

Output Feedback Robust Synchronization of Networked Lur'e Systems With Incrementally Passive Nonlinearities*

Fan Zhang^{1,3}, Harry L. Trentelman², Jacquélien M.A. Scherpen¹

Abstract—In this paper we deal with robust synchronization problems for uncertain dynamical networks of identical Lur'e systems diffusively interconnected by means of measurement outputs. In contrast to stabilization of one single Lur'e system with a passive static nonlinearity in the negative feedback loop, in the present paper the feedback nonlinearities are assumed to be incrementally passive. We assume that the interconnection topologies among these Lur'e agents are undirected and connected throughout this paper. A distributed dynamical protocol is proposed. We establish sufficient conditions for the existence of such protocol that robustly synchronizes the Lur'e dynamical network. The protocol parameter matrices are computed in terms of the system matrices defining the individual agent, but also the second smallest and largest eigenvalues of the Laplacian matrix associated with the interconnection topology.

I. INTRODUCTION

Since collective behaviors of multiple interconnected dynamical systems are widespread in nature, technology and human society, synchronization of complex dynamical networks has been an extremely appealing topic in multidisciplinary research communities over the last decade, see [4], [9], [12], [20], [24] to name just a few. This is due to the fact that complex dynamical networks have potential applications in a wide area such as spatiotemporal planning, cooperative multitasking and formation control [5], [19]. Furthermore, complex dynamical networks are being developed to be flexible, versatile and robust to communication latencies, intermittent losses of sensor measurements and asynchronous members etc., which work quite well in certain complex tasks.

Synchronization of linear multi-agent networks has been well studied, see [17], [23] and the references therein. Synchronization problems for nonlinear multi-agent networks have also been addressed since a long time ago, probably with model uncertainties, time delays, data dropouts and quantized communications etc. [1], [6], [12], [18]. In [16], a passivity-based group coordination framework was proposed, especially applicable to nonlinear multi-agent networks even if there exist communication latencies [18]. A similar idea was applied to deal with a network of static output coupled incrementally passive oscillators in [10].

*The work of Zhang was supported in part by the China Scholarship Council (CSC)

¹F. Zhang and J.M.A. Scherpen are with the Research Institute of Technology and Management, University of Groningen, 9747 AG Groningen, The Netherlands {fan.zhang, j.m.a.scherpen}@rug.nl

²H.L. Trentelman is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AV Groningen, The Netherlands h.l.trentelman@rug.nl

³F. Zhang is also with the Research Center of Satellite Technology, Harbin Institute of Technology, 150080 Harbin, China

However, without the assumption that each agent in the network is passive or incrementally passive, there is still no systematic approach to handle distributed coordination problems for nonlinear multi-agent networks. In [12], the authors discussed robust synchronization of linear multi-agent networks against additive perturbations of the agents' transfer matrices for both undirected as well as directed interconnections. It was shown that a radius of uncertainties is allowed, which is proportional to the quotient of the smallest and largest nonzero eigenvalues of the underlying graph Laplacian matrix. In short, networks of non-passive agents deserve more attention.

In this paper, we consider homogeneous nonlinear multi-agent networks in which the dynamics of the individual agent is represented by a Lur'e system, i.e. a nonlinear system consisting of the negative feedback interconnection of a nominal linear system with an uncertain static nonlinearity around it [14]. Besides Chua's circuits, many control system applications, e.g. aircrafts and flexible robotic arms, can be described by Lur'e systems. The feedback loop can represent different kinds of nonlinearities such as saturation and dead zone. In the present paper we assume the feedback nonlinearities to be incrementally passive. Incremental passivity is often used in nonlinear control systems [14], [2]. In contrast with our previous work [7], [8], here we study output feedback based distributed dynamical protocols. We stress that we do not make any assumption on passivity or incremental passivity of the agents in this paper.

To the best of our knowledge, the present work constitutes the first paper in which a treatment for output feedback based robust synchronization of Lur'e dynamical networks is given. In secure communication applications, output feedback based master-slave synchronization of two Lur'e systems has been extensively studied. In [13], they used dynamical output feedback to recover a message signal in master-slave synchronization of Lur'e systems while the measurement noise was considered. Synchronization criteria for two static output coupled Lur'e systems with time delays were derived in [21]. In addition, in [11], it was assumed that the feedback nonlinearities are slope-restricted but also precisely known. The assumption that the feedback nonlinearities are known is often employed in observer-based output feedback stabilization of Lur'e systems, see e.g. [15]. Thus the design of the above observer-based output feedback controllers does not deal with robustness, and hence addresses a different problem from the one addressed in our paper. Our dynamical protocol is provided by a general dynamical system, which receives the weighted relative measurements and the weighted relative

