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Adaptative Interval Observer with Application to the Estimation of Biofuel Production by Microalgae

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Abstract— **In this paper, the design of an interval observer with an adaptive dynamical gain is presented. The observer is formulated in the framework of robust state estimation of uncertain dynamical systems, where an interval that encloses the unknown state variables is provided. Here the observer is based on a change of coordinate that involves a time varying gain. We introduce a dynamics for the gain, whose trajectory converges toward a predefined optimal value (which maximizes the convergence rate of the observer). The observer performance is illustrated with the estimation of microalgal oil in the framework of biofuel production. The proposed observer design, when applied to experimental data of *Isochrysis affinis galbana*, appears to be a suitable robust estimation technique.**

I. Introduction

Robust state estimation of uncertain bioprocesses is a trending topic that has received increasing attention in the last decades. Observer techniques have been studied e.g. in [1], [2], proposing methods for the estimation of state variables, kinetics (reaction rates) and fundamental parameters. The high uncertain nature of this kind of processes and moreover the lack of specific instrumentation and sensors have made crucial the development of robust estimation techniques. Interval observers [3], [4] offer a way to deal with uncertainty (on the inputs–outputs and the dynamics) in a guaranteed state estimation framework.

Indeed, interval observers have become a leading state estimation technique. They are inserted in the class of guaranteed state estimation methods, providing a region of the state space where the unknown state variables are sure to be. They are constructed on the basis of positive differential systems [5] and have been successfully applied to the estimation of variables of biological systems [6], [7], [8], chaotic dynamics [9], [10], vehicle positioning [11], linear systems with additive disturbances [12], etc. Assuming that a guaranteed bound of the initial unknown state and bounds on the uncertainties (inputs, disturbances, parameters, ...) are provided, a framer basically consists in an auxiliary dynamical system whose trajectories always stay above or below (component by component) those of the original system. This definition is general and does not impose any further constraint on the qualitative behavior of the framer. Therefore, an interval observer is a stable framer (with bounded error dynamics).

One of the main advantages of interval observers is that the estimation performance can be evaluated online. Thanks to this property, several framers can be run in parallel [6] (bundle of framers) and then the best estimates can be selected. In [13] a strategy is proposed in order to improve the observer performance through the coupling of framers. The main advantage is that the stability of the framer envelope is guaranteed by a stable framer within the bundle [6].

In [7], an interval observer is designed based on a change of coordinate that involves a gain [14]. Considering a time-varying gain, it is shown that there exists an optimal gain providing the narrowest interval. However, this optimal gain is directly dependent on the (unknown) state variable. Moreover, its derivative is also needed to implement the observer. In [7] a strategy is proposed by bounding the gain interval and running several interval observers with constant gains in parallel. Here, we propose an alternative strategy, where a dynamics for the optimal gain is introduced (so we call it adaptive interval observer). The proposed adaption dynamics ensures high gain convergence of the observer gain towards the optimal value for which interval is the narrowest. Moreover, it is constrained to guarantee that some appropriate sign conditions on the gain stay valid.

The proposed approach is illustrated by a real case, for the estimation of the microalgal oil. Such process may become more and more used since microalgae are potentially considered as one of the main biofuel producers in the future [15], [16]. Nevertheless, the lack of on-line sensor for monitoring the nutrient status of the cell and its subsequent lipid accumulation makes it difficult to control and optimize lipid production. Some software sensors have been proposed to address this issue and estimate the internal nitrogen quota in microalgae [1], [8]. In the present paper we use a recently developed model [17] predicting both the nitrogen status of the microalgal cells and their lipid content to support an adaptive interval observer. The observer performance is illustrated with experimental data of *Isochrysis affinis galbana* (clone T-iso) cultures.

The paper is organized in four main sections. Section II is devoted to recalls on interval observer design. Then, after a presentation of the microalgal lipid model, the design of an interval observer for the estimation of microalgal

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lipid production is developed in Section III. In Section IV, an improved interval observer design is presented. Finally, Section V is devoted to the application of the method to the lipid interval observer, and its validation with experimental data.

II. Recall on interval observer design

A. Notations and definitions

We consider a differential system given by:

$$(\Sigma) : \begin{cases} \dot{x}(t) = \mathrm{f}(x(t), u(t), \lambda) \\ y(t) = \mathrm{h}(x(t)) \end{cases} \quad (1)$$

where $x(t) \in \Omega \subset \mathbb{R}^{n_x}$ is the vector of state variables with Ω a convex compact set of $\mathbb{R}^{n_x}$, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the vector of inputs with $\mathcal{U}$ the set of admissible controls and $\boldsymbol{\lambda} \in \boldsymbol{\Lambda} \in \mathbb{R}^{n_\lambda}$ is a parameter vector. f is a $\mathcal{C}^1$ mapping. The initial state vector $x(0)$ is in an admissible domain $\Omega_0 \subset \Omega$. The output y is related to the state via the $\mathcal{C}^1$ mapping h.

