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Input-output decoupling of discrete-time nonlinear systems by measurement feedback

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***Abstract*— This paper addresses the input-output decoupling problem for discrete-time nonlinear systems by measurement feedback. Necessary and sufficient conditions are given to solve the problem by static or dynamic measurement feedback, respectively. Since the dynamic measurement solution presented here depends on the solution of the input-output linearization problem, a sufficient condition for linearizability of certain functions is also given. Finally, the derived conditions are specified to solve the problem by output feedback.**

I. Introduction

The necessary and sufficient conditions for solvability of the input-output (i/o) decoupling problem by state feedback were given already in [10], [11] for continuous-time systems and in [7], [9] for discrete-time systems. The purpose of this paper is to consider a case when all the states are not available for measurement. Then one has to consider a different feedback, an output feedback or a measurement feedback, where only some functions of states are measured.

In the nonlinear case only few papers address the i/o decoupling problem by measurement or output feedback. Necessary and sufficient conditions have been given to solve the problem for continuous-time systems by static measurement feedback in [3] and for discrete-time systems by static output feedback in [8]. Theorem 1 below is the analogue of the conditions in [3], whereas in [8] the special case was studied when the controlled and measured outputs coincide. To our best knowledge, dynamic measurement feedback solution is looked for only in [12], where solvability conditions are given that depend explicitly on linearizability property of certain functions. These conditions are only sufficient since linearizability property is specified in the theorem via sufficient linearizability conditions.

The goal of this paper is to solve the i/o decoupling problem by dynamic measurement feedback for discrete-time systems. In this paper we use similar approach and the same mathemathical tools as in [3], [8], [12]. However, compared to [12], weaker linearizability conditions are used in this paper. Necessary and sufficient solvability conditions of i/o decoupling problem are given, which can be specified to the dynamic output feedback case. The problem of i/o linearization is also briefly mentioned in the paper, since it is an important part of the solution, given in this paper. Unlike in state feedback case, here one can not use all the states in feedback. Thus, we divide some shifts of controlled outputs into two parts: the one, which has to be compensated and the one, which does not. Finally, a feedback is found, that compensates the part which needs to be compensated, by linearizing certain functions, whenever it is possible. Notice that the obtained necessary and sufficient condition can be generalized to continuous-time systems very easily.

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II. Preliminaries

Consider a discrete-time nonlinear system, described by the equations

$$\begin{aligned} x(t+1) &= f(x(t),u(t)) \\ y_*(t) &= h_*(x(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the input, $y_*(t) \subset Y \in \mathbb{R}^m$ is the controlled output and $y(t) \subset Z \in \mathbb{R}^q$ is the measured output. It is assumed that the functions f, h_* and h are meromorphic. Also, we assume, that the system (1) is submersive, meaning that generically, i.e. everywhere except on a set of measure zero,

$$\operatorname{rank}\Big[\frac{\partial f}{\partial(x(t),u(t))}\Big] = n. \tag{2}$$

In this paper, the following notations are used. Instead of $x(t)$ and $x(t+k)$ $(k \geq 1)$ we use x and $x^{[k]}$, respectively. The same notations are used for the other variables. Throughout the paper it is assumed that $i = 1,\dots,m$.

Extend the map $f : (x,u) \mapsto x^{[1]}$ to the map $\tilde{f} : (x,u) \mapsto (x^{[1]}, z)$, where $z = \chi(x,u)$, $z \in \mathbb{R}^m$, such that $\tilde{f}$ is generically invertible. Let $\mathcal{K}$ be the field of meromorphic functions in finite number of variables from the set $\mathcal{C} = \{x, u^{[k]}, z^{[-l]}; k \geq 0, l > 0\}$. Introduce the forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$, defined by equations (1); in particular $\delta x = f(x,u)$. Moreover, $\delta u^{[k]} = u^{[k+1]}$ $(k \geq 0)$, $\delta z^{[-1]} = \chi(x,u)$, $\delta z^{[-l]} = z^{[-l+1]}$ $(l > 1)$ and

$$\begin{aligned}&\delta\varphi(x,u,\dots,u^{[k]},z^{[-1]},\dots,z^{[-l]}) = \\ &\quad \varphi(f(x,u),u^{[1]},\dots,u^{[k+1]},\chi(x,u),\dots,z^{[-l+1]}).\end{aligned}$$