protocol states, and uses these to determine the diffusive coupling inputs to the agents.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries and formulates the output feedback robust synchronization problem we are interested in. Our main results are presented in Section 3. Sufficient synchronization conditions are established and it is discussed how to compute a suitable dynamical protocol. Some concluding remarks together with suggestions for future work close the paper.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively. We denote by $\mathbb{R}^+ := [0, \infty)$. $\mathbb{R}^{m \times n}$ ($\mathbb{C}^{m \times n}$) denotes the space of m by n real (complex) matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript $(\cdot)^T$ denotes the transpose of a real matrix, and the superscript $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. We denote the block diagonal matrix with matrices M_i , $i = 1, 2, \dots, j$, on its diagonal by $\text{diag}(M_1, M_2, \dots, M_j)$. \star in a partitioned matrix means this block has no effect on the result we are interested in, and is left unspecified. The Kronecker product of the matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. An important property of the Kronecker product is $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4)$. We denote by $\mathbf{0}$ and I the zero and the identity matrices, respectively, of compatible dimensions. By $\mathbf{1}_N$ and $\mathbf{0}_N$ we denote the column vectors of dimension N with all the elements equal to one and zero, respectively.

In this paper, the interconnection topology of a network of bidirectionally interconnected dynamical systems is represented by an undirected graph \mathcal{G} that consists of a nonempty, finite node set $\mathcal{V} = \{1, 2, \dots, N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ with the property that $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ for all $i, j = 1, 2, \dots, N$ and $j \neq i$. We assume that the graph \mathcal{G} is simple, i.e. it does not contain any self-loop (i, i) and there is at most one undirected edge between any two different nodes. An undirected path connecting nodes i_0 and i_l is a sequence of undirected edges of the form (i_{p-1}, i_p) , $p = 1, \dots, l$. The graph \mathcal{G} is connected if there exists an undirected path between any pair of distinct nodes. The adjacency matrix \mathcal{A} associated with the graph \mathcal{G} is defined as $[\mathcal{A}]_{ij} = a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $[\mathcal{A}]_{ij} = 0$ otherwise, where a_{ij} is the edge weight of (j, i) . The degree of node i is given by $d_i = \sum_{j=1}^N a_{ij}$. $\mathcal{D} := \text{diag}(d_1, d_2, \dots, d_N)$ is the degree matrix of the graph \mathcal{G} . The Laplacian matrix of the graph \mathcal{G} is defined by $\mathcal{L} := \mathcal{D} - \mathcal{A}$. According to the Gershgorin circle theorem, all the eigenvalues of \mathcal{L} are nonnegative real. It is well known that $\mathcal{L}\mathbf{1}_N = \mathbf{0}_N$, i.e. $\mathbf{1}_N$ is an eigenvector associated with the Laplacian eigenvalue 0.

Let \mathcal{G} be an undirected graph with N nodes, where $N \geq 2$. The graph \mathcal{G} is connected if and only if its Laplacian eigenvalue 0 has geometric multiplicity one [17]. In this case, the eigenvalues of the Laplacian matrix \mathcal{L} associated with the graph \mathcal{G} can be ordered as $\lambda_1 = 0 < \lambda_2 \leq \dots \leq \lambda_N$. Furthermore, there exists an orthogonal matrix $\mathcal{U} = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & \mathcal{U}_2 \end{bmatrix}$, where $\mathcal{U}_2 \in \mathbb{R}^{N \times (N-1)}$, such that $\mathcal{U}^T \mathcal{L} \mathcal{U} =$

$\text{diag}(0, \lambda_2, \dots, \lambda_N)$. It is obvious that $\mathcal{U}_2^T \mathcal{U}_2 = I_{N-1}$ and $\mathcal{U}_2 \mathcal{U}_2^T = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T$. Denote $\Lambda := \text{diag}(0, \lambda_2, \dots, \lambda_N)$, which can be partitioned as $\Lambda = \begin{bmatrix} 0 & \mathbf{0}_{N-1}^T \\ \mathbf{0}_{N-1} & \Lambda \end{bmatrix}$, where $\bar{\Lambda} := \text{diag}(\lambda_2, \dots, \lambda_N)$.

The following lemma will play a crucial role in our main results.

Lemma 1: ([8]) For any two vectors $a = [a_1^T, a_2^T, \dots, a_N^T]^T$ and $b = [b_1^T, b_2^T, \dots, b_N^T]^T$, where $a_i, b_i \in \mathbb{R}^n$, $i = 1, 2, \dots, N$, $N \geq 2$, we have

$$a^T (\mathcal{U}_2 \mathcal{U}_2^T \otimes I_n) b = \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)^T (b_i - b_j),$$

where \mathcal{U}_2 is defined above.

Before moving on, we give the definition of minimal left annihilator of a given matrix.

Definition 1: ([22]) For a given matrix $B \in \mathbb{C}^{m \times n}$ with rank $r < \min\{m, n\}$, we denote by B^\perp any $\mathbb{C}^{(m-r) \times m}$ matrix of full row rank such that $B^\perp B = \mathbf{0}$. Any such matrix B^\perp is called a *minimal left annihilator* of B .

Note that the minimal left annihilator is only defined for matrices with linearly dependent rows. The set of all such matrices is given by $B^\perp = T U_2^*$, where T is any nonsingular matrix and U_2 is obtained from the singular value decomposition $B = [U_1 \ U_2] \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}$. Thus, for a given B , B^\perp is not unique. Throughout this paper, B^\perp will denote any choice from this set of matrices.

