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A matrix sign function based solution of parameter dependent Sylvester equations

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***Abstract*—This paper focuses on some parameter dependent Sylvester equations arising in systems and control theory. The matrices involved are assumed to be i) parameter dependent, ii) not necessarily of the same size, and iii) with possible common eigenvalues depending on the parameter value. A matrix sign function based solution is proposed, considering two main cases: the square coefficient matrices are diagonalizable or block-diagonalizable.**

I. Introduction

The Sylvester equation:

$$BX - XA = C \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ and $X \in \mathbb{R}^{m\times n}$, has been extensively considered in the literature. Standard algorithms to solve it are based on the Bartels-Stewart method [1] and the Hessenberg-Schur method ones [2]. One can quote also, the spectral decomposition based solution presented in [3] or the polynomial based solution given in [4]. One more is the matrix sign function based method first proposed in [5], in the case of the Sylvester equation is "stable" (i.e. when the spectrums of matrices $-A$ and B in of the form (1) lie in the open left half plane). Recently, a modified matrix sign function method was proposed in [6] for solving particular generalized Sylvester equations arising in analysis and control problems of descriptor systems.

In the present paper, a matrix sign function based method is proposed with the aim to deal with the parameter-dependent Sylvester equation:

$$B_\theta(\theta)X(\theta) - X(\theta)A_\theta(\theta) = C_\theta(\theta),\ \theta \in \Omega_\theta \subset \mathbb{R}^\gamma \tag{2}$$

Sylvester equation (1) has many applications in various fields, for instance, in control theory (control and filtering design, and model reduction; see for instance [7]) and signal processing. Specifically in control design, particular cases of great interest for the authors concern problems of asymptotic (output) regulation [8] and extended (comprehensive) control ([9], [10] and references cited therein) that lead to Sylvester equations of the form (1). It is well-established in the literature when and how this problem can be solved in a constant setting (i.e. with constant coefficient matrices, see previous references). The parametric case, however, is rarely mentioned in the literature except in [11] and thus could be an important application of the new matrix sign function based method proposed in this paper.

The idea underlying the proposed method lies upon two results: i) a matrix sign function based computational method for computing eigenvalues of a matrix [12], ii) and a solution to the Sylvester equation based on the spectral decomposition of matrices [3]. Note that several attempts to separate eigenvalues of a given matrix using the matrix sign function exist in the literature. See for instance [13] which give a different insight to the use of the matrix sign function in this context. The use of the result in [12] is dictated by the aim of proposing a method that can be extended to the parameter dependent case (2).

The remainder of this paper is organized as follows. Section II presents the matrix sign function. It also sketches the lines of the matrix sign algorithm for stable Sylvester equations. Section III is devoted to the main result, which is extended to the parametric case in Section IV. Finally, two examples are presented in order to demonstrate the efficiency of the approach proposed.

Notations: Hereafter $\otimes$ denotes the Kronecker product of matrices, $\mathbb{C}^*$ denotes the union of the open right-half ($\mathbb{C}_+$) and the open left-half ($\mathbb{C}_-$) complex plane, and I_n the $n\times n$ identity matrix. $\sigma^+(X)$ and $\sigma^-(X)$ denote respectively the set of eigenvalues, stable and anti-stable ones. Finally, *diag* is used for block-wise diagonal matrices.

II. Preliminaries

In this section, we introduce some preliminary definitions and results that are used extensively in the sequel.

A. The matrix sign function

Let us introduce the matrix sign function developed in [14]. It is well-known that the sign function is defined for $z \in \mathbb{C}^*$ by:

$$sign(z) = \begin{cases} 1, & \text{if } \mathrm{Re}(z) > 0 \\ -1, & \text{if } \mathrm{Re}(z) < 0 \end{cases} \tag{3}$$

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If $\mathrm{Re}(z)=0$ then $sign(z)$ is undefined. By analogy, the matrix sign function is restricted to square matrices with no eigenvalues on the imaginary axis. Let the matrix $Z\in\mathbb{R}^{n\times n}$ have a Jordan canonical form that is: $Z\triangleq TJT^{-1}$ with $J=diag(J_1,J_2)$ where the square matrices J_1 and J_2 have eigenvalues respectively in $\mathbb{C}^-$ and $\mathbb{C}^+$. Then the sign of the matrix $Z\in\mathbb{R}^{n\times n}$ can be defined as:

$$sign(Z)\triangleq T\begin{bmatrix}-I_{J_1} & 0\\ 0 & I_{J_2}\end{bmatrix}T^{-1} \tag{4}$$

where I_{J_1} and I_{J_2} are identity matrices of the same dimensions as J_1 and J_2 respectively. An iterative method to calculate the matrix sign is, for instance, the so-called Newton iterations [14], [15]:

$$\begin{aligned}&Z_0=Z\\&Z_{k+1}=\frac{1}{2}\left(Z_k+Z_k^{-1}\right),\ k=1,2,3,\ldots\\&sign(Z)=\lim_{k\to\infty}Z_k\end{aligned} \tag{5}$$