Since $\tilde{f}$ is invertible, one can also define inverse of operator δ, called backward-shift operator δ^{-1}, as $\delta^{-1}x = \tilde{f}^{-1}(x,z^{[-1]})$, $\delta^{-1}u^{[k]} = u^{[k-1]}$ $(k \geq 0)$, $\delta^{-1}z^{[-l]} = z^{[-l-1]}$ $(l > 1)$ and

$$\begin{aligned}&\delta^{-1}\varphi(x,u,\dots,u^{[k]},z^{[-1]},\dots,z^{[-l]}) = \\ &\quad \varphi(\tilde{f}^{-1}(x,z^{[-1]}),u,\dots,u^{[k-1]},z^{[-2]},\dots,z^{[-l-1]}).\end{aligned}$$

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Since the operator δ is an automorphism of the field $\mathcal{K}$, the pair $(\mathcal{K}, \delta)$ is an inversive difference field, see [1], and denoted simply by $\mathcal{K}$.

Introduce the set of symbols $\mathrm{d}\mathcal{C} = \{\mathrm{d}x, \mathrm{d}u^{[k]}, \mathrm{d}z^{[-l]}; k \geq 0, l > 0\}$. Let $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\mathcal{C}\}$ be the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$. The elements of $\mathcal{E}$, i.e.

$$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i + \sum_{k\geq 0}\sum_{j=1}^{m} b_{kj}\mathrm{d}u_j^{[k]} + \sum_{l>0}\sum_{\rho=1}^{m} c_{l\rho}\mathrm{d}z_\rho^{[-l]}$$

where only finite number of coefficients $a_i, b_{kj}, c_{l\rho} \in \mathcal{K}$ are nonzero, are called one-forms. A one-form ω is called exact if it is a differential of some function $\varphi \in \mathcal{K}$, i.e. $\omega = \mathrm{d}\varphi$. The operators δ and δ^{-1} are extended to $\mathcal{E}$ by the rules

$$\begin{aligned} \delta\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) &= \sum_j \delta(a_j)\mathrm{d}(\delta\varphi_j) \\ \delta^{-1}\Big(\sum_j a_j \mathrm{d}\varphi_j\Big) &= \sum_j \delta^{-1}(a_j)\mathrm{d}(\delta^{-1}\varphi_j), \end{aligned}$$

where $a_j, \varphi_j \in \mathcal{K}$. Let $y_* = (y_{*1}, \ldots, y_{*m})$ be the controlled output vector of the system (1). The relative degree r_i of an output y_{*i} is defined by $r_i := \min\{k \in \mathbb{N} \mid \mathrm{d}y_{*i}^{[k]} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}\}$. If there does not exist such integer k, then set $r_i := \infty$. We also define some subspaces of $\mathcal{E}$, i.e. $\mathcal{E}^k = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[k-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[k-1]}\}$ for every $k \in \mathbb{N}$ and $\mathcal{X} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$.

In general, a one-form ω is a linear combination over $\mathcal{K}$ of certain number of standard basis elements of $\mathcal{E}$, i.e. $\mathrm{d}\mathcal{C}$. However, it is often possible to find a linearly independent set of exact one-forms, with less elements than those basis elements of $\mathcal{E}$, in terms of which ω can be expressed.

Definition 1: A minimal number $\gamma \in \mathbb{N}$ so that there exist γ linearly independent exact one-forms such that ω is a linear combination of these exact one-forms, is called the rank of a one-form ω.

If the rank of a one-form $\omega \in \mathcal{E}$ is equal to 1, then by definition 1 there exist $\lambda, \varphi \in \mathcal{K}$ such that $\omega = \lambda\mathrm{d}\varphi$, i.e. the one-form ω is integrable.

We say that system (1) is right-invertible with respect to the output y_* if there exist $j_i \in \mathbb{N}$ such that

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(h_{*1}(x^{[j_1]}), \ldots, h_{*m}(x^{[j_m]}))^T}{\partial u} = m, \qquad (3)$$

where $h_*(x) = (h_{*1}(x), \ldots, h_{*m}(x))$, see [2] for more information. Also, let $j_{max} := \max\{j_1, \ldots, j_m\}$.

As in [12], we define for each output component y_{*i} a subspace Ω_i of $\mathcal{X}$ in the following way:

$$\begin{aligned} \Omega_i \;=\; & \{\omega \in \mathcal{X} \mid \forall k \in \mathbb{N} : \delta^k\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \\ & \ldots, \mathrm{d}y_{*i}^{[r_i+k-1]}\}\}. \end{aligned} \qquad (4)$$

The subspaces Ω_i are essential to solve the input-output decoupling problem. It is because the forward-shifts of the elements of Ω_i do not depend on the input u explicitly. Suppose $\Omega_i = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_l\}$. We define the forward-shift of subspace Ω_i elementwise by $\Omega_i^{[1]} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\theta_1^{[1]}, \ldots, \mathrm{d}\theta_l^{[1]}\}$. Denote $\Omega_i^{[0]} := \Omega_i$, and $\Omega_i^{[k]} := (\Omega_i^{[k-1]})^{[1]}$. The following lemma gives a procedure for computing the subspaces Ω_i.