In this paper, we consider a network of $N (\geq 2)$ identical Lur'e systems described by (see Fig. 1)

$$\begin{cases} \dot{x}_i = A_p x_i + B_p u_i + E_p d_i \\ z_i = C_p x_i \\ y_i = M_p x_i \\ d_i = -\phi(z_i, t) \end{cases}, \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$, $z_i(t) \in \mathbb{R}^p$ and $y_i(t) \in \mathbb{R}^q$ are the state to be synchronized, the diffusive coupling input, the system output and the measurement output of agent i , respectively. The equation $d_i = -\phi(z_i, t)$ represents a time-varying, memoryless, nonlinear negative feedback loop. The function $\phi(\cdot, t)$ from $\mathbb{R}^p \times \mathbb{R}^+$ to \mathbb{R}^p is uncertain and can be any function from a set of functions to be specified later. A_p, B_p, C_p, E_p and M_p are known constant matrices of compatible dimensions. Without loss of generality, we assume that the dimensions m and q of the diffusive coupling

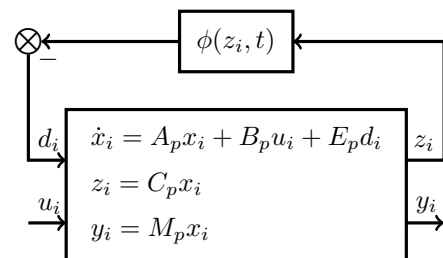


Fig. 1. Lur'e System

inputs and the measurement outputs, respectively, are strictly less than the state space dimension n . In this case the rows of B_p are linearly dependent and thus B_p^\perp exists. Similarly, $(M_p^T)^\perp$ exists as well. Furthermore, M_p is assumed to have full row rank. The interconnection topology among these agents is represented by the connected undirected graph \mathcal{G} which is fixed.

In our paper, the agents (1) in the network \mathcal{G} are assumed to be interconnected by means of a distributed dynamical protocol of the form

$$\begin{cases} \dot{w}_i = A_c w_i + B_c \sum_{j=1}^N a_{ij}(y_i - y_j) + D_c \sum_{j=1}^N a_{ij}(w_i - w_j) \\ u_i = C_c w_i \end{cases}, \quad i = 1, 2, \dots, N,$$

where $w_i(t) \in \mathbb{R}^{n_c}$ is the protocol state for agent i , A_c , B_c , C_c and D_c are the parameter matrices of the protocol, and $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of the graph \mathcal{G} . n_c , A_c , B_c , C_c and D_c need to be determined.

Remark 1: The dynamical protocol determines the information exchange among these agents, i.e. the copy of the protocol at agent i receives the weighted relative measurements and the weighted relative protocol states, uses these to determine the diffusive coupling input to agent i , and at the same time processes these quantities to determine the dynamics of its protocol state.

Definition 2: The network of agents (1) with the protocol (2) is robustly synchronized if $x_i(t) - x_j(t) \rightarrow \mathbf{0}$ and $w_i(t) - w_j(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, $\forall i, j = 1, 2, \dots, N$, for all initial conditions and all uncertain functions $\phi(\cdot, t)$ from a particular set of functions to be specified in the next section.

III. MAIN RESULTS

In this section, our main results are presented. We first establish sufficient conditions for the protocol (2) to robustly synchronize the network of agents (1). Subsequently we discuss how to compute a suitable protocol.

By interconnecting (1) and (2) we get the Lur'e dynamical network

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} I_N \otimes A_p & I_N \otimes B_p C_c \\ \mathcal{L} \otimes B_c M_p & I_N \otimes A_c + \mathcal{L} \otimes D_c \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ - \begin{bmatrix} I_N \otimes E_p \\ \mathbf{0} \end{bmatrix} \Phi(z, t) \\ z = (I_N \otimes C_p) x \end{cases}, \quad (3)$$

where $x = [x_1^T, x_2^T, \dots, x_N^T]^T$, $w = [w_1^T, w_2^T, \dots, w_N^T]^T$, $\Phi(z, t) = [\phi(z_1, t)^T, \phi(z_2, t)^T, \dots, \phi(z_N, t)^T]^T$, $z = [z_1^T, z_2^T, \dots, z_N^T]^T$, and \mathcal{L} is the Laplacian matrix of the graph \mathcal{G} .

In this paper, we assume the set of uncertain functions $\phi(\cdot, t)$ to consist of all functions that are incrementally passive. Incremental passivity for static systems of the form

$$d = \phi(z, t) \quad (4)$$

with input $z(t) \in \mathbb{R}^p$ and output $d(t) \in \mathbb{R}^p$ is defined as follows.

Definition 3: ([2]) The system (4) is called *incrementally passive* if the function $\phi(\cdot, t)$ satisfies

$$(z_1 - z_2)^T (\phi(z_1, t) - \phi(z_2, t)) \geq 0$$

for all $z_1, z_2 \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$.

