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Risk-Averse Feedback for Stochastic Fault-Tolerant Control Systems with Actuator Failure Accommodation

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Abstract— **In this paper, an innovative baseline control strategy for the linear-quadratic class of stochastic fault-tolerant control systems with performance risk aversion has been obtained. Moreover, an efficient paradigm of determining mathematical statistics associated with performance robustness is developed, upon which the model uncertainties of time-varying parameter perturbations and critical actuator failures of the stuck-type ones are also treated in the stochastic framework. The synthesis of risk-averse state feedback control law is shown feasible by the use of adapted dynamic programming approach to the optimal statistical control problem considered herein.**

I. Introduction

Over the last three decades, there have been an increasing need for fault-tolerant control systems to continue operating acceptably to fulfill specified performance following faults in the system being controlled or in the controller. Many important issues in fault-tolerant control systems can be found in [1] and references therein. Motivated by the growing demand for reliability and maintainability, such efforts on different fault detection and diagnosis techniques have led to the development of many reconfigurable control systems [2] and [3]. However, little attention has been paid to the problem of graceful performance degradation with multiple risk considerations. More recently, risk-averse feedback for stochastic control problems has begun to show promises and successes [4] and [5].

When the actuator failures of stuck-type ones occur, for example aircraft and robotic manipulators wherein the control surfaces or effectors would be stuck in certain positions, the command control inputs could not affect the actuator outputs. Hence, the total loss of control efficiency would result given the fact that there was very limited time and possibility to perform fault diagnosis. The main contributions of this research investigation are to show how to: i) integrate an element of automatic allocation representation of apportioning performance robustness and reliability requirements into the multi-attribute requirement of qualitative characteristics of expected performance and performance risks, once the random performance measure of the generalized chi-squared type associated with the linear-quadratic class of stochastic fault-tolerant control systems has been recognized and ii) adapt a risk-averse control strategy for the baseline controller in the stochastic fault-tolerant control system to retain some portion of its control integrity in the event of a specified set of possible severe actuator faults.

The structure of this paper is as follows. Section II contains the problem description and the notion of admissible control for the linear-quadratic class of stochastic fault-tolerant systems. In addition, the development of all the mathematical statistics for performance robustness is carefully discussed. The detailed problem statements and solution method in obtaining an optimal feedback controller with performance risk aversion are described in Sections III and IV. In Section V, some conclusions are also included.

II. Mathematical Statistics for Performance Robustness

In this section, some preliminaries are in order. Some spaces of random variables and stochastic processes are introduced: $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) \triangleq \{\wp : \Omega \mapsto \mathbb{R}^n$ such that $\wp$ is $\mathcal{F}_t$-measurable and $E\{||\wp||^2\} < \infty\}$. In addition, $L^2_{\mathcal{F}}([t_0,t_f];\mathbb{R}^k) \triangleq \{\hbar : [t_0,t_f] \times \Omega \mapsto \mathbb{R}^k$ such that $\hbar(\cdot)$ is $\mathbb{F} = \{\mathcal{F}_t\}_{t \ge t_0 \ge 0}$-adapted and $E\{\int_{t_0}^{t_f} ||\hbar(t)||^2\, dt\} < \infty\}$.

As for the setting, some classes of stochastic fault-tolerant control problems with actuator failure accommodation shall be investigated. A typical description for the faulty system being controlled on $[t_0, t_f]$ makes use of the concept of "unknown inputs" acting upon a nominal linear model of the system described by

$$dx(t) = [(A + \Delta A(t))x(t) + (B + \Delta B(t))Lu(t) + (B + \Delta B(t))(I - L)f_a(t)]dt + Gdw(t),\; x(t_0) = x_0. \quad (1)$$

With regards of the combined state and control effectiveness model (1), the nominal system coefficients $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $G \in \mathbb{R}^{n\times p}$ are the constant matrices. The process disturbances $w(t) \equiv w(t,\omega)$ assume the form of a stationary Wiener process with the correlations of independent increments for all $\tau_1, \tau_2 \in [t_0, t_f]$

$$E\left\{[w(\tau_1) - w(\tau_2)][w(\tau_1) - w(\tau_2)]^T\right\} = W|\tau_1 - \tau_2|$$

whose a-priori second-order statistic $W > 0$ is also known.

In addition, the parameter perturbations considered with robustness to minor model uncertainties are approximated as

$$\Delta A(t) = H_A F(t) E_A\,, \qquad \Delta B(t) = H_B F(t) E_B \quad (2)$$

whereby some a-priori knowledge about uncertainty are used for a direct derivation of known constant matrices H_A, H_B, E_A and E_B with appropriate dimensions. And $F(t)$ is an unknown real time-varying matrix with Lebesgue measurable elements satisfying the condition $F^T(t)F(t) \le I$ for all $t \in [t_0, t_f]$.

Here $x(t) \equiv x(t,\omega)$ is the controlled state process valued in $\mathbb{R}^n$ and $u(t) \equiv u(t,\omega)$ is the normal control process valued in an admissible set $U \subseteq L^2_{\mathcal{F}}([t_0,t_f];\mathbb{R}^m)$.

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Notice that the working condition of the actuators is described by the distribution matrix $L \triangleq diag(l_1, \ldots, l_m)$, whereby $l_i = 1$ and $i \in \overline{L} \triangleq \{1, \ldots, m\}$ denotes a healthy ith actuator and $l_i = 0$ corresponds to the total failure of the ith actuator. In this case, the ith actuator is stuck at certain position and thus, its actual control input is only an offset value. Such an offset is denoted as $f_a(t)$ and is also considered as the disturbances added to the system through the failed actuators. In retrospect with a passive approach to control input failures, the robust feedback controller proposed herein is designed to tolerate control channel failures.

In most situations, the system (1) is assumed to possess $(m-l)$ degree of actuator redundancy if (†) the system is complete state controllable with all m actuators functional and (‡) if any l actuators out of m actuators fail, with the remaining $(m-l)$ actuators, the system is still completely state controllable. See [6].

Associated with each $u \in U$ is a path-wise finite-horizon integral-quadratic form (IQF) performance measure with the generalized chi-squared random distribution $J : U \mapsto \mathbb{R}^+$, for which the passive fault-tolerant controller stabilizes the fault system with multiple degrees of performance robustness and closely tracks the reference trajectory $r \in \mathcal{C}(t_0, t_f; \mathbb{R}^n)$

$$J(u) = [x(t_f) - r(t_f)]^T Q_f [x(t_f) - r(t_f)] + \int_{t_0}^{t_f} [x^T(\tau) Q_1 x(\tau) + (x(\tau) - r(\tau))^T Q_2 (x(\tau) - r(\tau))] d\tau + \int_{t_0}^{t_f} \left[u^T(\tau) R_1 u(\tau) - f_a^T(\tau) R_2 f_a(\tau)\right] d\tau \,, \quad (3)$$

where the terminal penalty weighting $Q_f \in \mathbb{R}^{n\times n}$, the state weighting $Q_1 \in \mathbb{R}^{n\times n}$, the reference tracking weighting $Q_2 \in \mathbb{R}^{n\times n}$, control weighting $R_1 \in \mathbb{R}^{m\times m}$ and fault effect reduction weighting $R_2 \in \mathbb{R}^{m\times m}$ are continuous-time matrix functions with the properties of symmetry and positive semi-definiteness. In addition, R_1 is invertible.

