$$

and performance index in infinite horizon

$$J = \sum_{t=0}^{\infty} \underline{x}_t^T \left(\boldsymbol{Q}+\boldsymbol{F}^T\boldsymbol{\varrho}\boldsymbol{R}\boldsymbol{\varrho}\boldsymbol{F}\right)\underline{x}_t \leq \underline{x}_0^T\boldsymbol{P}\underline{x}_0 \tag{12}$$

In the case of no actuator failure taken into account, the state-feedback matrix $\boldsymbol{F}$ for (9) is given as the solution to the following:

$$\boldsymbol{F} = -\left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}+\boldsymbol{R}\right)^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}\,, \tag{13}$$

$$\boldsymbol{P} = \boldsymbol{Q}+\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F}-\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}\,(\boldsymbol{B}\boldsymbol{P}\boldsymbol{B}+\boldsymbol{R})^{-1}\,\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}\,, \tag{14}$$

and optimal value of the control index $J_{\boldsymbol{F}}$ (9), resulting from $\boldsymbol{F}$ wedug (13), (14), as the upper bound of (12) is

$$J_{\boldsymbol{F}} = \underline{x}_0^T\boldsymbol{P}\underline{x}_0\,. \tag{15}$$

5. OPTIMAL STATE-FEEDBACK IN THE CASE OF ACTUATOR FAILURE

The following must be given before the final algorithm is described (based on [3,4]):

$$\underline{u}_t^T = \left[u_{t,1}^F,\, u_{t,2}^F,\, \ldots,\, u_{t,m}^F\right]^T , \tag{16}$$

$$\boldsymbol{\varrho}_+ = \mathrm{diag}\left\{\varrho_{+,1},\, \varrho_{+,2},\, \ldots,\, \varrho_{+,m}\right\} , \tag{17}$$

$$\boldsymbol{\varrho}_- = \mathrm{diag}\left\{\varrho_{-,1},\, \varrho_{-,2},\, \ldots,\, \varrho_{-,m}\right\} , \tag{18}$$

$$\boldsymbol{\varrho} = \mathrm{diag}\left\{\varrho_1,\, \varrho_2,\, \ldots,\, \varrho_m\right\} . \tag{19}$$

On the basis of [3,4,2] the following algorithm for determination of $\boldsymbol{F}$ can be given:

1) solve (14) for $\boldsymbol{P}$, and choose the diagonal $\boldsymbol{R}_0$ satisfying
$$\boldsymbol{R}_0 \leq \left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}\right)^{-1} ; \tag{20}$$
2) solve
$$\boldsymbol{P} = \boldsymbol{Q} + \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{P}J_0\boldsymbol{P}\boldsymbol{A} \tag{21}$$
for stabilising $\boldsymbol{P}$ and check if
$$\boldsymbol{R}_0 \leq \left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}\right)^{-1} , \tag{22}$$
where
$$\boldsymbol{J}_0 = \boldsymbol{B}\left(\boldsymbol{I} - \boldsymbol{\Gamma}_0^2\right) \times \times\Big(\left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}\right)\left(\boldsymbol{I} - \boldsymbol{\Gamma}_0^2\right) + \boldsymbol{R}_0^{-1}\boldsymbol{\Gamma}_0^2\Big)^{-1}\boldsymbol{B}^T ; \tag{23}$$
3) if (20) is satisfied for $\boldsymbol{R}_0$ and $\boldsymbol{P}$, then increase $\boldsymbol{R}_0$ and go to step 2, otherwise decrease $\boldsymbol{R}_0$ and go to step 2;
4) if (20) is satisfied for $\boldsymbol{R}_0$ and $\boldsymbol{P}$, and stabilising $\boldsymbol{P}$ satisfies (21) for (21), (22) there is no positive definite solution for $\boldsymbol{R}_0'$, where $\boldsymbol{R}_0 \leq \boldsymbol{R}_0' \leq \left(\boldsymbol{B}^T\boldsymbol{P}^*\boldsymbol{P} + \boldsymbol{R}\right)^{-1}$, then stop the algorithm; the state-feedback matrix can be calculated in this case as
$$\boldsymbol{F} = -\boldsymbol{\Gamma}^{-1}\left(\boldsymbol{I} - \left(\boldsymbol{X}^{-1} - \boldsymbol{R}_0\right)\left(\left(\boldsymbol{I} - \boldsymbol{\Gamma}_0^2\right) + \boldsymbol{\Gamma}_0^2\boldsymbol{R}_0^{-1}\boldsymbol{X}^{-1}\right)^{-1}\boldsymbol{\Gamma}_0^2\boldsymbol{R}_0^{-1}\right)\boldsymbol{X}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} , \tag{24}$$
where $\boldsymbol{X} = \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}$.