In general, incremental passivity is stronger than the property of passivity, which is defined by $z^T \phi(z, t) \geq 0$ for all $z \in \mathbb{R}^p$ and $t \in \mathbb{R}^+$. Passivity implies incremental passivity for linear systems, and also for monotonically increasing static nonlinearities [10].

The following theorem gives a condition under which the distributed protocol (2) robustly synchronizes the network (1).

Theorem 1: Let $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$, $D_c \in \mathbb{R}^{n_c \times n_c}$. If there exists a positive definite matrix $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that

$$P(A + BH_i M) + (A + BH_i M)^T P < \mathbf{0} \quad (5)$$

and

$$PE = C^T \quad (6)$$

for all $i = 2, \dots, N$, where $A = \begin{bmatrix} A_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_c \times n_c} \end{bmatrix}$, $B = \begin{bmatrix} B_p & \mathbf{0} \\ \mathbf{0} & I_{n_c} \end{bmatrix}$, $H_i = \begin{bmatrix} \mathbf{0} & \lambda_i C_c \\ B_c & A_c + \lambda_i D_c \end{bmatrix}$, $M = \begin{bmatrix} M_p & \mathbf{0} \\ \mathbf{0} & I_{n_c} \end{bmatrix}$, $E = \begin{bmatrix} E_p \\ \mathbf{0}_{n_c \times p} \end{bmatrix}$, $C = [C_p \quad \mathbf{0}_{p \times n_c}]$, then the network of agents (1) with the protocol (2) is robustly synchronized, i.e. the Lur'e network (3) is synchronized for all incrementally passive $\phi(\cdot, t)$.

Proof. Let \mathcal{U} be an orthogonal matrix such that $\mathcal{U}^T \mathcal{L} \mathcal{U} = \Lambda$ as defined in Section 2. All notation introduced in Section 2 will be used without redefinitions or statements throughout this paper. Let

$$\begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \mathcal{U}^T \otimes I_n & \\ & \mathcal{U}^T \otimes I_{n_c} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \mathcal{U}_2^T \otimes I_n & \\ & \mathcal{U}_2^T \otimes I_{n_c} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \dots, \tilde{x}_N^T]^T$, $\tilde{w} = [\tilde{w}_1^T, \tilde{w}_2^T, \dots, \tilde{w}_N^T]^T$, $\tilde{x} = [\tilde{x}_2^T, \dots, \tilde{x}_N^T]^T$ and $\tilde{w} = [\tilde{w}_2^T, \dots, \tilde{w}_N^T]^T$. Denote $\bar{w} = (\bar{\Lambda}^{-1} \otimes I_{n_c}) \tilde{w}$. It follows from [12], Lemma 3.2 that $x_i(t) - x_j(t) \rightarrow \mathbf{0}$ and $w_i(t) - w_j(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, $\forall i, j = 1, 2, \dots, N$, if and only if $\tilde{x}(t) \rightarrow \mathbf{0}$ and $\bar{w}(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$. The dynamics of \tilde{x} and \bar{w} is given by

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes A_p & \bar{\Lambda} \otimes B_p C_c \\ I_{N-1} \otimes B_c M_p & I_{N-1} \otimes A_c + \bar{\Lambda} \otimes D_c \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} - \begin{bmatrix} \mathcal{U}_2^T \otimes E_p \\ \mathbf{0} \end{bmatrix} \Phi(z, t). \quad (7)$$

Hence the robust synchronization of x and w is equivalent to the global asymptotical stability of \tilde{x} and \bar{w} , respectively.

By Lemma 1, we have

$$\tilde{x}^T (\mathcal{U}_2^T \otimes C_p^T) \Phi(z, t)$$

$$\begin{aligned}
&= x^T (\mathcal{U}_2 \mathcal{U}_2^T \otimes C_p^T) \Phi(z, t) \\
&= x^T (I_N \otimes C_p^T) (\mathcal{U}_2 \mathcal{U}_2^T \otimes I_p) \Phi(z, t) \\
&= z^T (\mathcal{U}_2 \mathcal{U}_2^T \otimes I_p) \Phi(z, t) \\
&= \frac{1}{N} \sum_{1 \leq i < j \leq N} (z_i - z_j)^T (\phi(z_i, t) - \phi(z_j, t)) \geq 0.
\end{aligned}$$

Let $P > \mathbf{0}$ in (5) and (6) be appropriately partitioned as $P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$. Then (6) holds if and only if $P_1 E_p = C_p^T$ and $P_2^T E_p = \mathbf{0}$. Define a positive definite matrix P_4 by

$$P_4 = \begin{bmatrix} I_{N-1} \otimes P_1 & I_{N-1} \otimes P_2 \\ I_{N-1} \otimes P_2^T & I_{N-1} \otimes P_3 \end{bmatrix}.$$

Consider a quadratic Lyapunov function candidate

$$V(\bar{x}, \bar{w}) = \frac{1}{2} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T P_4 \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}.$$