Next, the notion of admissible controls is discussed. In the case of state feedback measurements, an admissible control would have the form, for some $\gamma(\cdot,\cdot)$

$$u(t) = \gamma(t, x(t))\,, \quad t \in [t_0, t]\,. \quad (4)$$

As shown in [7] and [8], the search for optimal control solutions to the optimal statistical control problem may be consistently and productively restricted to linear time-varying feedback laws generated from the accessible state $x(t)$ by

$$u(t) = K(t)x(t) + \ell(t)\,, \quad t \in [t_0, t_f] \quad (5)$$

with $K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times n})$ and $\ell \in \mathcal{C}([t_0, t_f]; \mathbb{R}^m)$ an admissible feedback gain and an affine vector whose further defining properties will be stated shortly.

Then, for the admissible $(K(\cdot), \ell(\cdot))$ and pair (t_0, x_0), it gives a sufficient condition for the existence of $x(t)$ in (1). In view of (5), the controlled system (1) with the initial condition $x(t_0) = x_0$ is rewritten as follows

$$dx(t) = ((A + \Delta A(t)) + (B + \Delta B(t))LK(t))x(t)dt + (B + \Delta B(t))L\ell(t)dt + Gdw(t) \quad (6)$$

subject to the performance measure (3)

$$J = x^T(t_f)Q_f x(t_f) - 2x^T(t_f)Q_f r(t_f) + r^T(t_f)Q_f r(t_f) + \int_{t_0}^{t_f} x^T(\tau)[Q_1 + Q_2 + K^T(\tau)R_1K(\tau)]x(\tau)d\tau + \int_{t_0}^{t_f} 2x^T(\tau)[K^T(\tau)R_1\ell(\tau) - Q_2 r(\tau)]d\tau + \int_{t_0}^{t_f} [r^T(\tau)Q_2 r(\tau) + \ell^T(\tau)R_1\ell(\tau) - f_a^T(\tau)R_2 f_a(\tau)]d\tau. \quad (7)$$

So far there are two types of information, i.e., process information (6) and goal information (7) have been given in advance to the controller (5). Since there is an external disturbance $w(\cdot)$ affecting the closed-loop performance, the controller now needs additional information about performance variations. This is *coupling information* and thus also known as *performance information*.

Regarding the structural constraints of the linear system dynamics (6) and finite-horizon integral-quadratic-form cost (7), the path-wise performance-measure (7) with which the control designer is risk averse is clearly a random variable of the generalized chi-squared type. Hence, the degree of uncertainty of the path-wise performance-measure (7) must be assessed via a complete set of higher-order statistics beyond the statistical mean or average. The essence of information about these higher-order performance-measure statistics in an attempt to describe or model performance uncertainty, is now considered as a source of information flow, which will affect perception of the problem and the environment at the risk-averse control designer. Next, the question of how to characterize and influence performance information is answered by modeling and management of cumulants (also known as semi-invariants) associated with (7) as shown in the following result.

Theorem 1: Cumulant-Generating Function.
Let $x(\cdot)$ be a state variable of the stochastic fault-tolerant system (6) with initial values $x(\tau) \equiv x_\tau$ and $\tau \in [t_0, t_f]$. Further let the moment-generating function be denoted by

$$\varphi(\tau, x_\tau, \theta) = \varrho(\tau, \theta) \exp\left\{x_\tau^T \Upsilon(\tau, \theta) x_\tau + 2x_\tau^T \eta(\tau, \theta)\right\} \quad (8)$$

$$\upsilon(\tau, \theta) = \ln\{\varrho(\tau, \theta)\}\,, \qquad \theta \in \mathbb{R}^+\,. \quad (9)$$

Then, the cumulant-generating function has the form of quadratic affine

$$\psi(\tau, x_\tau, \theta) = x_\tau^T \Upsilon(\tau, \theta) x_\tau + 2x_\tau^T \eta(\tau, \theta) + \upsilon(\tau, \theta) \quad (10)$$

where the scalar solution $\upsilon(\tau, \theta)$ solves the backward-in-time differential equation with the terminal-value condition $\upsilon(t_f, \theta) = r^T(t_f) Q_f r(t_f)$

$$\frac{d}{d\tau}\upsilon(\tau, \theta) = -\mathrm{Tr}\left\{\Upsilon(\tau, \theta) G W G^T\right\} - 2\eta^T(\tau, \theta)(B + \Delta B(\tau))L\ell(\tau) - \theta(r^T(\tau)Q_2 r(\tau) + \ell^T(\tau)R_1\ell(\tau) - f_a^T(\tau)R_2 f_a(\tau)) \quad (11)$$

and the matrix $\Upsilon(\tau,\theta)$ and vector $\eta(\tau,\theta)$ solutions satisfy the backward-in-time differential equations

$$\frac{d}{d\tau}\Upsilon(\tau,\theta) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK(\tau)]^T\Upsilon(\tau,\theta) - \Upsilon(\tau,\theta)[A+\Delta A(\tau)+(B+\Delta B(\tau))LK(\tau)] - 2\Upsilon(\tau,\theta)GWG^T\Upsilon(\tau,\theta) - \theta[Q_1+Q_2+K^T(\tau)R_1K(\tau)]\,,\quad \Upsilon(t_f,\theta)=\theta Q_f \quad (12)$$

$$\frac{d}{d\tau}\eta(\tau,\theta) = -[(A+\Delta A(\tau))+(B+\Delta B(\tau))LK(\tau)]^T\eta(\tau,\theta) - \Upsilon(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) - \theta[K^T(\tau)R_1\ell(\tau)-Q_2r(\tau)],\ \eta(t_f,\theta)=-\theta Q_f r(t_f). \quad (13)$$

Meanwhile, the scalar solution $\varrho(\tau)$ satisfies the backward-in-time differential equation with the terminal-value condition $\varrho(t_f,\theta)=\exp\{\theta r^T(t_f)Q_f r(t_f)\}$

$$\frac{d}{d\tau}\varrho(\tau,\theta) = -\varrho(\tau,\theta)\left[\mathrm{Tr}\left\{\Upsilon(\tau,\theta)GWG^T\right\} + 2\eta^T(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) + \theta(r^T(\tau)Q_2r(\tau)+\ell^T(\tau)R_1\ell(\tau)-f_a^T(\tau)R_2f_a(\tau))\right] \quad (14)$$

Proof: For notional simplicity, it is convenient to have $\varpi(\tau,x_\tau,\theta)\triangleq\exp\{\theta J(\tau,x_\tau)\}$ in which the performance measure (7) is rewritten as the cost-to-go function from an arbitrary state x_τ at a running time $\tau\in[t_0,t_f]$, that is,

$$J(\tau,x_\tau) = x^T(t_f)Q_fx(t_f) - 2x^T(t_f)Q_fr(t_f) + \int_\tau^{t_f} x^T(t)[Q_1+Q_2+K^T(t)R_1K(t)]x(t)dt + r^T(t_f)Q_fr(t_f) + \int_\tau^{t_f} 2x^T(t)[K^T(t)R_1\ell(t)-Q_2r(t)]dt + \int_\tau^{t_f}[r^T(t)Q_2r(t)+\ell^T(t)R_1\ell(t)-f_a^T(t)R_2f_a(t)]dt \quad (15)$$

subject to

$$dx(t) = ((A+\Delta A(t))+(B+\Delta B(t))LK(t))x(t)dt + (B+\Delta B(t))L\ell(t)dt + Gdw(t)\,,\quad x(\tau)=x_\tau\,. \quad (16)$$