B. Framers

In the framework of parameter uncertainties, we assume that the real inputs and parameters are unknown, but that they can be enclosed by known quantities[1] :

$$\forall t \geq 0,\ \underline{u}(t) \leq u(t) \leq \overline{u}(t),\ \underline{\lambda} \leq \lambda \leq \overline{\lambda} \quad (2)$$

A framer associated to the differential system (Σ) is a differential system whose solutions generate guaranteed bounds for the state variables of (Σ) [3].

Definition 1: A *framer* for system (1) where inputs and parameters satisfy equation (2) is a pair of coupled dynamical systems:

$$\begin{cases} \dot{\overline{z}} = \overline{f}(\overline{z}, \underline{z}, \overline{u}(t), \underline{u}(t), \overline{\lambda}, \underline{\lambda}, \theta(t), y), \\ \dot{\underline{z}} = \underline{f}(\overline{z}, \underline{z}, \overline{u}(t), \underline{u}(t), \overline{\lambda}, \underline{\lambda}, \theta(t), y), \\ \overline{x} = \overline{g}(\overline{z}, \underline{z}, \overline{u}(t), \underline{u}(t), \overline{\lambda}, \underline{\lambda}, \theta(t), y), \\ \underline{x} = \underline{g}(\overline{z}, \underline{z}, \overline{u}(t), \underline{u}(t), \overline{\lambda}, \underline{\lambda}, \theta(t), y) \\ \underline{x}(0) = \underline{x}_0;\ \overline{x}(0) = \overline{x}_0, \\ \overline{z}(0) = \overline{h}(\overline{x}_0, \underline{x}_0, \overline{u}(0), \underline{u}(0), \overline{\lambda}, \underline{\lambda}, y, \theta(0)), \\ \underline{z}(0) = \underline{h}(\overline{x}_0, \underline{x}_0, \overline{u}(0), \underline{u}(0), \overline{\lambda}, \underline{\lambda}, y, \theta(0)). \end{cases} \quad (3)$$

such that, for $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, and for any $\theta(t) \in \Theta$, we have $\forall t \geq 0$:

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

The applications $\overline{h}$ and $\underline{h}$ are such that (with shorten notations):

$$\underline{g}(\overline{h}(\overline{x}_0, \underline{x}_0), \underline{h}(\overline{x}_0, \underline{x}_0)) \leq \underline{x}_0 \text{ and } \overline{x}_0 \leq \overline{g}(\overline{h}(\overline{x}_0, \underline{x}_0), \underline{h}(\overline{x}_0, \underline{x}_0))$$

The time–varying parameter vector $\theta \in \Theta$ can be used in order to tune the framer performance.

Note that this definition is rather general, highlighting the fact that a framer is simply designed to give an upper and a lower bound of the unknown state. The following definition also includes stability properties.

Definition 2: A *framer* (3) with bounded predictions $\overline{x}$ and $\underline{x}$ is called an interval observer.

[1] All the inequalities in $\mathbb{R}^n$ must be understood component by component.

III. Application to microalgae lipid production

As an example, we apply the presented approach to estimate lipid content of microalgae for biofuel production.

A. Model presentation

In [17], a mathematical model which describes microalgal lipid production under nitrogen stresses has been presented. It is based on the Droop model, which represents the effect of a limited nutrient on phytoplankton growth [18], [19], [20]. Droop model considers that the growth of the biomass x is related to the limited nutrient quota q_n, while nutrient uptake depends on the external concentration of nutrient s (nitrogen). Mairet *et al.* [17] have introduced a simplified carbon metabolism: the organic carbon is split into a functional pool (proteins, nucleic acids, membranes) and two storage pools: sugar and neutral lipid. Carbon from CO_2 is first incorporated as sugar. These carbohydrates are mobilized to produce functional carbon (mainly proteins) when microalgae uptake nitrogen. In parallel, carbohydrates are used to produce neutral lipid l which can be stored or mobilized to produce functional carbon (membranes). Considering this simplified carbon metabolism, the Droop model is completed with the dynamic of the neutral lipid quota q_l. In a perfectly mixed reactor, the lipid model reads:

$$\begin{cases} \dot{s} = Ds_{in} - \rho(s)x - Ds \\ \dot{q_n} = \rho(s) - \mu(q_n)q_n \\ \dot{x} = \mu(q_n)x - Dx \\ \dot{q_l} = (\beta q_n - q_l)\mu(q_n) - \gamma\rho(s) \end{cases} \quad (4)$$

where D is the dilution rate and s_{in} the influent nitrogen concentration.