Obviously, V is positive definite and radially unbounded. The time derivative of V along the trajectories of the system (7) is given by

$$\begin{aligned}
\dot{V}(\bar{x}, \bar{w}) &= \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T P_4 \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{w}} \end{bmatrix} = \\
&\begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix}^T \left(\begin{array}{c|c} \begin{matrix} I_{N-1} \otimes P_1 A_p & \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 B_c M_p & + I_{N-1} \otimes P_2 A_c \\ \hline I_{N-1} \otimes P_2^T A_p & \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 B_c M_p & + I_{N-1} \otimes P_3 A_c \\ \hline I_{N-1} \otimes P_1 A_p & \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 B_c M_p & + I_{N-1} \otimes P_2 A_c \\ \hline I_{N-1} \otimes P_2^T A_p & \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 B_c M_p & + I_{N-1} \otimes P_3 A_c \\ \hline I_{N-1} \otimes P_1 A_p & \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 B_c M_p & + I_{N-1} \otimes P_2 A_c \\ \hline I_{N-1} \otimes P_2^T A_p & \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 B_c M_p & + I_{N-1} \otimes P_3 A_c \\ \hline I_{N-1} \otimes P_1 A_p & \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 B_c M_p & + I_{N-1} \otimes P_2 A_c \\ \hline I_{N-1} \otimes P_2^T A_p & \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 B_c M_p & + I_{N-1} \otimes P_3 A_c \end{matrix} & \begin{matrix} \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 A_c \\ \hline \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 A_c \\ \hline \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 A_c \\ \hline \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 A_c \\ \hline \bar{\Lambda} \otimes P_1 B_p C_c \\ + I_{N-1} \otimes P_2 A_c \\ \hline \bar{\Lambda} \otimes P_2^T B_p C_c \\ + I_{N-1} \otimes P_3 A_c \end{matrix} \end{array} \right) \\
&\cdot \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} - \begin{bmatrix} \mathcal{U}_2^T \otimes P_1 E_p \\ \mathcal{U}_2^T \otimes P_2^T E_p \end{bmatrix} \Phi(z, t) = \\
&\sum_{i=2}^N \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix}^T \left(\begin{array}{c|c} \begin{matrix} P_1 A_p & \lambda_i P_1 B_p C_c \\ + P_2 B_c M_p & + P_2 A_c \\ \hline P_2^T A_p & \lambda_i P_2^T B_p C_c \\ + \lambda_i P_3 B_c M_p & + P_3 A_c \\ \hline P_1 A_p & \lambda_i P_1 B_p C_c \\ + P_2 B_c M_p & + P_2 A_c \\ \hline P_2^T A_p & \lambda_i P_2^T B_p C_c \\ + \lambda_i P_3 B_c M_p & + P_3 A_c \\ \hline P_1 A_p & \lambda_i P_1 B_p C_c \\ + P_2 B_c M_p & + P_2 A_c \\ \hline P_2^T A_p & \lambda_i P_2^T B_p C_c \\ + \lambda_i P_3 B_c M_p & + P_3 A_c \end{matrix} & \begin{matrix} \lambda_i P_1 B_p C_c \\ + P_2 A_c \\ \hline \lambda_i P_2^T B_p C_c \\ + P_3 A_c \\ \hline \lambda_i P_1 B_p C_c \\ + P_2 A_c \\ \hline \lambda_i P_2^T B_p C_c \\ + P_3 A_c \end{matrix} \end{array} \right) \\
&\cdot \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix} - \bar{x}^T (\mathcal{U}_2^T \otimes C_p^T) \Phi(z, t) \leq \\
&\sum_{i=2}^N \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} A_p & \lambda_i B_p C_c \\ B_c M_p & A_c + \lambda_i D_c \end{bmatrix} \\
&\cdot \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix} = \sum_{i=2}^N \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix}^T P (A + B H_i M) \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix} = \frac{1}{2} \sum_{i=2}^N \\
&\begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix}^T [P(A + B H_i M) + (A + B H_i M)^T P] \begin{bmatrix} \tilde{x}_i \\ \tilde{w}_i \end{bmatrix},
\end{aligned}$$

which is negative definite. Thus the system (7) is globally asymptotically stable, i.e. the Lur'e network (3) is robustly synchronized. This completes the proof. \square

Below we will discuss conditions for the existence of protocol matrices A_c, B_c, C_c, D_c and a common solution $P > \mathbf{0}$ of (5) and (6) in Theorem 1. Our first theorem gives necessary conditions.