By definition, the moment-generating function or the first characteristic function is $\varphi(\tau,x_\tau,\theta)\triangleq E\{\varpi(\tau,x_\tau,\theta)\}$. Thus, the total time derivative of $\varphi(\tau,x_\tau,\theta)$ is obtained as

$$\frac{d}{d\tau}\varphi(\tau,x_\tau,\theta) = -\theta[x_\tau^T(Q_1+Q_2+K^T(\tau)R_1K(\tau))x_\tau + 2x_\tau^T(K^T(\tau)R_1\ell(\tau)-Q_2r(\tau)) + r^T(\tau)Q_2r(\tau) + \ell^T(\tau)R_1\ell(\tau) - f_a^T(\tau)R_2f_a(\tau)]\varphi(\tau,x_\tau,\theta)\,.$$

Using the standard Ito's formula, it follows

$$d\varphi(\tau,x_\tau,\theta) = E\{d\varpi(\tau,x_\tau,\theta)\} = E\Big\{\varpi_\tau(\tau,x_\tau,\theta)\,d\tau + \varpi_{x_\tau}(\tau,x_\tau,\theta)\,dx_\tau + \frac{1}{2}\mathrm{Tr}\left\{\varpi_{x_\tau x_\tau}(\tau,x_\tau,\theta)GWG^T\right\}d\tau\Big\}$$

In view of (16), it is shown that

$$d\varphi(\tau,x_\tau,\theta) = \varphi_\tau(\tau,x_\tau,\theta)d\tau + \varphi_{x_\tau}(\tau,x_\tau,\theta)[((A+\Delta A(\tau)) + (B+\Delta B(t))LK(\tau))x(\tau) + (B+\Delta B(\tau))L\ell(\tau)]d\tau + \frac{1}{2}\mathrm{Tr}\left\{\varphi_{x_\tau x_\tau}(\tau,x_\tau,\theta)GWG^T\right\}d\tau$$

which, under the assumption of $\varphi(\tau,x_\tau,\theta) = \varrho(\tau,\theta)\exp\left\{x_\tau^T\Upsilon(\tau,\theta)x_\tau + x_\tau^T\eta(\tau,\theta)\right\}$ and its partial derivatives, leads to the result

$$d\varphi(\tau,x_\tau,\theta) = \Big\{\frac{\frac{d}{d\tau}\varrho(\tau,\theta)}{\varrho(\tau,\theta)} + x_\tau^T\frac{d}{d\tau}\Upsilon(\tau,\theta)x_\tau + 2x_\tau^T\frac{d}{d\tau}\eta(\tau,\theta) + x_\tau^T((A+\Delta A(\tau))+(B+\Delta B(\tau))LK(\tau))^T\Upsilon(\tau,\theta)x_\tau + x_\tau^T\Upsilon(\tau,\theta)((A+\Delta A(\tau))+(B+\Delta B(\tau))LK(\tau))x_\tau + 2x_\tau^T\Upsilon(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) + 2x_\tau^T((A+\Delta A(\tau))+(B+\Delta B(\tau))LK(\tau))^T\eta(\tau,\theta) + 2\eta^T(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) + \mathrm{Tr}\{\Upsilon(\tau,\theta)GWG^T\} + 2x_\tau^T\Upsilon(\tau,\theta)GWG^T\Upsilon(\tau,\theta)x_\tau\Big\}\varphi(\tau,x_\tau,\theta)d\tau\,.$$

To have constant and quadratic terms being independent of arbitrary x_τ, it requires

$$\frac{d}{d\tau}\Upsilon(\tau,\theta) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK(\tau)]^T\Upsilon(\tau,\theta) - \Upsilon(\tau,\theta)[A+\Delta A(\tau)+(B+\Delta B(\tau))LK(\tau)] - 2\Upsilon(\tau,\theta)GWG^T\Upsilon(\tau,\theta) - \theta[Q_1+Q_2+K^T(\tau)R_1K(\tau)]\,,\quad \Upsilon(t_f,\theta)=\theta Q_f$$

$$\frac{d}{d\tau}\eta(\tau,\theta) = -[(A+\Delta A(\tau))+(B+\Delta B(\tau))LK(\tau)]^T\eta(\tau,\theta) - \Upsilon(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) - \theta[K^T(\tau)R_1\ell(\tau)-Q_2r(\tau)],\ \eta(t_f,\theta)=-\theta Q_fr(t_f)$$

$$\frac{d}{d\tau}\varrho(\tau,\theta) = -\varrho(\tau,\theta)\left[\mathrm{Tr}\left\{\Upsilon(\tau,\theta)GWG^T\right\} + 2\eta^T(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) + \theta(r^T(\tau)Q_2r(\tau)+\ell^T(\tau)R_1\ell(\tau) - f_a^T(\tau)R_2f_a(\tau))\right],\quad \varrho(t_f,\theta)=\exp\{\theta r^T(t_f)Q_fr(t_f)\}\,.$$

Finally, the backward-in-time differential equation satisfied by $\upsilon(\tau,\theta)$ is obtained

$$\frac{d}{d\tau}\upsilon(\tau,\theta) = -\mathrm{Tr}\left\{\Upsilon(\tau,\theta)GWG^T\right\} - 2\eta^T(\tau,\theta)(B+\Delta B(\tau))L\ell(\tau) - \theta(r^T(\tau)Q_2r(\tau)+\ell^T(\tau)R_1\ell(\tau) - f_a^T(\tau)R_2f_a(\tau)),\quad \upsilon(t_f,\theta)=\theta r^T(t_f)Q_fr(t_f)\,,$$

which completes the proof. ■

As will be seen in the following development, the shape and functional form of an utility function tell a great deal about the basic attitudes of control designers toward the uncertain outcomes or performance risks. Of particular, the new utility function or the so-called the generalized performance index, which is being proposed herein as a linear manifold defined by a finite number of higher-order statistics associated with (7) will provide a convenient allocation representation of apportioning performance robustness and reliability requirements into the multi-attribute requirement of qualitative characteristics of expected performance and performance risks. Subsequently, higher-order statistics that encapsulate the uncertain nature of (7) can now be generated via a Maclaurin series expansion of (10)

$$\psi(\tau, x_\tau, \theta) = \sum_{r=1}^{\infty} \frac{\partial^{(r)}}{\partial \theta^{(r)}} \psi(\tau, x_\tau, \theta)\bigg|_{\theta=0} \frac{\theta^r}{r!}, \tag{17}$$

from which all $\kappa_r \triangleq \frac{\partial^{(r)}}{\partial \theta^{(r)}} \psi(\tau, x_\tau, \theta)\Big|_{\theta=0}$ are called the mathematical statistics of (15). Moreover, the series expansion coefficients are computed by using the cumulant-generating function or the second characteristic function (10)

$$\frac{\partial^{(r)}}{\partial \theta^{(r)}} \psi(\tau, x_\tau, \theta)\bigg|_{\theta=0} = x_\tau^T \frac{\partial^{(r)}}{\partial \theta^{(r)}} \Upsilon(\tau, \theta)\bigg|_{\theta=0} x_\tau + 2x_\tau^T \frac{\partial^{(r)}}{\partial \theta^{(r)}} \eta(\tau, \theta)\bigg|_{\theta=0} + \frac{\partial^{(r)}}{\partial \theta^{(r)}} \upsilon(\tau, \theta)\bigg|_{\theta=0}. \tag{18}$$