Lemma 1: [4] The subspace Ω_i may be computed as the limit of the following algorithm:

$$\begin{aligned} \Omega_i^0 &= \mathcal{X} \\ \Omega_i^{k+1} &= \{\omega \in \Omega_i^k \mid \delta\omega \in \Omega_i^k + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{*i}^{[r_i]}\}\}. \end{aligned} \qquad (5)$$

III. Main results

A. Problem statement

One says that system (1) is i/o decoupled if every controlled output y_{*i} depends on exactly one input variable u_i, i.e. the relative degrees r_i are finite and

$$\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-r_i]}\} \quad k \geq r_i.$$

The next lemma gives the necessary and sufficient condition for a system to be i/o decoupled.

Lemma 2: Under the assumption that $r_i < \infty$, for $i = 1, \ldots, m$, the system (1) is input-output decoupled iff

$$\mathrm{d}y_{*i}^{[r_i]} \in \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i\}. \qquad (6)$$

Proof: Necessity. If the system (1) is input-output decoupled, then

$$\mathrm{d}y_{*i}^{[r_i]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i\}.$$

Thus, there exists $\omega_i \in \mathcal{X}$ and $\lambda_i \in \mathcal{K}$ such that $\mathrm{d}y_{*i}^{[r_i]} = \omega_i + \lambda_i\mathrm{d}u_i$. We will show that $\omega_i \in \Omega_i$. Note that

$$\delta^\sigma\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\} \qquad (7)$$

for every $\sigma \in \mathbb{N}$. Since $\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-r_i]}\}$ for $k \geq 0$, then

$$\begin{aligned} & \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[\sigma-1]}\} = \\ & \quad \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+\sigma-1]}\}. \end{aligned} \qquad (8)$$

Thus, (7) and (8) give

$$\delta^\sigma\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+\sigma-1]}\}$$

for every $\sigma \in \mathbb{N}$, which means that $\omega_i \in \Omega_i$.

Sufficiency. By Lemma 1 and (6), one gets

$$\Omega_i^{[1]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_{*i}^{[r_i]}\} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i\}.$$

Thus, $\Omega_i^{[k]} \subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k-1]}\}$ and therefore,

$$\begin{aligned} \mathrm{d}y_{*i}^{[r_i+k]} &\in \Omega_i^{[k]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i^{[k]}\} \\ &\subseteq \Omega_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\} \\ &\subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u_i, \ldots, \mathrm{d}u_i^{[k]}\}, \end{aligned}$$

which means, that system (1) is i/o decoupled. ■

The input-output decoupling problem can be formulated as follows. Find a regular dynamic measurement feedback of the form

$$\begin{aligned} \eta^{[1]} &= F(\eta, z, v) \\ u &= H(\eta, z, v), \end{aligned} \qquad (9)$$

where $v \subset V \in \mathbb{R}^m$ is the new input and $\eta \subset \Delta \in \mathbb{R}^\rho$ is the state of the feedback, such that the closed-loop system (1),(9) satisfies the following conditions:

(i) the relative degree $\bar{r}_i$ of output y_{*i} of the closed-loop system is finite;
(ii) $\mathrm{d}y_{*i}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}v_i, \ldots, \mathrm{d}v_i^{[k-\bar{r}_i]}\}$ for $k \geq \bar{r}_i$.

Condition (ii) guarantees that outputs y_{*i} are decoupled, i.e. different outputs y_{*i} are affected by different inputs v_i at every time instant $k \geq \bar{r}_i$. By regularity of feedback we mean that (9) defines generically the (z, η)-dependent one-to-one correspondence between the variables v and u. Feedback (9) is called static if $\rho := \dim \eta = 0$.

Since the main Theorem of this paper, given below, depends on the solution of the i/o linearization problem, we give first the problem statement for the i/o linearization. For more information, see [5].