Other iterative methods exist to compute the matrix sign function. The interested reader can refer to [14] for more details.

Note that the matrix sign function is undefined for matrices with either zeros or purely imaginary eigenvalues. Moreover, iterative computational schemes for computing the matrix sign function may become nonconvergent (or has a slow convergence) when Z has eigenvalues close to the imaginary axis. A way to overcome this difficulty is to consider a sufficiently small positive scalar ε such that $sign\left(Z\pm\varepsilon I_n\right)$ are both defined. Hence,

$$sign(Z)=\frac{1}{2}\left(sign(Z+\varepsilon I)+sign(Z-\varepsilon I)\right) \tag{6}$$

B. *Matrix sign based solution for stable Sylvester equation*

Consider the Sylvester equation of the form (1) and the associated Hamiltonian matrix given by:

$$H=\begin{bmatrix}-A & 0\\ C & -B\end{bmatrix} \tag{7}$$

Let us note the matrix sign function of H:

$$sign(H)=\begin{bmatrix}S_{11} & 0\\ S_{21} & S_{22}\end{bmatrix} \tag{8}$$

According to [15], the matrix H can be block-diagonalized by using the similarity transformation $\begin{bmatrix}I & 0\\ X & I\end{bmatrix}$. Thus the computation of the matrix sign function of H leads to:

$$sign(H)=\begin{bmatrix}I & 0\\ X & I\end{bmatrix}sign\left(\begin{bmatrix}-A & 0\\ 0 & -B\end{bmatrix}\right)\begin{bmatrix}I & 0\\ X & I\end{bmatrix}^{-1}$$

Hence,

$$\begin{bmatrix}S_{11} & 0\\ S_{21} & S_{22}\end{bmatrix}=\begin{bmatrix}-sign(A) & 0\\ -Xsign(A)+sign(B)X & -sign(B)\end{bmatrix} \tag{9}$$

And finally, $S_{21}=-Xsign(A)+sign(B)X$. This new Sylvester equation reduces to $S_{21}=-2X$ in the case of stable Sylvester equation. Unfortunately, this cannot be applied in the general case (1).

III. A Matrix Sign Function Based Algorithm to Solve a General Sylvester Equation

In this section, a new matrix sign function based method is proposed to solve the Sylvester equation (1). The method consists of three steps. The matrix sign function is first used to separate the eigenvalues of matrices A and B. Then the spectral decomposition of these matrices (i.e. computing projectors) is derived. Finally, an explicit solution X is derived based on the previous steps.

The idea underlying this method lies upon two results: a matrix sign function based computational method for computing eigenvalues of a matrix [12] and a solution to the Sylvester equation based on the spectral decomposition of matrices [3]. Our main goal here is to propose a method that can be extended to the parameter dependent case (2).

A. *Diagonalizable stationary coefficient matrices case*

Step 1: Spectrum splitting

Let's consider $Z\in\mathbb{R}^{n\times n}$, a square matrix whose Jordan canonical form is:

$$Z=ML_ZM^{-1} \tag{10}$$

with L_Z a triangular matrix of Jordan blocks associated to the eigenvalues of Z, and M the matrix of the eigenvectors of Z. Then, the matrix sign of Z is given by:

$$sign(Z)=Msign\left(L_Z\right)M^{-1} \tag{11}$$

In this subsection, the matrix Z is assumed to be diagonalizable with $\sigma^+(Z)\neq\varnothing$ and $\sigma^-(Z)\neq\varnothing$. $\sigma(Z)=\left\{\lambda_{Z_0},\ldots,\lambda_{Z_{n-1}}\right\}$ defines the spectrum of Z.