The following notation has been adopted in the algorithm:

$$\boldsymbol{\Gamma} = \mathrm{diag}\left\{\gamma_1,\, \gamma_2,\, \ldots,\, \gamma_m\right\} , \tag{25}$$

$$\boldsymbol{\Gamma}_0 = \mathrm{diag}\left\{\gamma_{0,1},\, \gamma_{0,2},\, \ldots,\, \gamma_{0,m}\right\} , \tag{26}$$

where:

$$\gamma_i = \frac{\varrho_{+,i} + \varrho_{-,i}}{2} , \tag{27}$$

$$\gamma_{0,i} = \frac{\varrho_{+,i} - \varrho_{-,i}}{\varrho_{+,i} + \varrho_{-,i}} . \tag{28}$$

The matrix $\boldsymbol{P}$ is a stabilising Ricatti equation solution:

$$\boldsymbol{P} = \boldsymbol{Q} + \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{B}\left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}\right)^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} , \tag{29}$$

and matrix $\boldsymbol{A} - \boldsymbol{B}\left(\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} + \boldsymbol{R}\right)^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A}$ has all its eigenvalues in a unit circle.

Furthermore, on the basis of (19), (25) and (26) we have:

$$\boldsymbol{\varrho} = \left(\boldsymbol{I} + \boldsymbol{\varrho}_0\right)\boldsymbol{\Gamma} , \tag{30}$$

$$\left|\boldsymbol{\varrho}_0\right| \leq \boldsymbol{\Gamma}_0 \leq \boldsymbol{I} , \tag{31}$$

where certain matrix $\boldsymbol{\varrho}_0 = \mathrm{diag}\left\{\varrho_{0,1},\, \varrho_{0,2},\, \ldots,\, \varrho_{0,m}\right\}$.

In the case of actuator failure, the cited algorithm assures that upper bound of the performance index is not exceeded during the operation of controller.

6. ACTUATOR FAILURE AND CONTROL WITH CONSTRAINTS

According to [5] control with imposed (amplitude) constraints, i.e. possible saturation of actuator output, can be treated as a kind of actuator failure. In such a case, we introduce the notation: $\gamma_i = \alpha_i$ (saturation level) for $i = 1, 2, \ldots, m$, and constrained control vector is formulated for $i = 1, 2, \ldots, m$ as

$$u_{i,t}^F = \mathrm{sat}\left(\underline{f}_i^T\underline{x}_t; \alpha_i\right) , \tag{32}$$

where sat is a cut-off saturation function for the i–th component, with the cut-off level $\pm\alpha_i$, and vector $\underline{f}_i$ is the i–th row of $\boldsymbol{F}$.

The case of actuator failure and single control signal on the basis of (8) and (32) is presented in the Figure 3. The assumed failure model can be written for all components of control vector ($i = 1, 2, \ldots, m$) as

$$u_{i,t}^F = \mathrm{sat}\left(\varrho_i v_{t,i}; \alpha_i\right) , \tag{33}$$

being the composition of (32) and (8) (hatched area in Fig. 3).

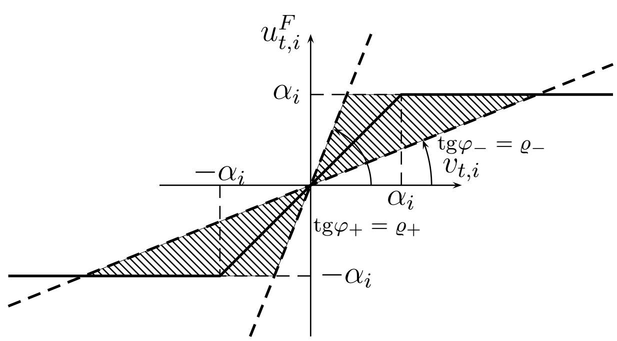

Fig. 3. Actuator failure and amplitude constraints

7. PERFORMANCE INDEXES

In order to evaluate the control quality, SAE and SSE (sums of absolute and squared errors) indexes have been introduced:

$$J_1 = \frac{1}{N} \sum_{t=0}^{N} \sum_{k=1}^{2} |r_{k,t} - y_{k,t}| \,, \qquad (34)$$

$$J_2 = \frac{1}{N} \sum_{t=0}^{N} \sum_{k=1}^{2} (r_{k,t} - y_{k,t})^2 \,, \qquad (35)$$

where N denotes simulation horizon.