In this model, the absorption rate $\rho(s)$ and growth rate $\mu(q_n)$ are respectively taken as Michaelis-Menten and Droop functions:

$$\begin{aligned} \rho(s) &= \bar{\rho}\frac{s}{s + K_s} \\ \mu(q_n) &= \bar{\mu}(1 - \frac{Q_0}{q_n}) \end{aligned} \quad (5)$$

where K_s is the half saturation constant for substrate uptake and Q_0 the minimal cell quota. $\bar{\rho}$ and $\bar{\mu}$ are the maximum inorganic nitrogen uptake rate and the maximum growth rate, respectively.

B. Framer design

For the sake of simplicity, the framer is first presented considering that s, q_n, and x are perfectly measured and that the specific growth ($\mu(q_n)$) and uptake ($\rho(s)$) rates are the only uncertain term in system (4):

Hypothesis 1: The specific growth rate is unknown but bounded by known functions $\underline{\mu}(q_n)$ and $\overline{\mu}(q_n)$:

$$\underline{\mu}(q_n) \leq \mu(q_n) \leq \overline{\mu}(q_n)$$

Let us consider the change of coordinate $z = (\theta - q_l)x + \gamma s$, where θ is a time–varying gain. The dynamics of z is given by:

$$\dot{z} = D(\gamma s_{in} - z) + (\theta - \beta q_n)\mu(q_n)x + \dot{\theta}x \quad (6)$$

which let us introduce the following property.

Proposition 1: Given $\theta(t) \geq \beta q_n$ and $\underline{z}_0$ and $\overline{z}_0$ such that $z(t_0) = z_0 \in [\underline{z}_0; \overline{z}_0]$, the system

$$\begin{cases} \dot{\overline{z}} = D(\gamma s_{in} - \overline{z}) + (\theta - \beta q_n)\overline{\mu}(q_n)x + \dot{\theta}x \\ \dot{\underline{z}} = D(\gamma s_{in} - \underline{z}) + (\theta - \beta q_n)\underline{\mu}(q_n)x + \dot{\theta}x \\ \overline{q_l} = \theta - (\underline{z} - \gamma s)/x \\ \underline{q_l} = \theta - (\overline{z} - \gamma s)/x \end{cases} \tag{7}$$

is a framer of system (4).

Proof. Let us introduce the comparisons $\overline{e}_{q_l} = \overline{q_l} - q_l$ and $\underline{e}_{q_l} = q_l - \underline{q_l}$. Their dynamics read:

$$\begin{array}{l} \dot{\overline{e}}_{q_l} = -\mu(q_n)\overline{e}_{q_l} + (\theta - \beta q_n)\left[\mu(q_n) - \underline{\mu}(q_n)\right]x \\ \dot{\underline{e}}_{q_l} = -\mu(q_n)\underline{e}_{q_l} + (\theta - \beta q_n)\left[\overline{\mu}(q_n) - \mu(q_n)\right]x \end{array} \tag{8}$$

At time t_0 it is possible to check that $\overline{e}_{q_l} \geq 0$. Now consider the time instant $t^\star$ such that $\overline{e}_{q_l}(t^\star) = 0$, from Equation (8) we have $\dot{\overline{e}}_{q_l}(t^\star) > 0$ and therefore the error will always stay positive, *i.e.* $\overline{q_l} \geq q_l$, $\forall t \geq t_0$. Similarly, one can easily check that $\underline{e}_{q_l}$ will also stay positive, so the system (7) is a framer of system (4). □

Note that it is not necessary to know the absorption rate $\rho(s)$ for implementing framer (7).

C. Optimal framers

As it has been stated in [7], the optimal pair of gains $\tilde{\theta}_1$ and $\tilde{\theta}_2$ respectively minimizes $\dot{\overline{e}}_{q_l}$ and $\dot{\underline{e}}_{q_l}$ at any time instant $t \geq t_0$, that is:

$$\begin{array}{l} \tilde{\theta}_1 = \arg \min\limits_{\theta \geq \beta q_n} \{J_1(\theta, q_n, x, \overline{e}_{q_l})\} \\ \\ \tilde{\theta}_2 = \arg \min\limits_{\theta \geq \beta q_n} \{J_2(\theta, q_n, x, \underline{e}_{q_l})\} \end{array} \tag{9}$$

with $J_1(\theta, q_n, x, \underline{e}_{q_l}) = \dot{\overline{e}}_{q_l}$ and $J_2(\theta, q_n, x, \overline{e}_{q_l}) = \dot{\underline{e}}_{q_l}$.