According to [12], a first spectrum splitting can be obtained with V_0 as follows:

$$V_0=sign(Z)+sign\left(L_Z\right),\ V_0^{-1}ZV_0=\begin{bmatrix}Z_1 & 0\\ 0 & Z_2\end{bmatrix}$$

The eigenvalues of Z_1 and Z_2 have positive and negative real parts respectively. This spectrum splitting can be carried forward by calculating the matrices $\overline{Z}_1$ and $\overline{Z}_2$ as follows:

$$\overline{Z}_k=Z_k-\left(\frac{trace\left(Z_k\right)}{\dim\left(Z_k\right)}\right)I_{\dim\left(Z_k\right)},\ k=1,2 \tag{12}$$

Then, based on the sign of the resulting matrices $\bar{Z}_1$ and $\bar{Z}_2$, one can partition Z_1 and Z_2 into block diagonal matrices with submatrices of lower order using:

$$V_k = sign\left(\bar{Z}_k\right) + sign\left(L_{\bar{Z}_k}\right),\ k = 1,2$$

The iterations can be continued until all the sub matrices are reduced to a Jordan form.

Remark 1: Defective matrices require an additional computation step. Indeed, if $sign(Z) = I_n$ it shows that eigenvalues of Z are equal or have same signs. To determine the repetition of the eigenvalues, we compute:

$$\bar{\bar{Z}} = Z - \left(\frac{trace(Z)}{n}\right) I_n \tag{13}$$

If $\bar{\bar{Z}}$ is singular it means that all eigenvalues are similar and thus, an additional step is necessary and will be described in the next subsection.

Step 2: Computing projectors

The matrices E_j of the spectral decomposition $Z = \sum_{j=0}^{n-1} \lambda_{Z_j} E_j$, where λ_{Z_j} are the distinct non-zero eigenvalues of the matrix Z, are such that:

$$E_j^2 = E_j,\ \sum_{j=0}^{n-1} E_j = I,\ E_j E_k = 0,\ j \neq k \tag{14}$$

A way to compute these projectors (see [12] for instance) is given by:

$$E_j = \left(\prod_{i=0}^{n-1}\left(1 - \frac{\lambda_{Z_j}}{\lambda_{Z_i}}\right)\right)^{-1} . \prod_{i=0}^{n-1}\left(I_n - \frac{Z}{\lambda_{Z_i}}\right), i \neq j \tag{15}$$

Remark 2: Note that equation (15) makes use of some matrix products and thus can be easily computed in the parameter dependent case, exploiting the particularity of the class of parametric matrices that will be considered in the next section.

Step 3: Towards the analytic solution of (1)

Let $A = \sum_{j=0}^{n-1} \lambda_{A_j} E_j$ and $B = \sum_{k=0}^{m-1} \lambda_{B_k} F_k$ be the spectral decomposition of the diagonalizable matrices A and B where λ_{A_j} and λ_{B_k} are respectively their distinct non-zero eigenvalues. The matrices E_j and F_k are idempotent and verify the properties given in (14).

Theorem 1: [3] The Sylvester equation (1) with stationary non-defective coefficient matrices admits a solution if and only if whenever there exist i and k such that $\lambda_{A_j} = \lambda_{B_k}$ then $F_k C E_j = 0$. This solution is then given by:

$$X = \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \frac{1}{\left(\lambda_{B_k} - \lambda_{A_j}\right)} F_k C E_j \tag{16}$$

B. Defective stationary coefficient matrices case

Step 1: Spectrum splitting

A defective matrix $Z \in \mathbb{R}^{n\times n}$ requires a modification of the proposed procedure as noted in Remark 1.

In this subsection, and without lost of generality, Z is supposed to have only a repeated eigenvalue that will be noted λ_Z. Otherwise a spectrum splitting can be performed as previously described in Step 1 of § III.A.

Since in this case $sign(Z) = \pm I_n$, then it is easy to see that

$$\lambda_Z = \frac{trace(Z)}{n} \tag{17}$$

Note that in this case $\bar{\bar{Z}}$, computed using equation (13), is singular and thus do not admit a matrix sign.

Step 2: Computing projectors

In this step, the matrix $Z \in \mathbb{R}^{n\times n}$ is supposed to be defective but not with full degeneracy. The matrices E_j of the spectral decomposition $Z = \sum_{j=0}^{n_1-1} \lambda_{Z_j} E_j + N_j$, where λ_{Z_j} are the non-zero eigenvalues matrix Z. $n_1 \le n$ is the number of distinct eigenvalues of Z, are idempotent and verify the properties (14). The matrices N_j are nilpotent and verify the following conditions

$$E_j N_j = N_j E_j = N_j,\ E_j N_k = N_j N_k = 0, \forall j \neq k \tag{18}$$

These matrices can be computed as follows:

$$N_j = \left(A - \lambda_{A_j} I_n\right) E_j \tag{19}$$

Note that computing λ_{Z_j} is possible according to the procedures presented in Step 1 in § III.A and § III.B.