B. Input-output linearization

In this section, we consider a discrete-time multi-input multi-output (MIMO) nonlinear system, described by the set of i/o difference equations

$$y_l^{[n_l]} = \Phi_l(y_\tau, \ldots, y_\tau^{[n_{l\tau}]}, u_i, \ldots, u_i^{[n_l-1]}) \tag{10}$$

for $l, \tau = 1, \ldots, q$, where Φ_l are supposed to be meromorphic functions of their arguments and the indices in (10) satisfy the relations

$$\begin{aligned} &n_1 \leq n_2 \leq \cdots \leq n_q, \quad && n_{l\tau} < n_\tau \\ &n_{l\tau} < n_l, && \tau \leq l \\ &n_{l\tau} \leq n_l, && \tau > l. \end{aligned} \tag{11}$$

Like above, we assume, that system (10) is submersive, which for the i/o model means that the map $\Phi = (\Phi_1, \ldots, \Phi_q)^T$ satisfies generically the condition

$$\mathrm{rank}\Big[\frac{\partial \Phi}{\partial(y, u)}\Big] = q,$$

where $y = (y_1, \ldots, y_q)$ and $u = (u_1, \ldots, u_m)$.

One says that equations (10) are linearizable by regular dynamic output feedback of the form (9), if the differentials $\mathrm{d}y_l^{[n_l]}$, defined by the input-output equations of the closed-loop system, satisfy the relations

$$\mathrm{d}y_l^{[n_l]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}y_\tau^{[n_{l\tau}]}, \ldots, \mathrm{d}y_\tau, \mathrm{d}v\} \tag{12}$$

for $l, \tau = 1, \ldots, q$. In the case, when

$$\mathrm{d}y_l^{[n_l]} \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}v\}$$

for $l = 1, \ldots, q$, equations (10) are said to be strictly linearizable.

One says that the set of functions $\varphi_i(z, \ldots, z^{[s-1]}, u, \ldots, u^{[s-1]})$, $i = 1, \ldots, m$, are linearizable (strictly linearizable) if the system of equations

$$y_i^{[s]} = \varphi_i(y, \ldots, y^{[s-1]}, u, \ldots, u^{[s-1]})$$

is linearizable (strictly linearizable).

C. Input-output decoupling

First, we give a solution to the input-output decoupling problem by static measurement feedback. Let $h_*(x) = (h_{*1}(x), \ldots, h_{*m}(x))$.

Theorem 1: Let the relative degrees r_i of outputs y_i be finite. Then the system (1) is input-output decoupable by static measurement feedback iff

(i)
$$\mathrm{rank}_{\mathcal{K}}\Big[\frac{\partial(h_{*1}(x^{[r_1]}), \ldots, h_{*m}(x^{[r_m]}))^T}{\partial u}\Big] = m;$$
(ii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}u\}$ with rank 1, such that $\mathrm{d}y_{*i}^{[r_i]} - \omega_i \in \Omega_i$.

The proof of Theorem 1 is given in [8] for the case when $y = y_*$. The proof of the general case is similar.

In Theorem 2 below, the necessary and sufficient conditions for solvability of the input-output decoupling problem by dynamic measurement feedback are given, relaxing the conditions of Theorem 1.

Theorem 2: The system (1) is input-output decoupable by dynamic measurement feedback (9) iff the following conditions are satisfied:

(i) the system (1) is right-invertible with respect to the controlled outputs y_{*i};
(ii) there exists $s \geq j_{max} - r_i + 1$ such that[1]
$$\begin{aligned} \mathrm{d}y_{*i}^{[r_i+s-1]} \in\ & \Omega_i + \cdots + \Omega_i^{[s-1]} \\ & + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}; \end{aligned}$$
(iii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ with rank 1 such that
$$\mathrm{d}y_{*i}^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]};$$
(iv) for $\omega_i = \lambda_i \mathrm{d}\varphi_i$, $\lambda_i, \varphi_i \in \mathcal{K}$, the functions $\varphi_i(y, \ldots, y^{[s-1]}, u, \ldots, u^{[s-1]})$ are strictly linearizable by dynamic feedback.

Proof. Necessity. Let $s \geq 1$ be such that in the closed-loop system the relative degree $\bar{r}_i$ of output y_{*i} is $\bar{r}_i = r_i + s - 1$. By Lemma 2 and the fact that the closed-loop system is i/o decoupled,

$$\mathrm{d}y_{*i}^{[\bar{r}_i]} \in \bar{\Omega}_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}, \tag{13}$$

where $\bar{\Omega}_i$ is the subspace Ω_i for the closed-loop system. Next, we show that $\bar{\Omega}_i = \Omega_i + \cdots + \Omega_i^{[s-1]}$. From the definition of the subspace Ω_i,

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y_{*i}^{[r_i]}, \ldots, \mathrm{d}y_{*i}^{[r_i+s-2]}\}.$$