Recalling Equations (8), it can be easily verified that the optimal gain values in $(\beta q_n, +\infty)$ are $\tilde{\theta}_1 = \tilde{\theta}_2 = \beta q_n$.

IV. Estimation of the optimal interval observer gain

A. Motivation

In the following, we assume that there exists some optimal interval observer gain, noted $\tilde{\theta}(t) \in \Theta$, which optimizes some observer performance (*e.g.* the interval decreasing rate, see section III-C or [7]). The constraint when using a time varying parameter in the design of the framer is that its derivative $\dot{\theta}(t)$ must be used to compute $\overline{z}$ and $\underline{z}$. When dealing with an optimal value of this parameter, the computation of this derivative may reveal delicate since it may depend on the (unknown) state variable of system (Σ) (see section III-C or [7] for an example of the computation of this optimal parameter). Here we propose to use bounds on this optimal gain, and to introduce a dynamics of adaptation so that the gain θ converges towards the optimal gain $\tilde{\theta}(t)$:

$$\dot{\theta} = \phi(\tilde{\theta}, \theta, \overline{\theta}, \underline{\theta}) \tag{10}$$

where adaptation dynamics of θ are driven by the mapping ϕ. Functions $\overline{\theta}$ and $\underline{\theta}$ are bounds to ensure that θ stays in the parametric domain Θ for which system (3) is guaranteed to be a framer. Moreover, we assume that there exists a positive constant ϵ such that, for any time t, $\underline{\theta}(t) + \epsilon < \tilde{\theta} < \overline{\theta}(t) - \epsilon$.

The mapping ϕ should be defined to guaranty that θ converges to $\tilde{\theta}$. In practice, this convergence must be fast compared to the original system (3), a high gain strategy in the adaptation dynamics has therefore been chosen.

Since θ must be bounded between two known bounds $\underline{\theta}$ and $\overline{\theta}$ to satisfy sign conditions required for the framer, a possible choice of ϕ can be (for each component θ_i)

$$\begin{cases} \dot{\theta}_i = K_i^a(\tilde{\theta}_i - \theta_i)\left(1 + \frac{\epsilon}{\overline{\theta_i} - \theta_i} + \frac{\epsilon}{\theta_i - \underline{\theta_i}}\right) \\ \\ \theta_i(t_0) \in]\underline{\theta}_i(t_0), \overline{\theta}_i(t_0)[\end{cases} \tag{11}$$

where K_i^a are positive adaptation gains and ϵ a small constant.

For sake of simplicity, since the dynamics of the observer gains θ_i are uncoupled, we will focus on one of the components and omit the subscript i in the following. Thus, we will write generically θ instead of θ_i.

Property 1: Assuming that the derivatives of $\underline{\theta}$ and $\overline{\theta}$ are bounded in norm by a constant M, and that there exists a positive constant ϵ such that $\underline{\theta}(t) + \epsilon < \tilde{\theta} < \overline{\theta}(t) - \epsilon$, then Equation (11) guarantees that for any $\theta(t_0)$ such that $\underline{\theta}(t_0) < \theta(t_0) < \overline{\theta}(t_0)$, we have $\underline{\theta}(t) < \theta(t) < \overline{\theta}(t)$, $\forall t > t_0$.

Proof. Let us consider $\overline{e}_\theta = \overline{\theta} - \theta$ (remember that these equations indeed refer to one of the gain components, and that indexes are omitted). Its dynamics is:

$$\dot{\overline{e}}_\theta = \dot{\overline{\theta}} - K^a(\tilde{\theta} - \overline{\theta} + \overline{e}_\theta)\left(1 + \frac{\epsilon}{\overline{e}_\theta} + \frac{\epsilon}{\overline{\theta} - \underline{\theta} - \overline{e}_\theta}\right)$$

We focus on the dynamics of $\overline{e}_\theta$ for $\overline{e}_\theta < \epsilon$ to evaluate the sign of $\dot{\overline{e}}_\theta$ when $\overline{e}_\theta$ cancels.