Theorem 2: Assume there exists a positive integer n_c , $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$, $D_c \in \mathbb{R}^{n_c \times n_c}$ and a positive definite matrix $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that (5) and (6) hold for all $i = 2, \dots, N$. Then there exist positive definite matrices X_p and Y_p of size $n \times n$ such that

$$B_p^\perp (A_p X_p + X_p A_p^T) B_p^{\perp T} < \mathbf{0}, \quad (8)$$

$$E_p = X_p C_p^T, \quad (9)$$

$$M_p^{T\perp} (Y_p A_p + A_p^T Y_p) (M_p^{T\perp})^T < \mathbf{0}, \quad (10)$$

$$Y_p E_p = C_p^T, \quad (11)$$

$$Y_p - X_p^{-1} \geq \mathbf{0}. \quad (12)$$

Proof. Define $X := P^{-1}$. We get

$$(A + B H_i M) X + X (A + B H_i M)^T < \mathbf{0},$$

for all $i = 2, \dots, N$ and thus

$$B^\perp (A X + X A^T) B^{\perp T} < \mathbf{0}.$$

Similarly, we have

$$M^{T\perp} (Y A + A^T Y) (M^{T\perp})^T < \mathbf{0},$$

where $Y := P$. Partition

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}, \quad Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}.$$

Note that $B^\perp = [B_p^\perp \quad \mathbf{0}]$, $M^{T\perp} = [M_p^{T\perp} \quad \mathbf{0}]$,

$$A X + X A^T = \begin{bmatrix} A_p X_p + X_p A_p^T & \star \\ \star & \star \end{bmatrix},$$

$$Y A + A^T Y = \begin{bmatrix} Y_p A_p + A_p Y_p^T & \star \\ \star & \star \end{bmatrix}.$$

Then we obtain (8) and (10). We also have $E = X C^T$ and $Y E = C^T$, which imply (9) and (11), respectively. Furthermore, $X Y = I$ implies that $X_p Y_p + X_{pc} Y_{pc}^T = I$ and $X_p Y_{pc} + X_{pc} Y_c = \mathbf{0}$. Thus

$$Y_p - X_p^{-1} = Y_{pc} Y_c^{-1} Y_{pc}^T \geq \mathbf{0},$$

i.e. (12) holds. \square

We will now show that the necessary conditions obtained in Theorem 2 above are almost sufficient. In fact, if we replace the inequality (12) by its strict version

$$Y_p - X_p^{-1} > \mathbf{0}, \quad (13)$$

we obtain sufficient conditions for the existence of A_c, B_c, C_c, D_c and $P > \mathbf{0}$ such that (5) and (6) hold for all $i = 2, \dots, N$.

Our following protocol design is inspired by the measurement feedback \mathcal{H}_∞ -optimization controller construction for general linear systems in [3].

Theorem 3: There exists a positive integer n_c and matrices $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$, $D_c \in \mathbb{R}^{n_c \times n_c}$, $P > \mathbf{0} \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that (5) and (6) hold for all $i = 2, \dots, N$ if there exist positive definite matrices X_p and Y_p of size $n \times n$ such that (8), (9), (10), (11) and

$$\begin{aligned}
& Z_p(-\lambda_i B_p C_c + A_c + \lambda_i D_c) + (-\lambda_i B_p C_c + A_c + \lambda_i D_c)^T Z_p \\
&= Z_p[-\lambda_i B_p F + A_p + Z_p^{-1} Y_p G M_p + \Delta_1 + \lambda_i (B_p F + \Delta_2)] + [-\lambda_i B_p F + A_p + Z_p^{-1} Y_p G M_p + \Delta_1 + \lambda_i (B_p F + \Delta_2)]^T Z_p \\
&= Z_p A_p + A_p^T Z_p + Y_p G M_p + M_p^T G^T Y_p + Z_p(\Delta_1 + \lambda_i \Delta_2) + (\Delta_1 + \lambda_i \Delta_2)^T Z_p \\
&= Z_p A_p + A_p^T Z_p + Y_p G M_p + M_p^T G^T Y_p + Z_p [\tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p) - \lambda_i Z_p^{-1} ((1 - \tau) F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)] \\
&\quad + [\tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p) - \lambda_i Z_p^{-1} ((1 - \tau) F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)]^T Z_p \\
&= Z_p A_p + A_p^T Z_p + Y_p G M_p + M_p^T G^T Y_p + 2\tau (A_p^T X_p^{-1} + X_p^{-1} A_p) \\
&\quad - \lambda_i (1 - \tau) F^T B_p^T X_p^{-1} + \lambda_i \tau X_p^{-1} B_p F - \lambda_i (1 - \tau) X_p^{-1} B_p F + \lambda_i \tau F^T B_p^T X_p^{-1} \\
&= (Y_p - X_p^{-1}) A_p + A_p^T (Y_p - X_p^{-1}) + Y_p G M_p + M_p^T G^T Y_p + 2\tau R_F^i - \lambda_i F^T B_p^T X_p^{-1} - \lambda_i X_p^{-1} B_p F \\
&= Y_P R_G Y_P - R_F^i + 2\tau R_F^i
\end{aligned}$$

(17)

(13) hold, respectively. Suitable A_c, B_c, C_c, D_c and P are computed as follows.