Since $\bar{r}_i = r_i + s - 1$, then in the closed-loop system

$$\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}. \tag{14}$$

Thus,

$$\begin{aligned} &\Omega_i + \cdots + \Omega_i^{[s-1]} = \{\bar{\omega} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \mid \forall k \in \mathbb{N} : \\ &\bar{\omega}^{[k]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}y_{*i}^{[r_i+s-1]}, \ldots, \mathrm{d}y_{*i}^{[r_i+s-k-2]}\}\} \\ &= \bar{\Omega}_i. \end{aligned}$$

[1]Note that, one can, in principle, search, instead of the joint index s, a separate s_i that satisfies $s_i \geq j_{max} - r_i + 1$. Then s can be taken as $s = \max_i\{s_i\}$.

The last equality comes from the definition (4) of the subspace $\bar{\Omega}_i$. Therefore, (13) becomes

$$\mathrm{d}y_{*i}^{[r_i+s-1]} \in \Omega_i + \cdots + \Omega_i^{[s-1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}.$$

Then one can define the one-forms $\omega_i = \lambda_i \mathrm{d}v_i$ such that $\mathrm{d}y_i^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]}$. Now, conditions (ii) and (iii) must be satisfied, since otherwise v_i would depend on x', where $\mathrm{d}x' \in \mathcal{X}$ and $\mathrm{d}x' \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y\}$, i.e. the feedback would not be measurement feedback. Since conditions (ii) and (iii) are satisfied, $\omega_i = \lambda_i \mathrm{d}\varphi_i(u, \ldots, u^{[s-1]}, y, \ldots, y^{[s-1]})$ for some functions φ_i. Note that under the feedback $\omega_i = \lambda_i \mathrm{d}v_i$, i.e. the functions φ_i are strictly linearizable.

Sufficiency. We show that the feedback that linearizes strictly functions φ_i in (iv), solves the i/o decoupling problem.

Since for the closed-loop system $\mathrm{d}\varphi_i = \mathrm{d}v_i$, the relative degree of output y_{*i} is $r_i + s - 1$. Thus

$$\mathrm{d}y_{*i}^{[r_i+j]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\} \tag{15}$$

for $j = 0, \ldots, s-2$. From the definition (4) of the subspace Ω_i one concludes $\Omega_i + \cdots + \Omega_i^{[s-1]} \subseteq \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}\eta\}$.

Now, as in the necessity part, one can show that $\Omega_i + \cdots + \Omega_i^{[s-1]} = \bar{\Omega}_i$, where $\bar{\Omega}_i$ is the subspace Ω_i for the closed-loop system. Therefore, by (ii), (iii) and (iv), $\mathrm{d}y_{*i}^{[r_i+s-1]} \in \bar{\Omega}_i + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_i\}$. By Lemma 2, system (1) is i/o decoupled. ■

Next, we give a simple sufficient condition for the strict linearizability of functions φ_i in (iv) of Theorem 2. For general input-output linearization problem, see [6].

Theorem 3: Functions φ_i in (iv) of Theorem 2 are strictly linearizable by dynamic measurement feedback if there exist functions $\phi_{i,j} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[j-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[j-1]}\}$ for $j = 1, \ldots, s$, such that

$$\begin{aligned} \mathrm{d}\varphi_i \;=\; & \mathrm{d}\phi_{i,s}(\cdot, \delta\phi_{\mu,\nu}, y, u; \nu = 1, \ldots, s-1) \\ & \circ\delta\phi_{i,s-1}(\cdot, \delta\phi_{\mu,\nu}, y, u; \nu = 1, \ldots, s-2) \circ \cdots \\ & \cdots \circ \delta\phi_{i,1}(y, u) \end{aligned} \tag{16}$$

where $\mu = 1, \ldots, m$ and

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,j}, 1 \le j \le j_i\}) = m. \tag{17}$$

Proof: By condition (i) of Theorem 2 the indices j_i, defined by (3), are finite. Then set[2]

$$\eta_{i,\tau} = \phi_{i,\tau}(\cdot) \tag{18}$$

$$v_i = \phi_{i,s}(\cdot) \tag{19}$$

for $\tau = j_i, \ldots, s-1$. Because of (17), one can find a dynamic measurement feedback by solving (18), (19) with respect to the variables u and $\eta_{i,\tau}^{[1]}$, $\tau = j_i, \ldots, s-1$. Then, from (19) and (16) one concludes that in the closed-loop system $\mathrm{d}\varphi_i = \mathrm{d}v_i$. Thus, the functions φ_i are strictly linearizable. ■

[2]Note that here $\tau = j_i, \ldots, s-1$. This is because functions $\phi_{i,j}$, $j = 1, \ldots, j_i - 1$ depend by (3) and (17) on functions ϕ_{i,j_i}.