Note that $\tilde{\theta} - \theta = \tilde{\theta} - \overline{\theta} + \overline{e}_\theta < -\epsilon + \overline{e}_\theta$. Since $\dot{\overline{\theta}} \geq -M$, we get

$$-M + K^a(\epsilon - \overline{e}_\theta)\left(1 + \frac{\epsilon}{\overline{e}_\theta} + \frac{\epsilon}{\overline{\theta} - \underline{\theta} - \overline{e}_\theta}\right) < \dot{\overline{e}}_\theta \tag{12}$$

If $\overline{e}_\theta$ tends to 0, then the left hand side of (12) tends to infinity. Therefore, there exists a positive η such that, when $\overline{e}_\theta < \eta$, $\dot{\overline{e}}_\theta > 0$. As a consequence, $\overline{e}_\theta$ cannot cancel when it has been positively initiated.□

In order to study the convergence of θ towards the optimal value $\tilde{\theta}$, let us define $\delta(t) = \theta(t) - \tilde{\theta}(t)$ whose dynamics is:

$$\dot{\delta} = -K^a\delta\left(1 + \frac{\epsilon}{\overline{\theta} - \tilde{\theta} - \delta} + \frac{\epsilon}{\delta + \tilde{\theta} - \underline{\theta}}\right) - \dot{\tilde{\theta}} \tag{13}$$

Given Property 1, we can restrict our analysis to the positively invariant (time-varying) set $\Delta = \{\delta \in \mathbb{R} | \underline{\theta} - \tilde{\theta} < \delta < \overline{\theta} - \tilde{\theta}\}$.

Property 2: In the case where, for any time, $\dot{\tilde{\theta}} = 0$, equation (13) admits a unique GAS equilibrium $\delta^* = 0$ in the positively invariant set Δ.

Proof. The roots of Equation (13) with $\dot{\tilde{\theta}} = 0$ are $\delta_a = 0$ and the solutions of the polynomial equation:

$$-\delta^2 + \delta(\overline{\theta} + \underline{\theta} - 2\tilde{\theta}) - (\overline{\theta} - \tilde{\theta})(\underline{\theta} - \tilde{\theta}) + \epsilon(\overline{\theta} - \underline{\theta}) = 0 \quad (14)$$

The discriminent of (14) reads:

$$\Delta = (\overline{\theta} - \underline{\theta})^2 + 4\epsilon(\overline{\theta} - \underline{\theta}) > 0,$$

so Equation (13) admits 2 other solutions:

$$\begin{array}{rcl} \delta_b & = & \frac{(\overline{\theta}+\underline{\theta}-2\tilde{\theta})+\sqrt{(\overline{\theta}-\underline{\theta})^2+4\epsilon(\overline{\theta}-\underline{\theta}))}}{2} \\ & > & \frac{(\overline{\theta}+\underline{\theta}-2\tilde{\theta})+\sqrt{(\overline{\theta}-\underline{\theta})^2}}{2} = \overline{\theta} - \tilde{\theta} \end{array}$$

and:

$$\begin{array}{rcl} \delta_c & = & \frac{(\overline{\theta}+\underline{\theta}-2\tilde{\theta})-\sqrt{(\overline{\theta}-\underline{\theta})^2+4\epsilon(\overline{\theta}-\underline{\theta}))}}{2} \\ & < & \frac{(\overline{\theta}+\underline{\theta}-2\tilde{\theta})-\sqrt{(\overline{\theta}-\underline{\theta})^2}}{2} = \underline{\theta} - \tilde{\theta} \end{array}$$

Therefore, $\delta = 0$ is the unique equilibrium in Δ.

Since it is locally stable in dimension 1, it is globally stable. □

We can now extend this property to the perturbation framework, i.e. where $\tilde{\theta}$ is time–varying.

Property 3: Considering $\dot{\tilde{\theta}}$ as a (bounded) perturbation input, the system (13) in the invariant set Δ is input-to-state stable (ISS), i.e. there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $\delta(t_0) \in \Delta$ and any bounded input $\dot{\tilde{\theta}}(t)$, the solution $\delta(t)$ satisfies:

$$\|\delta(t)\| \leq \beta(\|\delta(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\dot{\tilde{\theta}}(\tau)\|\right)$$

Moreover, the mapping $\gamma(r) = \frac{r}{(1-\alpha)K^a}$, (with $0 < \alpha < 1$) can be as small as desired on Δ by an appropriate choice of K^a.

Proof. Taking $V = \delta^2/2$ as an ISS-Lyapunov function candidate, the derivative of V with respect to Equation (13) is given by:

$$\dot{V} = -K^a\delta^2\left(1 + \frac{\epsilon}{\overline{\theta} - \tilde{\theta} - \delta} + \frac{\epsilon}{\delta + \tilde{\theta} - \underline{\theta}}\right) - \delta\dot{\tilde{\theta}}$$

Since $\delta \in \Delta$, the term in brackets is greater than one, so we get:

$$\dot{V} < -(1-\alpha)K^a\delta^2 - \alpha K^a\delta^2 - \delta\dot{\tilde{\theta}}$$

where $0 < \alpha < 1$. Let us denote M the upper bound of the perturbation: $|\dot{\tilde{\theta}}| \leq M$

$$\dot{V} < -(1-\alpha)K^a\delta^2 - \alpha K^a\delta^2 + M|\delta|$$

Then, $\forall|\delta| \geq \frac{M}{(1-\alpha)K^a}$, we have $\dot{V} < -\alpha K^a\delta^2$. Thus, applying Theorem 4.19 from [21], the system is ISS with $\gamma(r) = \frac{r}{(1-\alpha)K^a}$. □