Furthermore, to compute matrices E_j consider, for instance, the function

$$f(Z) = \sum_{j=0}^{n-1}\sum_{p=0}^{p_j-1} \frac{f^{(p)}}{p!}\left(Z - \lambda_{Z_j} I_n\right)^p E_j \tag{20}$$

where $p_j = index\left(\lambda_{Z_j}\right), j = 0,1\cdots(n_1 - 1)$. The spectral projectors E_j can be computed iteratively if some judicious choices of f are made such as the following set of polynomials:

$$\begin{aligned} f(\lambda) = \Big\{ & 1, \left(\lambda - \lambda_{A_1}\right)^{p_1}, \left(\lambda - \lambda_{A_1}\right)^{p_1}\left(\lambda - \lambda_{A_2}\right)^{p_2}, \ldots \\ & \ldots, \left(\lambda - \lambda_{A_1}\right)^{p_1} \cdots \left(\lambda - \lambda_{A_{n_1}}\right)^{p_{n_1}} \Big\} \end{aligned} \tag{21}$$

Step 3: Towards the analytic solution of (1)

Let $A = \sum_{j=0}^{n_1-1} \lambda_{A_j} E_j + N_j$ and $B = \sum_{k=0}^{m_1-1} \lambda_{B_k} F_k + M_k$ be the spectral decompositions of respectively matrices A and B, λ_{A_j} and λ_{B_k} respectively the non-zero eigenvalues of the defective matrices A and B. Let also $n_1 \le n$ and $m_1 \le m$ be the number of distinct eigenvalues respectively of matrices A and B.

Theorem 2: [3] The Sylvester equation (1) with stationary defective coefficient matrices admits

$$X = \sum_{k=0}^{m_1-1} \sum_{j=0}^{n_1-1} \frac{1}{\left(\lambda_{B_k} - \lambda_{A_j}\right)} F_k C E_j \tag{22}$$

as a solution if whenever there exist i and k such that $\lambda_{A_j} = \lambda_{B_k}$ then $F_k C N_j = M_k C E_j$, $F_k C E_j = 0$ and $\sum_{k=0}^{m_1-1} \sum_{j=0}^{n_1-1} \frac{1}{\left(\lambda_{B_k} - \lambda_{A_j}\right)} \left(F_k C N_j \text{ - } M_k C E_j\right) = 0$.

IV. A MATRIX SIGN FUNCTION SOLUTION OF THE PARAMETER DEPENDENT SYLVESTER EQUATION

In this section an extension to the parameter dependent case (2) of the previously proposed procedure is presented. To this aim, we introduce the following class of parameter dependent matrices:

$$Z(\theta) \triangleq Z_e\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n},\ \theta \in \Omega_\theta \tag{23}$$

with $Z_e \in \mathbb{R}^{n\times nw^\gamma}$, $\bar{\theta} = \left[\bar{\theta}_1 \otimes \cdots \otimes \bar{\theta}_\gamma\right]^T \in \mathbb{R}^{w^\gamma}$ and $\bar{\theta}_k = \left[\theta_k^{-l} \ \cdots \ \theta_k^{-l} \ 1 \ \theta_k^1 \ \cdots \ \theta_k^m\right]^T \in \mathbb{R}^w$, $\forall k \in [1,\gamma]$.

In the sequel, $Z(\theta)$ is supposed to be non singular. Basic arithmetic over the class of parameter-dependent matrices introduced here follows directly according to the proposed definition (23) and is summarized in the following lemma.