- Choose $r_2 > 0$ such that

$$A_p X_p + X_p A_p^T - 2r_2 \lambda_2 B_p B_p^T < \mathbf{0}; \quad (14)$$

- Choose $r_1 > 0$ such that

$$Y_p A_p + A_p^T Y_p - 2r_1 M_p^T M_p < \mathbf{0}; \quad (15)$$

- Define $F := -r_2 B_p^T X_p^{-1}$ and $G := -r_1 Y_p^{-1} M_p^T$;
- Define

$$R_F^i := (A_p + \lambda_i B_p F)^T X_p^{-1} + X_p^{-1} (A_p + \lambda_i B_p F),$$

$i = 2, \dots, N$, and

$$R_G := (A_p + G M_p) Y_p^{-1} + Y_p^{-1} (A_p + G M_p)^T;$$

- Choose a real number $\tau \in (0, 1)$ such that

$$Y_p R_G Y_p < (1 - \tau) R_F^N;$$

- Define

$$Z_p := Y_p - X_p^{-1},$$

$$\tilde{G} := Z_p^{-1} Y_p G,$$

$$\Delta_1 := \tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p),$$

$$\Delta_2 := -Z_p^{-1} ((1 - \tau) F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F);$$

- Choose

$$P := \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix},$$

$$A_c := A_p + \tilde{G} M_p + \Delta_1,$$

$$B_c := -\tilde{G},$$

$$C_c := F,$$

$$D_c := B_p F + \Delta_2.$$

Obviously, the protocol has the same state dimension as the agents, i.e. $n_c = n$.

Proof. Obviously, by (9) and (11), the proposed $P = \begin{bmatrix} Y_p & -Z_p \\ -Z_p & Z_p \end{bmatrix}$ satisfies (6). Next we will show that (5) also holds for all $i = 2, \dots, N$.

By Finsler's lemma [22], (8) and (10) imply that there exist $r_2 > 0$ and $r_1 > 0$ such that (14) and (15) hold, respectively. Thus we have

$$R_F^i \leq (A_p + \lambda_2 B_p F)^T X_p^{-1} + X_p^{-1} (A_p + \lambda_2 B_p F) < \mathbf{0}$$

for all $i = 2, \dots, N$, and $R_G < \mathbf{0}$. Since we choose $\tau \in (0, 1)$ such that $Y_p R_G Y_p < (1 - \tau) R_F^N$, and $R_F^N \leq R_F^{N-1} \leq \dots \leq R_F^2$, we get $Y_p R_G Y_p < (1 - \tau) R_F^i$ for all $i = 2, \dots, N$. Note that such τ always exists and the largest Laplacian eigenvalue is involved.

Denote $A_i := A + B H_i M$. Then (5) holds if and only if

$$\tilde{P} \tilde{A}_i + \tilde{A}_i^T \tilde{P} < \mathbf{0}, \quad i = 2, \dots, N, \quad (16)$$

where

$$\tilde{P} = S^T P S = \begin{bmatrix} X_p^{-1} & \mathbf{0} \\ \mathbf{0} & Z_p \end{bmatrix},$$

$$\tilde{A}_i = S^{-1} A_i S =$$

$$\begin{bmatrix} A_p + \lambda_i B_p C_c & -\lambda_i B_p C_c \\ A_p - B_c M_p + \lambda_i B_p C_c - A_c - \lambda_i D_c & -\lambda_i B_p C_c + A_c + \lambda_i D_c \end{bmatrix},$$

and $S = S^{-1} = \begin{bmatrix} I & \mathbf{0} \\ I & -I \end{bmatrix}$. By straightforward computation, the (1, 1) block of the left hand of (16) turns out to be R_F^i .

The (2, 1) block can be computed to be equal to

$$\begin{aligned}
& Z_p (A_p - B_c M_p + \lambda_i B_p C_c - A_c - \lambda_i D_c) - \lambda_i C_c^T B_p^T X_p^{-1} \\
&= Z_p \{ A_p + Z_p^{-1} Y_p G M_p + \lambda_i B_p F - \\
&\quad [A_p + Z_p^{-1} Y_p G M_p + \tau Z_p^{-1} (A_p^T X_p^{-1} + X_p^{-1} A_p)] \\
&\quad - \lambda_i [B_p F - Z_p^{-1} ((1 - \tau) F^T B_p^T X_p^{-1} - \tau X_p^{-1} B_p F)] \} \\
&\quad - \lambda_i F^T B_p^T X_p^{-1} = -\tau R_F^i.
\end{aligned}$$

The (2, 2) block can be computed to be equal to $Y_P R_G Y_P - R_F^i + 2\tau R_F^i$, see (17). Thus the left hand of (16) equals

$$\begin{bmatrix} R_F^i & -\tau R_F^i \\ -\tau R_F^i & Y_P R_G Y_P - R_F^i + 2\tau R_F^i \end{bmatrix}$$

for all $i = 2, \dots, N$. The latter equals

$$\begin{bmatrix} (1 - \tau) R_F^i & \mathbf{0} \\ \mathbf{0} & Y_P R_G Y_P - (1 - \tau) R_F^i \end{bmatrix} + \tau \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes R_F^i$$

for all $i = 2, \dots, N$. Obviously, the first term above is negative definite and the second one is negative semi-definite since $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq \mathbf{0}$ and $R_F^i < \mathbf{0}$. Therefore, (16) and also (5) hold for all $i = 2, \dots, N$. This completes the proof. \square

Remark 2: Note that there is a gap between the necessary conditions obtained in Theorem 2 and the sufficient conditions obtained in Theorem 3, i.e. we need the strict inequality (13) to hold instead of the non-strict one. The conditions in Theorem 2 are quite close to necessary and sufficient conditions for Theorem 3. However, at this moment it is unclear how to close this gap. This is an interesting problem for future research.