Properties 1, 2 and 3 ensure that θ will stay in the parametric domain Θ and will converge towards the optimal time-varying value $\tilde{\theta}$, the attraction domain can be as small as desired by the choice of a large gain K^a. In particular, from Property 3, $\delta(t)$ remains bounded for bounded input $\dot{\tilde{\theta}}$, with an ultimate bound which is a function of the input magnitude. Moreover, if the optimal gain $\tilde{\theta}$ is constant, then system (13) is globally asymptotically stable. In practice, ϵ can be chosen small enough such that the two rational functions of Equations (11) affect the dynamic only if θ is very close to a bound.

V. Application to microalgae lipid production (continued)

A. Optimal framer

The implementation of the optimal framer defined by equations (7) with $\theta = \beta q_n$ require derivative computation of q_n, or at least bounded estimations, which are difficult to provide accurately. Therefore, in application of the main idea presented in Section IV, framer (7) is modified introducing an adaptive dynamics for θ:

Proposition 2: Given $\underline{z}_0$, $\overline{z}_0$ such that $z(t_0) \in [\underline{z}_0, \overline{z}_0]$, the following system is a framer for system (4):

$$\left\{\begin{array}{l} \dot{\overline{z}} = D(\gamma s_{in} - \overline{z}) + (\theta - \beta q_n)\overline{\mu}(q_n)x + \dot{\theta}x \\ \dot{\underline{z}} = D(\gamma s_{in} - \underline{z}) + (\theta - \beta q_n)\underline{\mu}(q_n)x + \dot{\theta}x \\ \dot{\theta} = \phi\left(\beta q_n + \epsilon, \theta, +\infty, \beta q_n\right) \\ \theta(t_0) > \beta q_n(t_0) \\ \overline{q_l} = \theta - (\underline{z} - \gamma s)/x \\ \underline{q_l} = \theta - (\overline{z} - \gamma s)/x \end{array}\right. \quad (15)$$

where the mapping ϕ is defined by Equation (11).

Proof. From Property 1, the mapping ϕ ensures that $\theta(t) > \beta q_n(t)$ for any positive time. Therefore, in direct consequence of Proposition 1, system (15) is a framer for system (4). □

Note that in this case, the optimal gain should be chosen as close as possible to the lower bound. Nevertheless, to satisfy the hypotheses of Property 1, we select the near optimal gain: $\tilde{\theta} = \beta q_n + \epsilon$, for a small ϵ.

B. Including the uncertainties on parameters and the measurement noise

In the following, we consider a realistic framework where all model parameters suffer from uncertainties and only s and x are measured, with noise perturbation. First, we provide an interval estimation of q_l, and then, a new framer for system (4) is proposed using the estimation of q_l in the gain dynamics.

Hypothesis 2: Online measurements $y_s(t)$ and $y_x(t)$ are perturbed by noises $\delta_s(t)$ and $\delta_x(t)$. We assume that these perturbations are of multiplicative nature:

$$y_s(t) = s(t)(1 + \delta_s(t)) \quad \text{and} \quad y_x(t) = x(t)(1 + \delta_x(t))$$

Moreover, these noise signals are bounded such that $|\delta_s(t)| \leq \Delta_s < 1$ and $|\delta_x(t)| \leq \Delta_x < 1$.

We can define dynamic bounds for the substrate and the biomass:

$$\underline{y}_s(t) \leq s(t) \leq \overline{y}_s(t) \quad \text{and} \quad \underline{y}_x(t) \leq x(t) \leq \overline{y}_x(t)$$

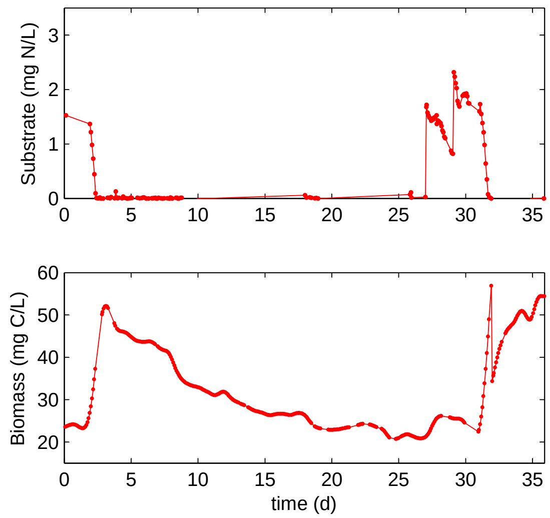

Fig. 1. Experimental measurements of *Isochrysis affinis galbana* (clone T-iso) cultures. Substrate (top) and biomass measurements (down) [23].