In addition, the following two performance indexes are taken into account:

$$J_\varphi = \frac{1}{N} \sum_{t=0}^{N} |\,\varphi(\underline{v}_t) - \varphi(\underline{u}_t)| \;\; [^\circ] \,, \qquad (36)$$

$$J_{\varphi^2} = \frac{1}{N} \sum_{t=0}^{N} \Big(\varphi(\underline{v}_t) - \varphi(\underline{u}_t) \Big)^2 \,, \qquad (37)$$

where $\varphi(\underline{v}_t)$ corresponds to the angle between $[1, 0, \ldots, 0]$ and $\underline{v}_t$, furthermore $\varphi(\underline{u}_t)$ is the angle between $[1, 0, \ldots, 0]$ and $\underline{u}_t$.

The index J_φ relates to mean absolute directional change.

8. SIMULATION TESTS

The following one-sample delay plant is taken into consideration:

$$\boldsymbol{A} = \left[\begin{array}{cc|c} -0.80 & 0.10 & \multirow{2}{*}{$\boldsymbol{I}$} \\ -0.40 & 1.00 & \\ \hline 0.49 & 0.10 & \multirow{2}{*}{$\boldsymbol{0}$} \\ -0.10 & -0.25 & \end{array} \right] \,, \quad \boldsymbol{B} = \begin{bmatrix} 1.0 & 0.3 \\ 0.5 & 0.8 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \,,$$

with two inputs and two outputs.

The designed controller is to assure tracking of a given reference vector (comprising square-wave signals of amplitude ± 3), by taking offset of control vector into account. The graphical presentation of the results is given in Figures 4 and 5 and Table 1 (together with plots of rates of changes of u_1 and u_2).

The simulations have been carried out for the following constraints, enabling asymptotic tracking of the reference vector with output vector $\alpha_1 = 1.5$, $\alpha_2 = 2.0$ (amplitude constraints), $\beta_1 = 2.0$, $\beta_2 = 2.0$ (rate constraints).

Table 1. Performance indexes (DP relates to direction-preserving control law)

	DP	DP+robust.	no DP	no DP+robust.
J_1	2.1409	1.7520	2.3045	1.6893
J_2	8.6222	7.9377	8.9025	7.8338
J_φ	0.0000	0.0000	11.2969	2.7354
J_{φ^2}	0.0000	0.0000	393.4586	68.2641
J	24.0636	21.2026	23.5887	20.9807

As it can be seen, introducing DP (direction-preserving) requirement results in performance indexes increase. Having introduced the robustness against actuator failure with no DP controller, leads to decrease of indexes both in DP and no DP case. What is interesting, introducing the robustness do the DP controller enables one to reduce performance indexes and introducing robustness to no DP controller, decreases mean angular deflection with respect to no DP with no modeled uncertainty is concerned. It is also visible, that improving the tracking performance does not mean that excessive directional change takes place.

9. CONCLUSIONS

Having examined the Figures 4 and 5, it can be said that introducing robustness against actuator failures leads to less frequent saturation of components of control vector, and its rates of changes. Nevertheless, it is to be borne in mind, that in some applications it is of priority not to alter the direction of computed control vector.

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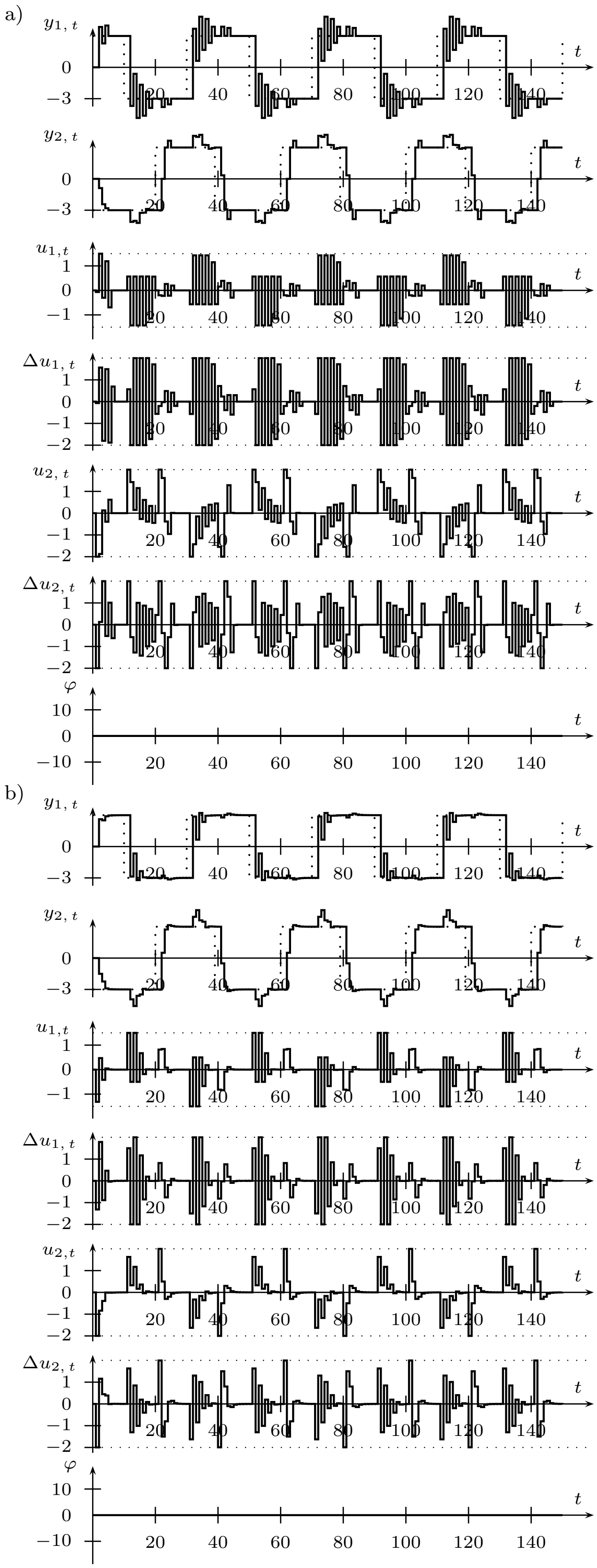