IV. CONCLUSIONS

In this paper we have discussed output feedback robust synchronization of homogeneous Lur'e networks with incrementally passive nonlinearities. Sufficient conditions for the existence of distributed dynamical protocols to robustly synchronize such multi-agent networks have been given. The protocol parameter matrices are computed by solving LMI's, which can be easily done by using the LMI Control Toolbox in Matlab. The robust output synchronization problem for such multi-agent networks could be a possible topic for future research.

REFERENCES

- [1] A. Diwadkar and U. Vaidya, "Robust synchronization in nonlinear network with link failure uncertainty," in *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, pp. 6325-6330, Orlando, FL, 2011.
- [2] A. Pavlov and L. Marconi, "Incremental passivity and output regulation," *Systems & Control Letters*, vol. 57, no. 5, pp. 400-409, 2008.
- [3] C. Scherer, *The Riccati Inequality and State-Space Hinfinity-Optimal Control*. PhD Dissertation, University of Wuerzburg, Germany, 1991.
- [4] C.W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems*. Singapore: World Scientific Publishing Co. Pte. Ltd., 2007.
- [5] D. Sun, *Synchronization and Control of Multiagent Systems*. Taylor & Francis Group: CRC Press, 2011.
- [6] F. Ceragioli, C. De Persis, and P. Frasca, "Discontinuities and hysteresis in quantized average consensus," *Automatica*, vol. 47, no. 9, pp. 1916-1928, 2011.
- [7] F. Zhang, H.L. Trentelman, and J.M.A. Scherpen, "Robust synchronization of Lur'e networks with incremental nonlinearities," in *Proc. of the 52nd IEEE Conference on Decision and Control*, pp. 3439-3444, Florence, Italy, 2013.
- [8] F. Zhang, H.L. Trentelman, and J.M.A. Scherpen, "Fully distributed robust synchronization of networked Lur'e systems with incremental nonlinearities," *Automatica*, to appear, 2014.
- [9] G. Antonelli, "Interconnected dynamic systems: an overview on distributed control," *IEEE Control Systems Magazine*, vol. 33, no. 1, pp. 76-88, 2013.
- [10] G.-B. Stan and R. Sepulchre "Analysis of interconnected oscillators by dissipativity theory," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 256-270, 2007.
- [11] H. Dimassi, A. Loria, and S. Belghith, "Adaptive observers-based synchronization of a class of Lur'e systems with delayed outputs for chaotic communications," in *Proc. of 2012 IFAC Conference on Analysis and Control of Chaotic Systems*, pp. 255-260, Cancún, México, 2012.
- [12] H.L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1511-1523, 2013.
- [13] J.A.K. Suykens, P.F. Curran, T. Yang, J. Vandewalle, and L.O. Chua, "Nonlinear \mathcal{H}_∞ synchronization of Lur'e systems: dynamic output feedback case," *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 44, no. 11, pp. 1089-1092, 1997.
- [14] M. Arcak and P. Kokotović, "Feasibility conditions for circle criterion designs," *Systems & Control Letters*, vol. 42, no. 5, pp. 405-412, 2001.
- [15] M. Arcak and P. Kokotović, "Observer-based control of systems with slope-restricted nonlinearities," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1146-1150, 2001.
- [16] M. Arcak, "Passivity as a design tool for group coordination," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1380-1390, 2007.
- [17] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton: Princeton University Press, 2010.
- [18] N. Chopra and M.W. Spong, "Output synchronization of nonlinear systems with time delay in communication," in *Proc. of the 45th IEEE Conference on Decision and Control*, pp. 4986-4992, San Diego, CA, 2006.
- [19] R.M. Murray, "Recent research in cooperative control of multivehicle systems," *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, pp. 571-583, 2007.
- [20] S.E. Tuna, "Synchronizing linear systems via partial-state coupling," *Automatica*, vol. 44, no. 8, pp. 2179-2184, 2008.
- [21] S.M. Lee, S.J. Choi, D.H. Ji, J.H. Park, and S.C. Won, "Synchronization for chaotic Lur'e systems with sector-restricted nonlinearities via delayed feedback control," *Nonlinear Dynamics*, vol. 59, no. 1-2, pp. 277-288, 2010.
- [22] T. Iwasaki and R.E. Skelton, "All controllers for the general \mathcal{H}_∞ control problem: LMI existence conditions and state space formulas," *Automatica*, vol. 30, no. 8, pp. 1307-1317, 1994.
- [23] W. Ren and R.W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. London: Springer-Verlag, 2008.
- [24] Z. Li, Z. Duan, and G. Chen, "Global synchronised regions of linearly coupled Lur'e systems," *International Journal of Control*, vol. 84, no. 2, pp. 216-227, 2011.