with

$$\begin{cases} \underline{y}_s(t) = \frac{y_s(t)}{(1+\Delta_s)} \\ \overline{y}_s(t) = \frac{y_s(t)}{(1-\Delta_s)} \end{cases} \quad \text{and} \quad \begin{cases} \underline{y}_x(t) = \frac{y_x(t)}{(1+\Delta_x)} \\ \overline{y}_x(t) = \frac{y_x(t)}{(1-\Delta_x)} \end{cases}$$

1) Interval estimation of q_n: An asymptotic interval estimator [22] of the nitrogen quota q_n is designed using a change of variable to eliminate the reaction rates $\rho(s)$ and $\mu(q)$

$$\zeta = s + q_n x$$

whose dynamics is $\dot{\zeta} = D(s_{in} - \zeta)$.

Property 4: Given $\underline{s_{in}}$ and $\overline{s_{in}}$ such that $s_{in} \in [\underline{s_{in}}; \overline{s_{in}}]$, and $\underline{\zeta}_0$ and $\overline{\zeta}_0$ such that $\zeta_0 \in [\underline{\zeta}_0; \overline{\zeta}_0]$, the following framer will provide bounds for the nitrogen quota q_n:

$$\begin{cases} \dot{\overline{\zeta}} = D(\overline{s_{in}} - \overline{\zeta}) \\ \dot{\underline{\zeta}} = D(\underline{s_{in}} - \underline{\zeta}) \\ \overline{q}_n = (\overline{\zeta} - \underline{y}_s)/\underline{y}_x \\ \underline{q}_n = (\underline{\zeta} - \overline{y}_s)/\overline{y}_x \end{cases} \tag{16}$$

Proof. Computing the dynamics of the errors $\overline{e}_\zeta = \overline{\zeta} - \zeta$ and $\underline{e}_\zeta = \zeta - \underline{\zeta}$, it is straightforward to show that they stay positive after a positive initialization. Then, we have:

$$\begin{aligned} \overline{q}_n &= (\overline{\zeta} - \underline{y}_s)/\underline{y}_x \geq (\zeta - y_s)/y_x = q_n \\ \underline{q}_n &= (\underline{\zeta} - \overline{y}_s)/\overline{y}_x \leq (\zeta - y_s)/y_x = q_n \end{aligned} \tag{17}$$

which concludes the proof. □

2) Interval estimation of q_l: The framer equations (15) have been modified in order to take into account all the uncertainties:

Property 5: Given $\underline{\zeta}_0$ and $\overline{\zeta}_0$ such as $\zeta_0 \in [\underline{\zeta}_0; \overline{\zeta}_0]$ and $\underline{z}_0$ and $\overline{z}_0$ such as $z_0 \in [\underline{z}_0; \overline{z}_0]$, the following framer will provide bounds for the lipid quota q_l:

$$\begin{cases} \dot{\overline{z}} = D(\overline{\gamma s_{in}} - \overline{z}) + (\theta - \underline{\beta q_n})\overline{\mu}(\overline{q_n})\overline{y}_x \\ \qquad\qquad + \dot{\theta}(\sigma\overline{y_x} + (1-\sigma)\underline{y_x}) \\ \dot{\underline{z}} = D(\underline{\gamma s_{in}} - \underline{z}) + (\theta - \overline{\beta}\overline{q_n})\underline{\mu}(\underline{q_n})\underline{y}_x \\ \qquad\qquad + \dot{\theta}(\sigma\underline{y_x} + (1-\sigma)\overline{y_x}) \\ \dot{\overline{\zeta}} = D(\overline{s_{in}} - \overline{\zeta}) \\ \dot{\underline{\zeta}} = D(\underline{s_{in}} - \underline{\zeta}) \\ \\ \dot{\theta} = \phi\left(\overline{\beta}\overline{q_n} + \epsilon, \theta, +\infty, \overline{\beta}\overline{q_n}\right) \\ \theta(t_0) > \overline{\beta}\overline{q}_n(t_0) \\ \\ \overline{q_l} = \theta - (\underline{z} - \overline{\gamma}\overline{y}_s)/\overline{y}_x \\ \underline{q_l} = \theta - (\overline{z} - \underline{\gamma}\underline{y}_s)/\underline{y}_x \\ \\ \overline{q}_n = (\overline{\zeta} - \underline{y}_s)/\underline{y}_x \\ \underline{q}_n = (\underline{\zeta} - \overline{y}_s)/\overline{y}_x \end{cases} \tag{18}$$

where the mapping ϕ is defined by Equation (11) and $\sigma = \begin{cases} 1 & \text{if } \dot{\theta} \geq 0 \\ 0 & \text{otherwise} \end{cases}$, uses the sign of $\dot{\theta}$ in order to provide the correct bounding.

