

Affine Feedback Transformation Group for Nonlinear SISO Systems

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Abstract—This paper describes a feedback transformation group for the class of nonlinear single-input, single-output systems that can be represented in terms of Chen-Fliess functional expansions. There is no a priori requirement that these input-output systems have a state space realization, so the results presented here are independent of any particular state space coordinate system or state space embedding when a realization is available. The group is referred to as *affine* since it can always represent the input-output feedback linearization law of any control affine state space realization having a well defined relative degree in the usual sense. It is further shown in a coordinate free fashion that the invariants of this transformation group correspond exactly with transfer functions of Brunovsky forms when the group acts on a generating series having a well defined relative degree (defined purely from an input-output point of view). This represents a generalization of the author’s previous work where a significantly smaller subgroup of this transformation group was considered.

Index Terms—Nonlinear control systems, formal power series, Chen-Fliess series, transformation groups

AMS Subject Classifications—93C10, 16T30

I. INTRODUCTION

Let G be a group and S a given set. G is said to act as a *transformation group* on the right of S if there exists a mapping $\phi : S \times G \rightarrow S : (h, g) \mapsto hg$ such that:

- i. $h1 = h$, 1 is the identity element of G ;
- ii. $h(g_1g_2) = (hg_1)g_2$ for all $g_1, g_2 \in G$.

The action is said to be *free* if $hg = h$ implies that $g = 1$. Transformation groups have been used extensively in system theory since its inception. The early work of Brockett, Krener and others in the case of linear systems [3], [4] and nonlinear state space systems [2], [19] has been important in understanding the role of invariance under feedback and coordinate transformations. More recently in [13], a feedback transformation group was presented for the class of nonlinear single-input, single-output (SISO) systems that can be represented in terms of Chen-Fliess functional expansions, so called *Fliess operators* [7], [8]. There was no a priori requirement that these input-output systems have a state space realization, so the results presented there are independent of any particular state space coordinate system or state space embedding when a realization is available. In particular, it was shown that this feedback transformation group leaves a certain subseries of an operator’s generating series invariant. The order, r , of this invariant subseries corresponds to the notion of relative degree (defined purely from an input-output point of view) when it is well defined. Such a subseries does *not*, however, coincide with the transfer function of the Brunovsky form, $1/s^r$, unless the generating series has a stronger notion of relative degree referred

to as *extended* relative degree. It was mentioned in [13] that this fact hints at the possibility of a larger feedback transformation group for this class of input-output systems whose invariants *do* correspond exactly to Brunovsky forms. In this paper, that group is presented. It is referred to as *affine* since it can always represent the input-output feedback linearization law of any control affine state space realization having a well defined relative degree in the usual sense [18]. The generalization requires a nontrivial extension of the approach applied in [13], as well as the combinatorial tools used in [11]–[17] to characterize system interconnections. The focus here will be exclusively on describing the group and its invariants when it acts on series having well defined relative degree. A characterization of the Hopf algebra underlying its dual, as was done for the smaller transformation group in [9]–[12] will be deferred to a later publication. Such Hopf algebras provide explicit computational tools to do, for example, analytical system inversion as demonstrated in [15], [17].

The paper is organized as follows. In the next section, a few key preliminary concepts are briefly outlined and some required tools related to the shuffle algebra are presented. In Section III the new transformation group is described in detail. In the subsequent section, the invariance theory is presented. The paper’s conclusions are given in the final section.

II. PRELIMINARIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \dots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X , $\eta = x_{i_1} \dots x_{i_k}$, is called a *word* over X . The *length* of η , $|\eta|$, is the number of letters in η . The set of all words with length k is denoted by X^k . The set of all words including the empty word, \emptyset , is designated by X^* . It forms a monoid under catenation. The set ηX^* is comprised of all words with the prefix η . Any mapping $c : X^* \rightarrow \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c . Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. If the *constant term* $(c, \emptyset) = 0$ then c is said to be *proper*. The *support* of c , $\text{supp}(c)$, is the set of all words having nonzero coefficients. The *order* of c , $\text{ord}(c)$, is the length of the shortest word in its support. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$. It forms an associative \mathbb{R} -algebra under the catenation product and a commutative and associative \mathbb{R} -algebra under the shuffle product, denoted here by \sqcup [1], [7].

A. Fliess Operators and Their Interconnections

One can formally associate with any series $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ a causal m -input, ℓ -output operator, F_c , in the following manner. Let $p \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue

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measurable function $u : [t_0, t_1] \rightarrow \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p -norm for a measurable real-valued function, u_i , defined on $[t_0, t_1]$. Let $L_p^m[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_p$ norm and $B_p^m(R)[t_0, t_1] := \{u \in L_p^m[t_0, t_1] : \|u\|_p \leq R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L_1^m[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L_1^m[t_0, t_1] \rightarrow C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^t u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0) d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0)$$

[7], [8].