In view of the definition (17), the rth performance-measure statistic therefore follows

$$\kappa_r = x_\tau^T \frac{\partial^{(r)}}{\partial \theta^{(r)}} \Upsilon(\tau, \theta)\bigg|_{\theta=0} x_\tau + 2x_\tau^T \frac{\partial^{(r)}}{\partial \theta^{(r)}} \eta(\tau, \theta)\bigg|_{\theta=0} + \frac{\partial^{(r)}}{\partial \theta^{(r)}} \upsilon(\tau, \theta)\bigg|_{\theta=0}. \tag{19}$$

for any finite $1 \le r < \infty$. For notational convenience, the following change of notations

$$H_r(\tau) \triangleq \frac{\partial^{(r)} \Upsilon(\tau, \theta)}{\partial \theta^{(r)}}\bigg|_{\theta=0}; \quad \breve{D}_r(\tau) \triangleq \frac{\partial^{(r)} \eta(\tau, \theta)}{\partial \theta^{(r)}}\bigg|_{\theta=0}$$
$$D_r(\tau) \triangleq \frac{\partial^{(r)} \upsilon(\tau, \theta)}{\partial \theta^{(r)}}\bigg|_{\theta=0} \tag{20}$$

is introduced so that the next theorem provides an effective and accurate capability for forecasting all the higher-order characteristics associated with performance uncertainty.

Theorem 2: Performance-Measure Statistics.
Let the stochastic fault-tolerant system be described by (6) and (7) whereby it is assumed to possess $(m-l)$ degree of actuator redundancy. For $k \in \mathbb{N}$ fixed, the kth cumulant of performance measure (7) is

$$\kappa_k = x_0^T H_k(t_0) x_0 + 2x_0^T \breve{D}_k(t_0) + D_k(t_0), \tag{21}$$

whereby the supporting variables $\{H_r(\tau)\}_{r=1}^k$, $\{\breve{D}_r(\tau)\}_{r=1}^k$ and $\{D_r(\tau)\}_{r=1}^k$ evaluated at $\tau = t_0$ satisfy the matrix/vector/scalar-valued differential equations (with the dependence of $H_r(\tau)$, $\breve{D}_r(\tau)$ and $D_r(\tau)$ upon $K(\tau)$ and $\ell(\tau)$ suppressed)

$$\frac{d}{d\tau} H_1(\tau) = -[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)]^T H_1(\tau) - H_1(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)] - [Q_1 + Q_2 + K^T(\tau) R_1 K(\tau)], \quad H_1(t_f) = Q_f \tag{22}$$

$$\frac{d}{d\tau} H_r(\tau) = -[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)]^T H_r(\tau) - H_r(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)] - \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} H_s(\tau) G W G^T H_{r-s}(\tau), \; H_r(t_f) = 0 \tag{23}$$

$$\frac{d}{d\tau} \breve{D}_1(\tau) = -[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)]^T \breve{D}_1(\tau) - H_1(\tau)(B + \Delta B(\tau)) L \ell(\tau) - K^T(\tau) R_1 \ell(\tau) + Q_2 r(\tau), \quad \breve{D}_1(t_f) = -Q_f r(t_f) \tag{24}$$

$$\frac{d}{d\tau} \breve{D}_r(\tau) = -[A + \Delta A(\tau) + (B + \Delta B(\tau)) L K(\tau)]^T \breve{D}_r(\tau) - H_r(\tau)(B + \Delta B(\tau)) L \ell(\tau), \quad \breve{D}_r(t_f) = 0 \tag{25}$$

$$\frac{d}{d\tau} D_1(\tau) = -\mathrm{Tr}\{H_1(\tau) G W G^T\} - 2\breve{D}_1^T(\tau)(B + \Delta B(\tau)) L \ell(\tau) - r^T(\tau) Q_2 r(\tau) - \ell^T(\tau) R_1 \ell(\tau), \quad D_1(t_f) = r^T(t_f) Q_f r(t_f) \tag{26}$$

$$\frac{d}{d\tau} D_r(\tau) = -\mathrm{Tr}\{H_r(\tau) G W G^T\} - 2\breve{D}_r^T(\tau)(B + \Delta B(\tau)) L \ell(\tau), \quad D_r(t_f) = 0. \tag{27}$$

Proof: The expression of performance-measure statistics described in (21) is readily justified by using result (19) and definition (20). What remains is to show that the solutions $H_r(\tau)$, $\breve{D}_r(\tau)$ and $D_r(\tau)$ for $1 \le r \le k$ indeed satisfy the backward-in-time differential equations (22)-(27). Notice that these equations (22)-(27) are satisfied by the solutions $H_r(\tau)$, $\breve{D}_r(\tau)$ and $D_r(\tau)$ and can be obtained by successively taking derivatives with respect to θ of the supporting equations (11)-(13) together with the assumptions of $(m-l)$ degree of actuator redundancy on $[t_0, t_f]$. ■

III. Problem Statements

The purpose of this section is to make use of increased insight into the roles played by performance-measure statistics on the generalized chi-squared performance measure (7) for risk-averse feedback strategies with actuator failure accommodation. The optimal statistical control of linear stochastic fault-tolerant systems here is distinguished by the fact that the evolution in time of all mathematical statistics (21) associated with the random performance measure (7) of the generalized chi-squared type are described by means of matrix differential equations (22)-(27).

For such problems it is important to have a compact statement of the optimal statistical control so as to aid mathematical manipulation. To make this more precise, one may think of the k-tuple state variables $\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot), \ldots, \mathcal{H}_k(\cdot))$, $\breve{\mathcal{D}}(\cdot) \triangleq (\breve{\mathcal{D}}_1(\cdot), \ldots, \breve{\mathcal{D}}_k(\cdot))$ and $\mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot), \ldots, \mathcal{D}_k(\cdot))$ whose continuously differentiable states $\mathcal{H}_r \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n \times n})$, $\breve{\mathcal{D}}_r \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^n)$ and $\mathcal{D}_r \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ having the representations $\mathcal{H}_r(\cdot) \triangleq H_r(\cdot)$, $\breve{\mathcal{D}}_r(\cdot) \triangleq \breve{D}_r(\cdot)$ and $\mathcal{D}_r(\cdot) \triangleq D_r(\cdot)$ with the right members satisfying the dynamics (22)-(27) are defined on $[t_0, t_f]$.