Lemma 1: Given $Z(\theta) = Z_e\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n}$, $V(\theta) = V_e\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n}$, a partition of Z_e given by $Z_e = \left[Z_1 \ \cdots \ Z_w\right]$, $Z_{i\in[1,w]} \in \mathbb{R}^{n\times n}$ and $\bar{\theta} = \left[\bar{\theta}_1 \otimes \cdots \otimes \bar{\theta}_\gamma\right]^T$ then the sum $Z(\theta) + V(\theta)$, the product $Z(\theta)V(\theta)$ and the trace of $Z(\theta)$ can be computed by:

$$Z(\theta) + V(\theta) = \left(Z_e + V_e\right)\left(\bar{\theta} \otimes I_n\right) \tag{24}$$

$$Z(\theta) \cdot V(\theta) = Z_e\left(I_{m^\gamma} \otimes V_e\right)\left(\left(\bar{\theta} \otimes \bar{\theta}\right) \otimes I_n\right) \tag{25}$$

$$trace(Z(\theta)) = \left[trace\left(Z_1\right) \ \ldots \ trace\left(Z_w\right)\right]\bar{\theta} \tag{26}$$

In this section, instead of the Newton iterations which give a rational approximation of $sign\left(Z(\theta)\right)$ an integral definition is given for the computation of $sign\left(Z(\theta)\right)$ leading to an exact expression of $sign\left(Z(\theta)\right)$.

Theorem 3: Consider a parametric matrix $Z(\theta)$ given by (23). The matrix sign of $Z(\theta)$, denoted $sign\left(Z(\theta)\right)$, is given by:

$$sign\left(Z(\theta)\right) = \frac{2}{\pi} Z(\theta) . \sum_{k=1}^{n} \int_0^\infty -\frac{p_{k-1}}{p_n}\left(\sum_{i=0}^{n-k}\binom{n-k}{i} y^{2(n-k-i)} Z(\theta)^{2i}\right) dy \tag{27}$$

where $\begin{cases} p_k = -\frac{1}{k}\sum_{j=1}^{k} p_{k-j}\left(\sum_{i=0}^{j}\binom{j}{i} y^{2(j-i)} trace\left(Z(\theta)^{2i}\right)\right). \\ p_0 = 1 \end{cases}$

Proof: The proof of this Theorem can be found in [17].

Assumption (A1): it is supposed that the matrix $Z(\theta)$, $\theta \in \Omega_\theta$ is either diagonalizable or block diagonalizable $\forall \theta \in \Omega_\theta$.

(A1) means that $Z(\theta)$ keeps the same class of spectral decomposition $\forall \theta \in \Omega_\theta$. Furthermore, under assumption (A1), the computation of $Z(\theta)$ spectral decomposition will follow the same routines as described in section III.

Let us consider now the Sylvester equation (2) with matrices $A_\theta(\theta), B_\theta(\theta)$ and $C_\theta(\theta)$ of the form (23).

Theorem 4: Under the assumption (A1), the extended parametric Sylvester equation (2) admits a solution given by:

$$X(\theta) = \sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \frac{1}{\left(\lambda_{B_{\theta_k}}(\theta) - \lambda_{A_{\theta_j}}(\theta)\right)} F_k(\theta) C_\theta(\theta) E_j(\theta) \tag{28}$$

if and only if whenever there exist j,k and $\theta_0 \in \Omega_\theta$ such that $\lambda_{A_{\theta_j}}(\theta_0) = \lambda_{B_{\theta_k}}(\theta_0)$ then, $F_k(\theta_0) C_\theta(\theta_0) E_j(\theta_0) = 0$ with $E_j(\theta)$, $F_k(\theta)$ verifying conditions (14) and $\lambda_{A_{\theta_j}}, \lambda_{B_{\theta_k}}$ are the parameter dependent non-zero eigenvalues of matrices A_θ and B_θ.

Proof: (Necessity) Suppose that $X(\theta)$ is a solution of equation (2) and that there exist j,k and $\theta_0 \in \Omega_\theta$ such that $\lambda_{A_{\theta_j}}(\theta_0) = \lambda_{B_{\theta_k}}(\theta_0)$ then

$$\sum_{k=0}^{m-1}\lambda_{B_{\theta_k}}(\theta_0)F_k(\theta_0)X(\theta_0)-\sum_{j=0}^{n-1}\lambda_{A_{\theta_j}}(\theta_0)X(\theta_0)E_j(\theta_0)=C_\theta(\theta_0)$$

Using orthogonality properties of F_k and E_j matrices we have

$$\lambda_{B_{\theta_k}}(\theta_0)F_k(\theta_0)X(\theta_0)E_j(\theta_0)-\lambda_{A_{\theta_j}}(\theta_0)F_k(\theta_0)X(\theta_0)E_j(\theta_0)=F_k(\theta_0)C_\theta(\theta_0)E_j(\theta_0)$$

Then, $\lambda_{A_{\theta_j}}(\theta_0)=\lambda_{B_{\theta_k}}(\theta_0)$ implies that $F_k(\theta_0)C_\theta(\theta_0)E_j(\theta_0)=0$. The sufficiency part follows directly by simple calculations.