When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ are interconnected in a cascade fashion, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c \circ d}$, where $c \circ d$ denotes the *composition product* of c and d as described in [5], [6]. This product is associative and \mathbb{R} -linear in its left argument c . In the event that two Fliess operators are interconnected to form a feedback system, the closed-loop system has a Fliess operator representation whose generating series is the *feedback product* of c and d , denoted by $c @ d$ [5], [16]. Consider, for example, the SISO case where $X = \{x_0, x_1\}$ and $\ell = 1$. Define the set of operators

$$\mathcal{F}_\delta = \{I + F_c : c \in \mathbb{R} \langle\langle X \rangle\rangle\},$$

where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for \mathcal{F}_δ is denoted by $\mathbb{R} \langle\langle X_\delta \rangle\rangle$. \mathcal{F}_δ forms a group under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + d + c \tilde{\circ} d$, and $\tilde{\circ}$ denotes the *modified composition product* [16].¹ It is of central importance that the corresponding group $(\mathbb{R} \langle\langle X_\delta \rangle\rangle, \circ, \delta)$ has a dual that forms a Faà di Bruno type Hopf algebra with antipode, α , satisfying

$$c_\delta^{\circ^{-1}} = \delta + c^{\circ^{-1}} = \delta + \sum_{\eta \in X^*} (\alpha a_\eta)(c) \eta,$$

where $c^{\circ^{-1}}$ denotes the composition inverse of c ,

$$a_\eta : \mathbb{R} \langle\langle X \rangle\rangle \rightarrow \mathbb{R} : c \mapsto (c, \eta)$$

is the coordinate function for $\eta \in X^*$, and $a_\delta(c_\delta) := 1$ [11]. The antipode can be computed by a recursive algorithm [14]. It is central to describing the feedback product as shown in the following theorem.

Theorem 2.1: [11] For any $c, d \in \mathbb{R} \langle\langle X \rangle\rangle$ it follows that

$$c @ d = c \tilde{\circ} (-d \circ c)^{\circ^{-1}} = c \circ (\delta - d \circ c)^{\circ^{-1}}.$$

¹The same symbol will be used for composition on $\mathbb{R} \langle\langle X \rangle\rangle$ and $\mathbb{R} \langle\langle X_\delta \rangle\rangle$. As elements in these two sets have a distinct notation, i.e., c versus c_δ , respectively, it will always be clear which product is at play.

B. Shuffle Product Operations on Ultrametric Spaces

The \mathbb{R} -vector space $\mathbb{R} \langle\langle X \rangle\rangle$ with the distance between two series defined as $\text{dist}(c, d) = \sigma^{\text{ord}(c-d)}$ for some arbitrary but fixed $0 < \sigma < 1$ is a complete ultrametric space [1]. In this section two lemmas are presented which describe how the distance between two series are altered by operations involving the shuffle product. These results are employed in the next section to prove the existence of a group inverse.

Lemma 2.1: For any series $c_i, d_i \in \mathbb{R} \langle\langle X \rangle\rangle$

$$\text{dist}(c_1 \sqcup d_1, c_2 \sqcup d_2) \leq \max(\sigma^{\text{ord}(c_1)} \text{dist}(d_1, d_2), \sigma^{\text{ord}(d_2)} \text{dist}(c_1, c_2)).$$

Proof: First observe that

$$\begin{aligned} \text{dist}(c_i \sqcup d_1, c_i \sqcup d_2) &= \sigma^{\text{ord}(c_i \sqcup (d_1 - d_2))} \\ &= \sigma^{\text{ord}(c_i) + \text{ord}(d_1 - d_2)} \\ &= \sigma^{\text{ord}(c_i)} \text{dist}(d_1, d_2). \end{aligned}$$

In which case, from the ultrametric triangle inequality it follows that

$$\begin{aligned} \text{dist}(c_1 \sqcup d_1, c_2 \sqcup d_2) &\leq \max(\text{dist}(c_1 \sqcup d_1, c_1 \sqcup d_2), \\ &\quad \text{dist}(c_1 \sqcup d_2, c_2 \sqcup d_2)) \\ &= \max(\sigma^{\text{ord}(c_1)} \text{dist}(d_1, d_2), \\ &\quad \sigma^{\text{ord}(d_2)} \text{dist}(c_1, c_2)). \end{aligned}$$

■

Corollary 2.1: For a fixed $c \in \mathbb{R} \langle\langle X \rangle\rangle$, the mapping $d \mapsto c \sqcup d$ is an ultrametric contraction if c is proper and an isometry on $(\mathbb{R} \langle\langle X \rangle\rangle, \text{dist})$ otherwise.