In the remainder of the development, the convenient mappings are introduced as follows

$$\mathcal{F}_r : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\breve{\mathcal{G}}_r : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n})^k \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \mapsto \mathbb{R}^n$$
$$\mathcal{G}_r : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n})^k \times \mathbb{R}^m \mapsto \mathbb{R}$$

where the rules of action are given by

$$\begin{aligned}\mathcal{F}_1(\tau, \mathcal{H}, K) \triangleq &-[Q_1 + Q_2 + K^T(\tau) R_1 K(\tau)] \\ &- [A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]^T \mathcal{H}_1(\tau) \\ &- \mathcal{H}_1(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]\end{aligned}$$

$$\begin{aligned}\mathcal{F}_r(\tau, \mathcal{H}, K) \triangleq &-\sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_s(\tau) GWG^T \mathcal{H}_{r-s}(\tau) \\ &- [A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]^T \mathcal{H}_r(\tau) \\ &- \mathcal{H}_r(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]\end{aligned}$$

$$\begin{aligned}&\breve{\mathcal{G}}_1(\tau, \mathcal{H}, \breve{\mathcal{D}}, K, \ell) \\ &\triangleq -[A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]^T \breve{\mathcal{D}}_1(\tau) \\ &- \mathcal{H}_1(\tau)(B + \Delta B(\tau))L\ell(\tau) - K^T(\tau) R_1 \ell(\tau) + Q_2 r(\tau)\end{aligned}$$

$$\begin{aligned}\breve{\mathcal{G}}_r(\tau, \mathcal{H}, \breve{\mathcal{D}}, K, \ell) \triangleq &-\mathcal{H}_r(\tau)(B + \Delta B(\tau))L\ell(\tau) \\ &- [A + \Delta A(\tau) + (B + \Delta B(\tau))LK(\tau)]^T \breve{\mathcal{D}}_r(\tau)\end{aligned}$$

$$\begin{aligned}\mathcal{G}_1(\tau, \mathcal{H}, \breve{\mathcal{D}}, \ell) \triangleq &-r^T(\tau) Q_2 r(\tau) - \ell^T(\tau) R_1 \ell(\tau) \\ &- \mathrm{Tr}\{\mathcal{H}_1(\tau) GWG^T\} - 2\breve{\mathcal{D}}_1^T(\tau)(B + \Delta B(\tau))L\ell(\tau)\end{aligned}$$

$$\begin{aligned}\mathcal{G}_r(\tau, \mathcal{H}, \breve{\mathcal{D}}, \ell) \triangleq &-2\breve{\mathcal{D}}_r^T(\tau)(B + \Delta B(\tau))L\ell(\tau) \\ &- \mathrm{Tr}\{\mathcal{H}_r(\tau) GWG^T\}\,.\end{aligned}$$

The product mappings that follow are necessary for a compact formulation

$$\mathcal{F} : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times \mathbb{R}^{m\times n} \mapsto (\mathbb{R}^{n\times n})^k$$
$$\breve{\mathcal{G}} : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n})^k \times \mathbb{R}^{m\times n} \times \mathbb{R}^m \mapsto (\mathbb{R}^n)^k$$
$$\mathcal{G} : [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n})^k \times \mathbb{R}^m \mapsto \mathbb{R}^k$$

whereby the corresponding notations $\mathcal{F} \triangleq \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$, $\breve{\mathcal{G}} \triangleq \breve{\mathcal{G}}_1 \times \cdots \times \breve{\mathcal{G}}_k$ and $\mathcal{G} \triangleq \mathcal{G}_1 \times \cdots \times \mathcal{G}_k$ are used. Thus, the dynamic equations of motion (22)-(27) can be rewritten

$$\frac{d}{d\tau}\mathcal{H}(\tau) = \mathcal{F}(\tau, \mathcal{H}(\tau), K(\tau)), \quad \mathcal{H}(t_f) \tag{28}$$
$$\frac{d}{d\tau}\breve{\mathcal{D}}(\tau) = \breve{\mathcal{G}}(\tau, \mathcal{H}(\tau), \breve{\mathcal{D}}(\tau), K(\tau), \ell(\tau)), \quad \breve{\mathcal{D}}(t_f) \tag{29}$$
$$\frac{d}{d\tau}\mathcal{D}(\tau) = \mathcal{G}(\tau, \mathcal{H}(\tau), \breve{\mathcal{D}}(\tau), \ell(\tau)), \quad \mathcal{D}(t_f) \tag{30}$$

whereby the k-tuple terminal-value conditions $\mathcal{H}(t_f) \triangleq \mathcal{H}_f = (Q_f, 0, \ldots, 0)$, $\breve{\mathcal{D}}(t_f) \triangleq \breve{\mathcal{D}}_f = (-Q_f r(t_f), 0, \ldots, 0)$ and $\mathcal{D}(t_f) \triangleq \mathcal{D}_f = (r^T(t_f) Q_f r(t_f), \ldots, 0)$.

Remarks: Notice that the product system uniquely determines the state matrices $\mathcal{H}$, $\breve{\mathcal{D}}$ and $\mathcal{D}$ once the admissible feedback K and ℓ are specified. Henceforth, these state variables will be considered as $\mathcal{H} \equiv \mathcal{H}(\cdot, K)$, $\breve{\mathcal{D}} \equiv \breve{\mathcal{D}}(\cdot, K, \ell)$ and $\mathcal{D} \equiv \mathcal{D}(\cdot, \ell)$.

For the given terminal data $(t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$, the classes of admissible feedback gains are next defined.

Definition 1: Admissible Feedback Gains.
Let compact subsets $\overline{L} \subset \mathbb{R}^m$ and $\overline{K} \subset \mathbb{R}^{m\times n}$ be the sets of allowable affine inputs and feedback gain values. For the given $k \in \mathbb{N}$ and sequence $\mu = \{\mu_r \geq 0\}_{r=1}^k$ with $\mu_1 > 0$, the set of admissible affine inputs $\mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}, \mathcal{D}_f;\mu}$ and feedback gains $\mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}, \mathcal{D}_f;\mu}$ are assumed to be the class of $\mathcal{C}([t_0, t_f]; \mathbb{R}^m)$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times n})$ with values $\ell(\cdot) \in \overline{L}$ and $K(\cdot) \in \overline{K}$ for which solutions to the dynamic equations (28)-(30) with the terminal-value conditions $\mathcal{H}(t_f) = \mathcal{H}_f$, $\breve{\mathcal{D}}(t_f) = \breve{\mathcal{D}}_f$ and $\mathcal{D}(t_f) = \mathcal{D}_f$ exist on the interval of optimization $[t_0, t_f]$.
It is now crucial to plan for robust decisions and performance reliability from the start because it is going to be much more difficult and expensive to add reliability to the process later. To be used in the design process, performance-based reliability requirements must be verifiable by analysis; in particular, they must be measurable, like all higher-order performance-measure statistics, as evidenced in the previous section. These higher-order performance-measure statistics become the test criteria for the requirement of performance-based reliability. What follows is risk-value aware performance index in the optimal statistical control. It naturally contains some tradeoffs between performance values and risks for the subject class of stochastic control problems.

On $\mathcal{K}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D};\mu} \times \mathcal{L}_{t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f;\mu}$ the performance index with risk-value considerations in the optimal statistical control is subsequently defined as follows.