The main difficulty in checking the conditions of Theorem 3 is related to finding the functions $\phi_{i,j}$, $j = 1, \ldots, s$. Below an algorithm is given for searching such functions whenever they exist. The algorithm is based on the input-output linearization algorithm, introduced in [5]. The main difference is that the one-forms $\bar{\omega}_{i,p}$ are defined differently to make the Algorithm more transparent, and the number of functions φ_i equals to the number of inputs u_i.

Algorithm.

Step 0. Find the one-forms ω_i, defined in condition (iii) and (ii) of Theorem 2.

Step 1. Let

$$\begin{aligned} \bar{\omega}_{i,1} \;=\; & \sum_{\mu=1}^{q} \alpha_{i,1,\mu} \mathrm{d}y_\mu^{[s-1]} + \sum_{j=1}^{m} \beta_{i,1,j} \mathrm{d}u_j^{[s-1]} \\ & \alpha_{i,1,\mu}, \beta_{i,1,j} \in \mathcal{K} \end{aligned}$$

be such that $\omega_i - \bar{\omega}_{i,1} \in \mathcal{E}^{s-1}$. Check whether $\gamma_{i,1} := \mathrm{rank}\ \bar{\omega}_{i,1} = 1$. If not, then stop, the conditions of Theorem 3 are not satisfied. Otherwise, let $\phi_{i,1}$ be such that $\bar{\omega}_{i,1} = \pi_i \mathrm{d}(\delta^{s-1}\phi_{i,1})$ for some $\pi_i \in \mathcal{K}$.

Step p. $(p = 2, \ldots, s-1)$ Let

$$\begin{aligned} \bar{\omega}_{i,p} \;=\; & \sum_{\mu=1}^{q} \alpha_{i,p,\mu} \mathrm{d}y_\mu^{[s-p]} + \sum_{j=1}^{m} \beta_{i,p,j} \mathrm{d}u_j^{[s-p]} + \sum_{l=1}^{p-1} \bar{\omega}_{i,l}, \\ & \alpha_{i,p,\mu}, \beta_{i,p,j} \in \mathcal{K} \end{aligned}$$

be such that $\omega_i - \bar{\omega}_{i,p} \in \mathcal{E}^{s-p}$. Check whether

$$\mathrm{d}\bar{\omega}_{i,p} \wedge \bar{\omega}_{i,p} \wedge \mathrm{d}(\delta^k \phi_{j,l}) = 0,$$

where $j = 1, \ldots, m$, $l = 1, \ldots, p-1$ and $k = s-p, \ldots, s-1$. If not, then stop. Otherwise, there exist $\gamma, \theta_{j,l} \in \mathcal{K}$ such that

$$\hat{\omega}_{i,p} := \gamma\big(\bar{\omega}_{i,p} + \sum_{j=1}^{m}\sum_{l=1}^{p-1} \theta_{j,l} \mathrm{d}(\delta^k \phi_{j,l})\big)$$

is exact $(k = s-p, \ldots, s-1)$. Then define $\phi_{i,p}$ such that $\hat{\omega}_{i,p} = \mathrm{d}(\delta^{s-p}\phi_{i,p})$.

Step s. Define $\phi_{i,s} = \varphi_i$, i.e. such that $\omega_i = \lambda_i \mathrm{d}\phi_{i,s}$ for some $\lambda_i \in \mathcal{K}$. End of the algorithm.

In the case when the controlled output y_* is measurable, i.e. $y_* = y$, one gets from Theorems 2 and 3 the following two corollaries.

Corollary 1: Under the assumption that the relative degrees r_i of outputs y_{*i} are finite, the system of the form (1) is i/o decoupable by dynamic output feedback iff the following conditions are satisfied

(i) the system (1) is right-invertible with respect to controlled outputs y_{*i};
(ii) there exists $s \ge j_{max} - r_i + 1$ such that
$$\begin{aligned} \mathrm{d}y_{*i}^{[r_i+s-1]} \in\; & \Omega_i + \cdots + \Omega_i^{[s-1]} \\ & + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}; \end{aligned}$$
(iii) there exist one-forms $\omega_i \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ with rank 1 such that
$$\mathrm{d}y_{*i}^{[r_i+s-1]} - \omega_i \in \Omega_i + \cdots + \Omega_i^{[s-1]};$$

(iv) for $\omega_i = \lambda_i \mathrm{d}\varphi_i$, the functions $\varphi_i(y_*, \ldots, y_*^{[s-1]}, u, \ldots, u^{[s-1]})$ are strictly linearizable by dynamic feedback.