Example 1: Consider the matrix equation (2) with the following data:

$$A_\theta=1,\ B_\theta(\theta)=\begin{bmatrix}\theta & 1\\ -1 & -\theta\end{bmatrix},\ C_\theta=\begin{bmatrix}0\\1\end{bmatrix},\Omega_\theta=]1,+\infty[$$

Step 1: Spectrum splitting

The computation of $sign(B_\theta(\theta))$ using theorem 3 leads to

$$sign(B_\theta(\theta))=\frac{2}{\pi}B_\theta(\theta)\int_0^\infty\frac{1}{y^2+\theta^2-1}I_2dy$$

Hence, $sign(B_\theta(\theta))=\left(1\big/\sqrt{\theta^2-1}\right)\begin{bmatrix}\theta & 1\\ -1 & -\theta\end{bmatrix}$.

Therefore, following the lines of §III, $V_0(\theta)$ is given by

$$V_0(\theta)=sign(B_\theta(\theta))+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and $V_0(\theta)^{-1}B_\theta(\theta)V_0(\theta)=\sqrt{\theta^2-1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

Now, the eigenvalues of $B_\theta(\theta)$ are directly $\lambda_{B_{\theta_{1,2}}}=\pm\sqrt{\theta^2-1}$ and $\lambda_{B_{\theta_1}}\neq\lambda_{B_{\theta_2}}$ for all $\theta\in\Omega_\theta$.

Step 2: Computing projectors

The computation of E_j and F_k using equation (15) leads to

$$E_0=1,\ F_0(\theta)=\frac{1}{2}\left(I_2+\frac{1}{\sqrt{\theta^2-1}}B_\theta(\theta)\right)$$

and $F_1(\theta)=\frac{1}{2}\left(I_2-\frac{1}{\sqrt{\theta^2-1}}B_\theta(\theta)\right)$

Step 3: Towards the analytic solution of (2)

For $\theta=\sqrt{2}$, we have $\lambda_{B_{\theta_0}}=\lambda_{A_{\theta_0}}=1$ and $F_0(\sqrt{2})C_\theta E_0=\begin{bmatrix}1.207\\ -0.5\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$. Finally, according to Theorem 4, the parameter dependent equation admits the following solution

$$X_\theta(\theta)=\frac{1}{\sqrt{\theta^2-1}-1}F_0(\theta)C_\theta E_0-\frac{1}{\sqrt{\theta^2-1}+1}F_1(\theta)C_\theta E_0$$

$$X_\theta(\theta)=\begin{bmatrix}\dfrac{1}{\theta^2-2}\\ \dfrac{1-\theta}{\theta^2-2}\end{bmatrix},\ \forall\theta\in\Omega_\theta\,/\,\theta\neq\sqrt{2}.$$

The following result extends those obtained in Theorem 2.

Theorem 5: Under the assumption (A1), the extended parametric Sylvester equation (2) admits a solution given by:

$$X(\theta)=\sum_{k=0}^{m_1-1}\sum_{j=0}^{n_1-1}\frac{1}{\left(\lambda_{B_{\theta_k}}(\theta)-\lambda_{A_{\theta_j}}(\theta)\right)}F_k(\theta)C_\theta(\theta)E_j(\theta)\quad(29)$$

if whenever there exist j,k and $\theta_0\in\Omega_\theta$ such that $\lambda_{A_{\theta_j}}(\theta_0)=\lambda_{B_{\theta_k}}(\theta_0)$ then the following conditions hold

$$\begin{aligned}&F_k(\theta_0)C_\theta(\theta_0)N_j(\theta_0)=M_k(\theta_0)C_\theta(\theta_0)E_j(\theta_0)\\&F_k(\theta_0)C_\theta(\theta_0)E_j(\theta_0)=0\end{aligned}\quad(30)$$

and

$$\sum_{k=0}^{m_1-1}\sum_{j=0}^{n_1-1}\frac{1}{\left(\lambda_{B_{\theta_k}}(\theta)-\lambda_{A_{\theta_j}}(\theta)\right)}\cdot\left(F_k(\theta)C_\theta(\theta)N_j(\theta)-M_k(\theta)C_\theta(\theta)E_j(\theta)\right)=0\quad(31)$$

with $E_j(\theta)$, $F_k(\theta)$ verifying conditions (14), $N_j(\theta)$, $M_k(\theta)$ verifying conditions (18) and $\lambda_{A_{\theta_j}},\lambda_{B_{\theta_k}}$ are the parameter dependent non-zero eigenvalues of matrices A_θ and B_θ.