Definition 2: Risk-Value Aware Performance Index.
Fix $k \in \mathbb{N}$ and the sequence of scalar coefficients $\mu = \{\mu_r \geq 0\}_{r=1}^k$ with $\mu_1 > 0$. Then for the given x_0, the risk-value aware performance index

$$\phi_0 : \{t_0\} \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$$

pertaining to the optimal statistical control of the stochastic fault-tolerant system with actuator failure accommodation over $[t_0, t_f]$ is defined by

$$\begin{aligned}&\phi_0(t_0, \mathcal{H}(t_0), \breve{\mathcal{D}}(t_0), \mathcal{D}(t_0)) \\ &\triangleq \underbrace{\mu_1 \kappa_1}_{\text{Value Measure}} + \underbrace{\mu_2 \kappa_2 + \cdots + \mu_k \kappa_k}_{\text{Risk Measures}} \\ &= \sum_{r=1}^{k} \mu_r [x_0^T \mathcal{H}_r(t_0) x_0 + 2x_0^T \breve{\mathcal{D}}_r(t_0) + \mathcal{D}_r(t_0)]\,,\end{aligned} \tag{31}$$

whereby additional design freedom by means of μ_r's utilized by the control designer with risk-averse attitudes are sufficient to meet and exceed different levels of performance-based reliability requirements, for instance, mean (i.e., the average of performance measure), variance (i.e., the dispersion of values of performance measure around its mean), skewness (i.e., the anti-symmetry of the density of performance measure), kurtosis (i.e., the heaviness in the density tails of performance measure), etc., pertaining to closed-loop performance variations and uncertainties while the supporting solutions $\{\mathcal{H}_r(\tau)\}_{r=1}^k$, $\{\breve{\mathcal{D}}_r(\tau)\}_{r=1}^k$ and $\{\mathcal{D}_r(\tau)\}_{r=1}^k$ evaluated at $\tau = t_0$ satisfy the dynamical equations (28)-(30). Given that the terminal time t_f and states $(\mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$, the other end condition involved the initial time t_0 and state pair $(\mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0)$ are specified by a target set requirement.

Definition 3: Target Set.
$(t_0, \mathcal{H}_0, \breve{\mathcal{D}}_0, \mathcal{D}_0) \in \mathcal{M}$, where the target set $\mathcal{M}$ is a closed subset of $[t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k$.
Now, the optimization problem is to minimize the risk-value aware performance index (31) over all admissible feedback gains $K = K(\cdot)$ and affine inputs $\ell = \ell(\cdot)$ in $\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ and $\mathcal{L}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$.

Definition 4: Optimization Problem.
Fix $k \in \mathbb{N}$ and the sequence of scalar coefficients $\mu = \{\mu_r \geq 0\}_{r=1}^k$ with $\mu_1 > 0$. The optimization problem of the stochastic fault-tolerant system with actuator failure accommodation is defined by the minimization of the risk-value aware performance index (31) over $\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D};\mu} \times \mathcal{L}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ and subject to the dynamical equations (28)-(30) on $[t_0, t_f]$.
It is important to recognize that the optimization considered here is in Mayer form and can be solved by applying an adaptation of the Mayer form verification theorem of dynamic programming as given in [9]. To embed the aforementioned optimization into a larger optimal control problem, the terminal time and states $(t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$ are parameterized as $(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$. Thus, the value function for this optimization problem is now depending on the terminal condition parameterizations.

Definition 5: Value Function.
Suppose that $(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k$ is given and fixed. Then, the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ is defined by the greatest lower bound of the performance index $\phi_0(t_0, \mathcal{H}(t_0, K), \breve{\mathcal{D}}_r(t_0, K, \ell), \mathcal{D}(t_0, \ell))$ for all $K = K(\cdot)$ and $\ell = \ell(\cdot)$ in $\mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$ and $\mathcal{L}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$.
For convention, $\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \triangleq \infty$ when $\mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu} \times \mathcal{L}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$ is empty. To avoid cumbersome notation, the dependence of trajectory solutions on $K(\cdot)$ and $\ell(\cdot)$ is suppressed. Next, some candidates for the value function are constructed with the help of the concept of reachable set.

Definition 6: Reachable Set.
Let the reachable set $\mathcal{Q} \triangleq \{(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k\}$ such that $\mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu} \times \mathcal{L}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$ is not empty.
Notice that $\mathcal{Q}$ contains a set of points $(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$, from which it is possible to reach the target set $\mathcal{M}$ with some trajectory pairs corresponding to a continuous feedback gain. Furthermore, the value function must satisfy both a partial differential inequality and an equation at each interior point of the reachable set, at which it is differentiable.

Theorem 3: Hamilton-Jacobi-Bellman (HJB) Equation.
Let $(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ be any interior point of the reachable set $\mathcal{Q}$, at which the scalar-valued function $\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ is differentiable. Then $\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ satisfies the partial differential inequality, for all $K \in \overline{K}$ and $\ell \in \overline{L}$

$$
\begin{aligned}
0 \geq\; & \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\mathcal{F}(\varepsilon, \mathcal{Y}, K)) \\
&+ \frac{\partial}{\partial \text{vec}(\breve{\mathcal{Z}})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\breve{\mathcal{G}}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, K, \ell)) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\mathcal{G}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \ell))
\end{aligned} \tag{32}
$$

wherein $\text{vec}(\cdot)$ the vectorizing operator of enclosed entities.

If there is optimal feedback gain K^* in $\mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$ and affine input ℓ^* in $\mathcal{L}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$, then the partial differential equation of dynamic programming

$$
\begin{aligned}
0 = \min_{K\in\overline{K}, \ell\in\overline{L}} \Big\{ & \frac{\partial}{\partial \varepsilon}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Y})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\mathcal{F}(\varepsilon, \mathcal{Y}, K)) \\
&+ \frac{\partial}{\partial \text{vec}(\breve{\mathcal{Z}})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\breve{\mathcal{G}}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, K, \ell)) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})\text{vec}(\mathcal{G}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \ell))\Big\}
\end{aligned} \tag{33}
$$

is satisfied. The minimum in (33) is achieved by the optimal feedback gain $K^*(\varepsilon)$ and affine input $\ell^*(\varepsilon)$ at ε.

Proof: Interested readers are referred to the mathematical details in [10]. ■

The verification theorem in the optimal statistical control notation is stated as follows.

Theorem 4: Verification Theorem.
Fix $k \in \mathbb{N}$ and let $\mathcal{W}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ be a continuously differentiable solution of the HJB equation (33), which satisfies the boundary $\mathcal{W}(t_0, \mathcal{H}(t_0), \breve{\mathcal{D}}(t_0), \mathcal{D}(t_0)) = \phi_0(t_0, \mathcal{H}(t_0), \breve{\mathcal{D}}(t_0), \mathcal{D}(t_0))$ for some $(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z}) \in \mathcal{M}$. Let $(t_f, \mathcal{H}_f, \breve{\mathcal{D}}_f, \mathcal{D}_f)$ be a point of $\mathcal{Q}$, let (K, ℓ) in $\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu} \times \mathcal{L}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$, and let $\mathcal{H}(\cdot)$, $\breve{\mathcal{D}}(\cdot)$ and $\mathcal{D}(\cdot)$ be the corresponding solutions of the equations (28)-(30). Then, $\mathcal{W}(\tau, \mathcal{H}(\tau), \breve{\mathcal{D}}(\tau), \mathcal{D}(\tau))$ is a non-increasing time-backward function of τ.