Fig. 4. Direction-preserving control law, a) without modeled actuator failure, b) with modeled actuator failure

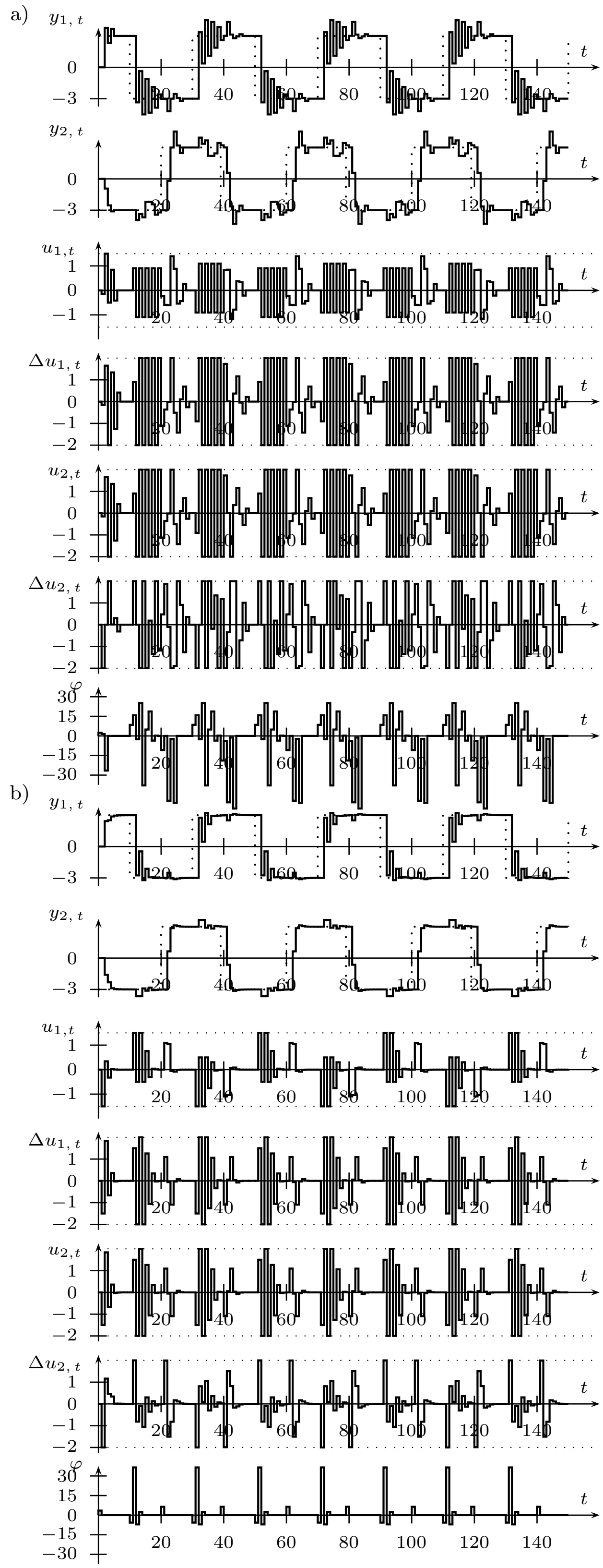

Fig. 5. Standard control law with cut-off, a) without modeled actuator failure, b) with modeled actuator failure