Corollary 2: Functions φ_i in (iv) of Corollary 1 are strictly linearizable by dynamic output feedback if there exist functions $\phi_{i,j} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_*, \ldots, \mathrm{d}y_*^{[j-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[j-1]}\}$, $j = 1, \ldots, s$, such that

$$\begin{aligned} \mathrm{d}\varphi_i &= \mathrm{d}\phi_{i,s}(\cdot, \delta\phi_{\mu,\nu}, y_*, u; \nu = 1, \ldots, s-1) \\ &\quad \circ \delta\phi_{i,s-1}(\cdot, \delta\phi_{\mu,\nu}, y_*, u; \nu = 1, \ldots, s-2) \circ \cdots \\ &\quad \cdots \circ \delta\phi_{i,1}(y_*, u) \end{aligned}$$

where $\mu = 1, \ldots, m$ and

$$\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{i,j}, 1 \le j \le j_i\}) = m.$$

Consider the system (1) without measured output y. In [7], it has been shown that such system can be i/o decoupled by state feedback if and only if it is right invertible, i.e condition (3) is satisfied. Next, we explain briefly that, when we take $y = x$, the conditions of Theorem 2 become necessary and sufficient condition for i/o decoupling problem by state feedback. For this, we show, that in the case of $y = x$, the conditions (ii), (iii) and (iv) of Theorem 2 are always satisfied.

Note that, when $y = x$, then $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \ldots, \mathrm{d}y^{[s-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ and since $\mathrm{d}y_{*i}^{[r_i+s-1]} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u, \ldots, \mathrm{d}u^{[s-1]}\}$ for $i = 1, \ldots, m$, the condition (ii) is always satisfied. Also, one can take $\omega_i = \mathrm{d}y_{*i}^{[r_i+s-1]}$, then condition (iii) of Theorem 2 is satisfied. It also means that $\varphi_i = y_{*i}^{[r_i+s-1]}$ in condition (iv) of Theorem 2 and these functions are linearizable if and only if the given system is right-invertible. Thus, condition (iv) of Theorem 2 is always satisfied if system (1) with $y = x$ is right-invertible.

IV. EXAMPLE

Consider a system described by the difference equations

$$\begin{aligned} x_1^{[1]} &= (x_3 + x_4)u_1 - x_2 \\ x_2^{[1]} &= \frac{u_1 x_5}{x_4} + x_1 \\ x_3^{[1]} &= x_1 x_3 \\ x_4^{[1]} &= (x_3 + x_4)u_1 x_5 \\ x_5^{[1]} &= \frac{u_2 x_5}{x_4} \\ y_{*1} &= x_1, \quad y_{*2} = x_4 \\ y_1 &= x_3 + x_4 \quad y_2 = \frac{x_5}{x_4}. \end{aligned} \tag{20}$$

We check if the conditions of Theorem 2 are satisfied for system (20). First, note that the relative degrees of outputs y_1 and y_2 are $r_1 = r_2 = 1$. Since

$$\begin{aligned} y_{*1}^{[1]} &= (x_3 + x_4)u_1 - x_2 \\ y_{*2}^{[2]} &= \left(y_{*1}^{[2]} + x_1 + \frac{u_1 x_5}{x_4}\right)\frac{u_2 x_5}{x_4}, \end{aligned}$$

one gets $\mathrm{rank}_{\mathcal{K}} \frac{\partial (y_{*1}^{[1]}, y_{*2}^{[2]})^T}{\partial u} = 2$. Therefore, the system (20) is right-invertible and $j_1 = 1$, $j_2 = 2$. The subspaces Ω_i are, according to Lemma 1, $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_3\}$ and $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_4\}$.