If (K^*, ℓ^*) is in $\mathcal{K}_{t_f,\mathcal{H}_f,\mathcal{D}_f;\mu} \times \mathcal{L}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ with the corresponding solutions $\mathcal{H}^*(\cdot)$, $\breve{\mathcal{D}}^*(\cdot)$ and $\mathcal{D}^*(\cdot)$ of the equations (28)-(30) such that, for $\tau \in [t_0, t_f]$,

$$
\begin{aligned}
0 =\; & \frac{\partial}{\partial \varepsilon}\mathcal{W}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), \mathcal{D}^*(\tau)) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Y})}\mathcal{W}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), \mathcal{D}^*(\tau))\cdot \\
&\quad \text{vec}(\mathcal{F}(\tau, \mathcal{H}^*(\tau), K^*(\tau))) \\
&+ \frac{\partial}{\partial \text{vec}(\breve{\mathcal{Z}})}\mathcal{W}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), \mathcal{D}^*(\tau))\cdot \\
&\quad \text{vec}(\breve{\mathcal{G}}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), K^*(\tau), \ell^*(\tau))) \\
&+ \frac{\partial}{\partial \text{vec}(\mathcal{Z})}\mathcal{W}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), \mathcal{D}^*(\tau))\cdot \\
&\quad \text{vec}(\mathcal{G}(\tau, \mathcal{H}^*(\tau), \breve{\mathcal{D}}^*(\tau), \ell^*(\tau))),
\end{aligned} \tag{34}
$$

then both K^* and ℓ^* in $\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu} \times \mathcal{L}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ are optimal. Moreover, it follows that $\mathcal{W}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}) = \mathcal{V}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$, where $\mathcal{V}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$ is the value function.

Proof: The detailed analysis can be found in the work by the author [10]. ■

IV. Optimal Risk-Averse Control Strategy

Recall that the optimization problem being considered herein is in Mayer form, which can be solved by an adaptation of the Mayer form verification theorem. Thus, the terminal time and states $(t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f)$ are parameterized as $(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$ for a family of optimization problems. For instance, the states (28)-(30) defined on the interval $[t_0,\varepsilon]$ now have terminal values denoted by $\mathcal{H}(\varepsilon)\equiv\mathcal{Y}$, $\breve{\mathcal{D}}(\varepsilon)\equiv\breve{\mathcal{Z}}$ and $\mathcal{D}(\varepsilon)\equiv\mathcal{Z}$, where $\varepsilon\in[t_0,t_f]$. Furthermore, with $k\in\mathbb{N}$ and $(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$ in $\mathcal{Q}$, the following real-valued candidate:

$$\mathcal{W}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}) = x_0^T\sum_{r=1}^{k}\mu_r(\mathcal{Y}_r+\mathcal{E}_r(\varepsilon))\,x_0 + 2x_0^T\sum_{r=1}^{k}\mu_r(\breve{\mathcal{Z}}_r+\breve{\mathcal{T}}_r(\varepsilon)) + \sum_{r=1}^{k}\mu_r(\mathcal{Z}_r+\mathcal{T}_r(\varepsilon))\,, \quad (35)$$

whereby the time parametric functions $\mathcal{E}_r \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n\times n})$, $\breve{\mathcal{T}}_r \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^n)$ and $\mathcal{T}_r\in\mathcal{C}^1([t_0,t_f];\mathbb{R})$ are yet to be determined.

Moreover, it can be easily shown that the derivative of $\mathcal{W}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$ with respect to ε as

$$\frac{d}{d\varepsilon}\mathcal{W}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}) = x_0^T\sum_{r=1}^{k}\mu_r(\mathcal{F}_r(\varepsilon,\mathcal{Y},K)+\frac{d}{d\varepsilon}\mathcal{E}_r(\varepsilon))\,x_0 + 2x_0^T\sum_{r=1}^{k}\mu_r(\breve{\mathcal{G}}_r(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,\ell)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_r(\varepsilon)) + \sum_{r=1}^{k}\mu_r(\mathcal{G}_r(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\ell)+\frac{d}{d\varepsilon}\mathcal{T}_r(\varepsilon)) \quad (36)$$

provided that $\ell\in\overline{L}$ and $K\in\overline{K}$. Trying this candidate for the value function (35) into the HJB equation (33) yields

$$0 \equiv \min_{\ell\in\overline{L},\,K\in\overline{K}}\Big\{x_0^T\sum_{r=1}^{k}\mu_r(\mathcal{F}_r(\varepsilon,\mathcal{Y},K)+\frac{d}{d\varepsilon}\mathcal{E}_r(\varepsilon))\,x_0 + 2x_0^T\sum_{r=1}^{k}\mu_r(\breve{\mathcal{G}}_r(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,\ell)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_r(\varepsilon)) + \sum_{r=1}^{k}\mu_r(\mathcal{G}_r(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\ell)+\frac{d}{d\varepsilon}\mathcal{T}_r(\varepsilon))\Big\}\,. \quad (37)$$

Since the initial condition x_0 is an arbitrary vector, the necessary condition for an extremum of (33) on $[t_0,\varepsilon]$ is obtained by differentiating the expression within the bracket of (37) with respect to the control parameters ℓ and K as follows

$$\ell = -R_1^{-1}[(B+\Delta B(\varepsilon))L]^T\sum_{r=1}^{k}\hat{\mu}_r\breve{\mathcal{Z}}_r\,, \quad \hat{\mu}_r \triangleq \frac{\mu_i}{\mu_1} \quad (38)$$

$$K = -R_1^{-1}[(B+\Delta B(\varepsilon))L]^T\sum_{r=1}^{k}\hat{\mu}_r\mathcal{Y}_r \quad (39)$$

whereby convex bounded parametric uncertainties $\Delta A(\varepsilon)$ and $\Delta B(\varepsilon)$ as well as the actuator channels L are known whenever model mismatches and control input outages occur.

Replacing (38) and (39) into the HJB equation (37) leads to the value of the minimum. What remains is to exhibit the time parametric functions for the candidate function $\mathcal{W}(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z})$ of the value function, i.e., $\{\mathcal{E}_r(\cdot)\}_{r=1}^k$, $\{\breve{\mathcal{T}}_r(\cdot)\}_{r=1}^k$, and $\{\mathcal{T}_r(\cdot)\}_{r=1}^k$ which yield a sufficient condition to have the left-hand side of (34) being zero for any $\varepsilon\in[t_0,t_f]$, when $\{\mathcal{Y}_r\}_{r=1}^k$, $\{\breve{\mathcal{Z}}_r\}_{r=1}^k$ and $\{\mathcal{D}_r\}_{r=1}^k$ are evaluated along the solutions of the dynamical equations (28)-(30). With a careful examination of (37), one can infer that $\{\mathcal{E}_r(\cdot)\}_{r=1}^k$, $\{\breve{\mathcal{T}}_r(\cdot)\}_{r=1}^k$ and $\{\mathcal{T}_r(\cdot)\}_{r=1}^k$ are chosen to satisfy the time-forward differential equations as follows

$$\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon) = [A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)]^T\mathcal{H}_1(\varepsilon) + \mathcal{H}_1(\varepsilon)[A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)] + [Q_1+Q_2+K^T(\varepsilon)R_1K(\varepsilon)]\,, \quad \mathcal{E}_1(t_0)=0 \quad (40)$$

$$\frac{d}{d\varepsilon}\mathcal{E}_r(\varepsilon) = [A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)]^T\mathcal{H}_r(\varepsilon) + \mathcal{H}_r(\varepsilon)[A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)] + \sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}_s(\varepsilon)GWG^T\mathcal{H}_{r-s}(\varepsilon),\ \mathcal{E}_r(t_0)=0 \quad (41)$$

$$\frac{d}{d\varepsilon}\breve{\mathcal{T}}_1(\varepsilon) = [A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)]^T\breve{\mathcal{D}}_1(\varepsilon) + \mathcal{H}_1(\varepsilon)(B+\Delta B(\varepsilon))L\ell(\varepsilon) + K^T(\varepsilon)R_1\ell(\varepsilon) - Q_2r(\varepsilon)\,, \quad \breve{\mathcal{T}}_1(t_0)=0 \quad (42)$$