Sketch of the proof: Let us show that when the conditions (30) hold for a common eigenvalue then $X(\theta)$ given by (29) is a solution of equation (2).

Simple calculations lead to

$$A_\theta X-XB_\theta=\sum_{k=0}^{m_1-1}\sum_{j=0}^{n_1-1}\frac{1}{\left(\lambda_{B_{\theta_k}}-\lambda_{A_{\theta_j}}\right)}F_kC_\theta E_j+\sum_{k=0}^{m_1-1}\sum_{j=0}^{n_1-1}\frac{1}{\left(\lambda_{B_{\theta_k}}-\lambda_{A_{\theta_j}}\right)}\left(F_kC_\theta N_j-M_kC_\theta E_j\right)$$

$$A_\theta X-XB_\theta=\sum_{k=0}^{m_1-1}F_kC_\theta\sum_{j=0}^{n_1-1}E_j=C_\theta.$$

Example 2: Consider the matrix equation (2) with the following data:

$$A_\theta(\theta)=\begin{bmatrix}\dfrac{1}{\theta}-2 & 1\\ -4 & \dfrac{1}{\theta}+2\end{bmatrix},\ B_\theta=2,\ C_\theta=\begin{bmatrix}2(\theta-1)\\ 1-\theta\end{bmatrix}^T,$$

$$\Omega_\theta=]0,+\infty[$$

Step 1: Spectrum splitting

The computation of $sign\left(A_\theta\left(\theta\right)\right)$ using theorem 3 leads to

$$sign\left(A_\theta\left(\theta\right)\right) = \frac{2}{\pi} A_\theta\left(\theta\right).$$

$$\int_0^\infty \frac{\theta^2}{\left(\theta^2 y^2+1\right)^2}\begin{bmatrix}\theta^2 y^2+4\theta+1 & -2\theta \\ 8\theta & \theta^2 y^2-4\theta+1\end{bmatrix}dy$$

Hence, $sign\left(A_\theta\left(\theta\right)\right) = sign\left(\theta\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ as $\theta > 0$ and $\bar{A}_\theta\left(\theta\right) = A_\theta\left(\theta\right) - \frac{trace\left(A_\theta\left(\theta\right)\right)}{2} = \begin{bmatrix}-2 & 1\\ -4 & 2\end{bmatrix}$ is a singular matrix. The eigenvalues of $A_\theta\left(\theta\right)$ write directly:

$$\lambda_{A_{\theta_{1,2}}}\left(\theta\right) = \frac{trace\left(A_\theta\left(\theta\right)\right)}{2} = \frac{1}{\theta},\ \theta \in \Omega_\theta$$

Step 2: Computing projectors

The computation of E_j and F_k using (19), (20) and (21) leads to $F_0 = 1$, $E_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $N_0 = \begin{bmatrix}-2 & 1\\ -4 & 2\end{bmatrix}$.

Step 3: Establishing an expression for the solution of (1)

For $\theta = 1$, we have $\lambda_{B_{\theta_0}} = \lambda_{A_{\theta_0}} = 1$ and $F_0 C_\theta\left(1\right) N_0 = M_0 C_\theta\left(1\right) E_0$, $F_0 C_\theta\left(1\right) E_0 = 0$, verifying the sufficient condition (31).

Finally, $X_\theta\left(\theta\right) = \left(\frac{1}{2-1/\theta}\right) F_0 C_\theta E_0 = \begin{bmatrix}\frac{2\theta}{\theta^2+\theta+1} & \frac{-\theta^2+\theta}{\theta^3-1}\end{bmatrix}$

is a possible solution for $\theta \in \Omega_\theta / \theta \neq 1$.

V. Concluding Remarks

In this paper, a matrix sign function based method for solving Sylvester equations is presented. The proposed method consists in three steps: splitting the spectrum, computing projectors and deriving explicit expression of a solution. Moreover, the method proposed in this paper is extended, by construction, to the parametric case. The efficiency of this method has been illustrated by some numerical examples.

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