Theorem 2.2: [15], [17] The set of non proper series $\mathbb{R}_{np} \langle\langle X \rangle\rangle \subset \mathbb{R} \langle\langle X \rangle\rangle$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is

$$c^{\sqcup^{-1}} = ((c, \emptyset)(1 - c'))^{\sqcup^{-1}} := (c, \emptyset)^{-1} (c')^{\sqcup^*},$$

where $c' := 1 - c/(c, \emptyset)$ is proper and $(c')^{\sqcup^*} := \sum_{k=0}^{\infty} (c')^{\sqcup k}$.

Lemma 2.2: The shuffle inverse is an isometry for any $c, d \in \mathbb{R}_{np} \langle\langle X \rangle\rangle$ having identical constant terms.

Proof: For any $c, d \in \mathbb{R} \langle\langle X \rangle\rangle$ with $(c, \emptyset) = (d, \emptyset) \neq 0$ observe

$$\begin{aligned} \text{ord}(c^{\sqcup^{-1}} - d^{\sqcup^{-1}}) &= \text{ord}\left(\sum_{k=1}^{\infty} (c')^{\sqcup k} - (d')^{\sqcup k}\right) \\ &= \text{ord}(c' - d') = \text{ord}(c - d), \end{aligned}$$

and hence the lemma is proved. ■

III. TRANSFORMATION GROUP $\mathbb{R}_{np} \langle\langle X_\delta \rangle\rangle$

It will be assumed henceforth that $X = \{x_0, x_1\}$ and $\ell = 1$, which corresponds to a SISO system. The first step in building the affine group is to redefine $\mathbb{R} \langle\langle X_\delta \rangle\rangle$ in terms of *pairs* of series and then to generalize the modified composition product in a consistent fashion.

Definition 3.1: Consider a pair of series $d_\delta = (d_L, d_R) \in \mathbb{R} \langle\langle X \rangle\rangle \times \mathbb{R} \langle\langle X \rangle\rangle =: \mathbb{R} \langle\langle X_\delta \rangle\rangle$. Define the *mixed composition product* mapping $\mathbb{R} \langle\langle X \rangle\rangle \times \mathbb{R} \langle\langle X_\delta \rangle\rangle$ into $\mathbb{R} \langle\langle X \rangle\rangle$ as

$$c \tilde{\circ} d_\delta = \phi_d(c)(1) = \sum_{\eta \in X^*} (c, \eta) \phi_d(\eta)(1),$$

where ϕ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X \rangle\rangle$ to $\text{End}(\mathbb{R}\langle\langle X \rangle\rangle)$ uniquely specified by $\phi_d(x_i\eta) = \phi_d(x_i) \circ \phi_d(\eta)$ with

$$\phi_d(x_0)(e) = x_0e, \quad \phi_d(x_1)(e) = x_1(d_L \sqcup e) + x_0(d_R \sqcup e)$$

for any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where $\phi_d(\emptyset)$ denotes the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$.

It is immediate that the mixed composition product reduces to the modified composition product where $d_\delta = d_L\delta + d_R|_{d_L=1}$ in the notation of Section II. Some fundamental properties of this product are given next.

Lemma 3.1: The mixed composition product on $\mathbb{R}\langle\langle X \rangle\rangle \times \mathbb{R}\langle\langle X_\delta \rangle\rangle$

- (1) is left \mathbb{R} -linear;
- (2) satisfies $c \tilde{\circ} (1, 0) = c$;
- (3) satisfies $c \tilde{\circ} d_\delta = k \in \mathbb{R}$ for any fixed d_δ if and only if $c = k$;²
- (4) satisfies $(x_0c) \tilde{\circ} d_\delta = x_0(c \tilde{\circ} d_\delta)$ and $(x_1c) \tilde{\circ} d_\delta = x_1(d_L \sqcup (c \tilde{\circ} d_\delta)) + x_0(d_R \sqcup (c \tilde{\circ} d_\delta))$;
- (5) distributes to the left over the shuffle product.

Proof:

(1) This fact follows directly from the definition of the mixed composition product.

(2) The claim is immediate since $\phi_{(1,0)}(\eta)(1) = \eta$.

(3) The only non trivial assertion is that $c \tilde{\circ} d_\delta = k$ implies $c = k$. This claim is best handled in a Hopf algebra setting. So this part of the proof will be deferred till those tools are presented in a subsequent publication. However, the special case where $d_L = 1$ is treated as such in [14].