$$\frac{d}{d\varepsilon}\breve{\mathcal{T}}_r(\varepsilon) = [A+\Delta A(\varepsilon)+(B+\Delta B(\varepsilon))LK(\varepsilon)]^T\breve{\mathcal{D}}_r(\varepsilon) + \mathcal{H}_r(\varepsilon)(B+\Delta B(\varepsilon))L\ell(\varepsilon)\,, \quad \breve{\mathcal{T}}_r(t_0)=0 \quad (43)$$

$$\frac{d}{d\varepsilon}\mathcal{T}_1(\varepsilon) = \mathrm{Tr}\{\mathcal{H}_1(\varepsilon)GWG^T\} + r^T(\varepsilon)Q_2r(\varepsilon) + \ell^T(\varepsilon)R_1\ell(\varepsilon) + 2\breve{\mathcal{D}}_1^T(\varepsilon)(B+\Delta B(\varepsilon))L\ell(\varepsilon), \quad \mathcal{T}_1(t_0)=0 \quad (44)$$

$$\frac{d}{d\varepsilon}\mathcal{T}_r(\varepsilon) = \mathrm{Tr}\{\mathcal{H}_r(\varepsilon)GWG^T\} + 2\breve{\mathcal{D}}_r^T(\varepsilon)(B+\Delta B(\varepsilon))L\ell(\varepsilon)\,, \quad \mathcal{T}_r(t_0)=0\,. \quad (45)$$

The affine control input and feedback gain specified in (38) and (39) are now applied along the solution trajectories of the time-backward Riccati-type equations (28)-(30) such that the sufficient condition of (34) of the verification theorem is satisfied. Hence, the optimal linear input (38) and feedback gain (39) minimizing (31) become optimal

$$\ell^*(\varepsilon) = -R_1^{-1}[(B + \Delta B(\varepsilon))L]^T \sum_{r=1}^{k} \hat{\mu}_r \breve{\mathcal{D}}_r^*(\varepsilon)$$

$$K^*(\varepsilon) = -R_1^{-1}[(B + \Delta B(\varepsilon))L]^T \sum_{r=1}^{k} \hat{\mu}_r \mathcal{H}_r^*(\varepsilon)\,.$$

Theorem 5: Risk-Averse Strategy for Actuator Failures. Let the stochastic fault-tolerant system be described by (6) and (7), whereby it is assumed to possess $(m - l)$ degree of actuator redundancy. Assume $k \in \mathbb{N}$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ fixed. The baseline control strategy $u(t)$ will tolerate to certain actuator failures while robustly tracking the desired trajectory $r(t)$ with multiple levels of performance reliability, is given by

$$u^*(t) = K^*(t)x^*(t) + \ell^*(t)\,, \quad t = t_0 + t_f - \tau \qquad (46)$$

$$K^*(\tau) = -R_1^{-1}[(B + \Delta B(\tau))L]^T \sum_{r=1}^{k} \hat{\mu}_r \mathcal{H}_r^*(\tau) \qquad (47)$$

$$\ell^*(\tau) = -R_1^{-1}[(B + \Delta B(\tau))L]^T \sum_{r=1}^{k} \hat{\mu}_r \breve{\mathcal{D}}_r^*(\tau) \qquad (48)$$

whereby the normalized weightings $\hat{\mu}_r \triangleq \mu_i/\mu_1$ emphasize on different design freedom of shaping the probability density function of the chi-squared performance-measure (7).

The optimal control design solutions $\{\mathcal{H}_r^*(\tau)\}_{r=1}^k$, and $\{\breve{\mathcal{D}}_r^*(\tau)\}_{r=1}^k$ respectively satisfy the time-backward matrix-valued differential equations

$$\frac{d}{d\tau}\mathcal{H}_1^*(\tau) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK^*(\tau)]^T \mathcal{H}_1^*(\tau) - \mathcal{H}_1^*(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau))LK^*(\tau)] - [Q_1 + Q_2 + K^{*T}(\tau)R_1K^*(\tau)]\,, \quad \mathcal{H}_1^*(t_f) = Q_f \qquad (49)$$

$$\frac{d}{d\tau}\mathcal{H}_r^*(\tau) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK^*(\tau)]^T \mathcal{H}_r^*(\tau) - \mathcal{H}_r^*(\tau)[A + \Delta A(\tau) + (B + \Delta B(\tau))LK^*(\tau)] - \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{H}_s^*(\tau)GWG^T\mathcal{H}_{r-s}^*(\tau),\ \mathcal{H}_r^*(t_f) = 0 \qquad (50)$$

and the time-backward vector-valued differential equations

$$\frac{d}{d\tau}\breve{\mathcal{D}}_1^*(\tau) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK^*(\tau)]^T \breve{\mathcal{D}}_1^*(\tau) - \mathcal{H}_1^*(\tau)(B + \Delta B(\tau))L\ell^*(\tau) - K^{*T}(\tau)R_1\ell^*(\tau) + Q_2 r(\tau),\ \breve{\mathcal{D}}_1^*(t_f) = -Q_f r(t_f) \qquad (51)$$

$$\frac{d}{d\tau}\breve{\mathcal{D}}_r^*(\tau) = -[A+\Delta A(\tau)+(B+\Delta B(\tau))LK^*(\tau)]^T \breve{\mathcal{D}}_r^*(\tau) - \mathcal{H}_r^*(\tau)(B + \Delta B(\tau))L\ell^*(\tau)\,, \quad \breve{\mathcal{D}}_r^*(t_f) = 0\,. \qquad (52)$$

Remarks. One of future research directions is to have a timely estimation capability for abrupt reduction of control effectiveness in stochastic fault-tolerant systems. A Kalman-like observer will be proposed to gain the knowledge of the amount of loss in the control effectiveness $f_a(\cdot)$. For example, such abrupt changes and rates of changes in the control effectiveness of the class of faulty systems here can be described by a continuous white noise acceleration model

$$d\dot{f}_a(t) = V^{\frac{1}{2}}dv(t) \quad \text{and} \quad df_a(t) = \dot{f}_a(t)dt \qquad (53)$$

whereby the time derivative of $f_a(t)$ is $\dot{f}_a(t)$ and the probability of fault occurrence in each control channel is governed by the additive stationary Wiener process $v(t) \equiv v(t, \omega)$ with the correlations of independent increments $\forall\, \tau_1, \tau_2 \in [t_0, t_f]$

$$E\left\{[v(\tau_1) - v(\tau_2)][v(\tau_1) - v(\tau_2)]^T\right\} = V|\tau_1 - \tau_2|$$

whose a-priori second-order statistic $V > 0$ should be designed appropriately for random actuator faults.

V. Conclusions

This paper proposed a novel paradigm of designing feedback controls for the linear-quadratic class of stochastic fault-tolerant systems to not only track reference trajectories but also accommodate certain actuator failures, in accordance of the risk-value aware performance index. This performance index with risk aversion consists of multiple selective performance-measure statistics beyond the traditional statistical average. In addition, a numerical procedure of calculating these mathematical statistics associated with the chi-squared performance-measure is also developed. Finally, the complexity of the risk-averse feedback controller may increase, depending on how many performance-measure statistics of the desired probability density function are to be optimized.

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