Since s has to satisfy the inequalities $s \ge j_{max} - r_i + 1$ for $i = 1, 2$, the first choice for s is $s = 2$. Compute

$$\begin{aligned} \mathrm{d}y_{*1}^{[2]} &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} - y_2\mathrm{d}u_1 - u_1\mathrm{d}y_2 - \mathrm{d}x_1 \\ &\in \Omega_1 + \Omega_1^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y, \mathrm{d}u^{[1]}, \mathrm{d}y^{[1]}\} \\ \mathrm{d}y_{*2}^{[2]} &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]} + y_2 u_1^{[1]} y_1^{[1]}\mathrm{d}u_2 \\ &\quad + u_2 u_1^{[1]} y_1^{[1]}\mathrm{d}y_2 \\ &\in \Omega_2 + \Omega_2^{[1]} + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}y, \mathrm{d}u^{[1]}, \mathrm{d}y^{[1]}\} \end{aligned}$$

and really, condition (ii) of Theorem 2 is satisfied. Choosing

$$\begin{aligned} \omega_1 &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} - y_2\mathrm{d}u_1 - u_1\mathrm{d}y_2 \\ &= \mathrm{d}(y_1^{[1]}u_1^{[1]} - y_2 u_1) \\ \omega_2 &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]} + y_2 u_1^{[1]} y_1^{[1]}\mathrm{d}u_2 \\ &\quad + u_2 u_1^{[1]} y_1^{[1]}\mathrm{d}y_2 = \mathrm{d}(u_1^{[1]} y_1^{[1]} u_2 y_2), \end{aligned}$$

then the condition (iii) is also satisfied. To check the condition (iv), we use Theorem 3. For that, apply Algorithm to the one forms ω_1 and ω_2.

Step 1. Take

$$\begin{aligned} \bar{\omega}_{1,1} &= u_1^{[1]}\mathrm{d}y_1^{[1]} + y_1^{[1]}\mathrm{d}u_1^{[1]} \\ \bar{\omega}_{2,1} &= u_2 y_2 y_1^{[1]}\mathrm{d}u_1^{[1]} + u_2 y_2 u_1^{[1]}\mathrm{d}y_1^{[1]}. \end{aligned}$$

It is obvious that the ranks of these one-forms are 1, because $\bar{\omega}_{1,1} = \mathrm{d}(y_1^{[1]}u_1^{[1]})$ and $\bar{\omega}_{2,1} = y_2 u_2 \mathrm{d}(y_1^{[1]}u_1^{[1]})$. Thus, one takes $\phi_{1,1} = \phi_{2,1} = y_1 u_1$.

Step 2. Since this is the last step, one takes $\phi_{1,2} = y_1^{[1]}u_1^{[1]} - y_2 u_1 = \delta\phi_{1,1} - u_1 y_2$ and $\phi_{2,2} = u_1^{[1]} y_1^{[1]} u_2 y_2 = \delta\phi_{2,1} u_2 y_2$.

It is easy to check that $\dim(\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_{1,1}, \mathrm{d}\phi_{2,1}, \mathrm{d}\phi_{2,2}\}) = 2 = m$. Therefore, conditions of Theorem 3 are satisfied.

Using the functions $\phi_{i,j}$ from Algorithm 1, one can find the feedback that decouples the system, as suggested in the proof of Theorem 3. For that, take

$$\begin{aligned} \eta_{1,1} &= \phi_{1,1} = y_1 u_1 \\ v_1 &= \phi_{1,2} = \eta_{1,1}^{[1]} - u_1 y_2 \\ v_2 &= \phi_{2,2} = \eta_{1,1}^{[1]} u_2 y_2. \end{aligned} \tag{21}$$

Solving equations (21) in terms of u_1, u_2 and $\eta_{1,1}^{[1]}$, one gets the decoupling feedback

$$\begin{aligned} \eta_{1,1}^{[1]} &= v_1 + \frac{\eta_{1,1} y_2}{y_1} \\ u_1 &= \frac{\eta_{1,1}}{y_1} \\ u_2 &= \frac{v_2 y_1}{y_2(y_1 v_1 + \eta_{1,1} y_2)}. \end{aligned} \tag{22}$$

For the closed-loop system one gets $\bar{r}_1 = \bar{r}_2 = r_i + s - 1 = 2$ and

$$\begin{aligned} \mathrm{d}y_{*1}^{[2]} &= \mathrm{d}v_1 - \mathrm{d}x_1 \in \bar{\Omega}_1 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_1\} \\ \mathrm{d}y_{*2}^{[2]} &= \mathrm{d}v_2 \in \bar{\Omega}_2 + \mathrm{span}_{\mathcal{K}}\{\mathrm{d}v_2\}, \end{aligned}$$

which means that by Lemma 2, the closed-loop system is i/o decoupled.

V. Conclusion

A necessary and sufficient conditions for the solvability of i/o decoupling problem by static and dynamic measurement feedback was given in this paper. The solution, given in this paper, depends on the linearization of certain functions by output feedback. A sufficient condition was given to linearize a set of functions, defined in the solution of the i/o decoupling problem. The dynamic output feedback solution was also given, based on the dynamic measurement feedback solution.

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