Proof. First, from Property 4, we have $\underline{q}_n(t) \leq q_n(t) \leq \overline{q}_n(t)$. Using the dynamics of the errors $\overline{e}_z = \overline{z} - z$ and $\underline{e}_z = z - \underline{z}$, one can show that they stay positive after a positive initialization. Then, considering model (4), we have $q_l < \beta q_n$, and therefore $z - \gamma s = (\theta - q_l) > 0$. We can finally conclude the proof:

$$\overline{q}_l = \theta - (\underline{z} - \overline{\gamma}\overline{y}_s)/\overline{y}_x \geq \theta - (z - \gamma s)/x = q_l$$

$$\underline{q}_l = \theta - (\overline{z} - \underline{\gamma}\underline{y}_s)/\underline{y}_x \leq \theta - (z - \gamma s)/x = q_l$$

□

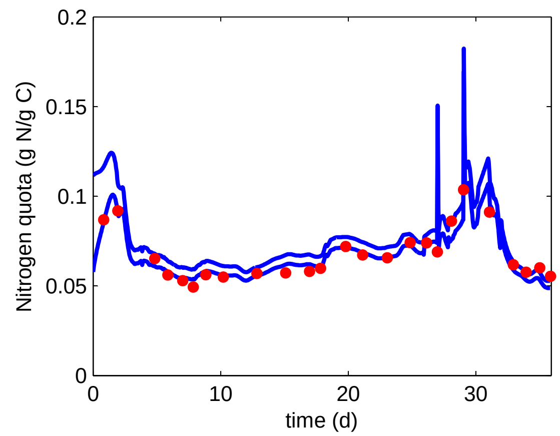

Fig. 2. Estimation of the nitrogen quota using the asymptotic interval framer (16) .

C. Experimental validation

The framer performances are assessed with experimental data of *Isochrysis affinis galbana* (clone T-iso) cultures. The

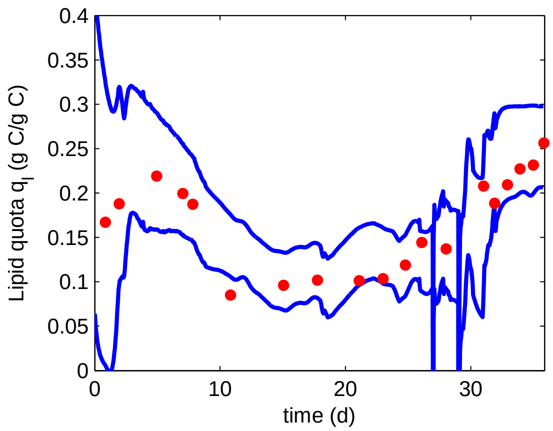

Fig. 3. Interval estimations of the neutral lipid quota provided by framer (18) with a dynamic gain.

experiment consists in imposing various nitrogen limitations through a succession of dilution rates changes. More details about the experiment and model parameter values can be found in [17], [23]. We consider a $\pm 10\%$ uncertainty for the maximum specific growth rate $\bar{\mu}$, a $\pm 2\%$ uncertainty for the other model parameters and a multiplicative noise on the measurements up to a 2%. We take for adaptation gain value $K^a = 10$. Figure 1 presents the substrate and biomass measurements $y_s(t)$ and $y_x(t)$ which are used to estimate the nitrogen and lipid quotas. The framer (18) provides an accurate interval estimation of the nitrogen and lipid quotas (see Figures 2 and 3). Note that we can observe perturbations at days 27 and 29 due to nitrogen feeding impulses. After that, the framer (18) rapidly provides an accurate estimation.

VI. Conclusions

We have proposed a new design for near optimal interval estimation. The idea relies on a high gain adaptive dynamics ensuring tracking of the optimal gain and of its derivative. It is worth noting that, because of the uncoupling between the gain dynamics, and since the optimal gain is maintained within a bounded interval, no strong peaking phenomenon is observed. A crucial point is that, even if the observer may not be optimal during a first transient of the adaptive gain, the conditions for framer design are always guaranteed and the predicted interval is always encompassing the real state.

The application of the proposed design to the real case of lipid production by microalgae in photobioreactors demonstrates the efficiency of the approach on a non trivial case. The proposed estimates of lipid content tend to be very accurate and may complement or even replace, the very difficult and expensive analytical measurements.

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