(4) Observe:

$$\begin{aligned} (x_0c) \tilde{\circ} d_\delta &= \phi_d(x_0c)(1) = \phi_d(x_0) \circ \phi_d(c)(1) = x_0(c \tilde{\circ} d_\delta) \\ (x_1c) \tilde{\circ} d_\delta &= \phi_d(x_1c)(1) = \phi_d(x_1) \circ \phi_d(c)(1) \\ &= x_1(d_L \sqcup (c \tilde{\circ} d_\delta)) + x_0(d_R \sqcup (c \tilde{\circ} d_\delta)). \end{aligned}$$

(5) One can define a shuffle product within $\text{End}(\mathbb{R}\langle\langle X \rangle\rangle)$ via

$$\begin{aligned} \phi_e(x_i\eta) \sqcup \phi_e(x_j\xi) &= \phi_e(x_i) \circ [\phi_e(\eta) \sqcup \phi_e(x_j\xi)] + \\ &\quad \phi_e(x_j) \circ [\phi_e(x_i\eta) \sqcup \phi_e(\xi)]. \end{aligned}$$

In which case, ϕ_e acts as an algebra map between the shuffle algebra on $\mathbb{R}\langle\langle X \rangle\rangle$ and the shuffle algebra within $\text{End}(\mathbb{R}\langle\langle X \rangle\rangle)$. That is, $\phi_e(c \sqcup d) = \phi_e(c) \sqcup \phi_e(d)$. Hence, $(c \sqcup d) \tilde{\circ} e_\delta = \phi_e(c \sqcup d)(1) = \phi_e(c)(1) \sqcup \phi_e(d)(1) = (c \tilde{\circ} e_\delta) \sqcup (d \tilde{\circ} e_\delta)$. ■

It is easily checked that

$$\text{dist}(c_\delta, d_\delta) := \max(\text{dist}(c_L, d_L), \text{dist}(c_R, d_R))$$

is an ultrametric on $\mathbb{R}\langle\langle X_\delta \rangle\rangle$.³ The following lemma states that the mixed composition product acts as an ultrametric contraction on this space.

Lemma 3.2: For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ and $d_{\delta,1}, d_{\delta,2} \in \mathbb{R}\langle\langle X_\delta \rangle\rangle$ it follows that

$$\text{dist}(c \tilde{\circ} d_{\delta,1}, c \tilde{\circ} d_{\delta,2}) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{\delta,1}, d_{\delta,2}),$$

²For notational convenience, $c = k\emptyset$ is written as $c = k$.

³Using dist for both the ultrametric on $\mathbb{R}\langle\langle X \rangle\rangle$ and $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ causes minimal confusion since their arguments are distinct.

where $c = (c, \emptyset)\emptyset + c'$. In which case, the mixed composition product acts as a contraction from $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \text{dist})$ to $(\mathbb{R}\langle\langle X \rangle\rangle, \text{dist})$.

Proof: For a fixed d_R , consider the map $d_L \mapsto c \tilde{\circ} (d_L, d_R)$. Likewise, for a fixed d_L there is a companion map $d_R \mapsto (c \tilde{\circ} (d_L, d_R))$. It is first shown that on the ultrametric space $(\mathbb{R}\langle\langle X \rangle\rangle, \text{dist})$:

$$\text{dist}(c \tilde{\circ} (d_{L,1}, d_R), c \tilde{\circ} (d_{L,2}, d_R)) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{L,1}, d_{L,2}) \quad (1a)$$

$$\text{dist}(c \tilde{\circ} (d_L, d_{R,1}), c \tilde{\circ} (d_L, d_{R,2})) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{R,1}, d_{R,2}). \quad (1b)$$

The first step is to verify that (1a) holds when $c = \eta \in X^*$. It is shown by induction on the length of η that

$$\text{ord}(\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1)) \geq |\eta| + \text{ord}(d_{L,1} - d_{L,2}), \quad (2)$$

where $d_{\delta,i} = (d_{L,i}, d_R)$ and $d_{L,i} \neq 0$ (the nondegenerate case). The claim is trivial when η is empty or a single letter. Assume the inequality holds for words up to length $k \geq 0$. For any $x_0\eta$ with $\eta \in X^k$, inequality (2) follows directly from the induction hypothesis. The case for $x_1\eta$ is handled as follows:

$$\begin{aligned} &\text{ord}(\phi_{d_1}(x_1\eta)(1) - \phi_{d_2}(x_1\eta)(1)) \\ &= \text{ord}(x_1[d_{L,1} \sqcup \phi_{d_1}(\eta)(1) - d_{L,2} \sqcup \phi_{d_2}(\eta)(1)] + \\ &\quad x_0[d_R \sqcup (\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1))]) \\ &= \text{ord}(x_1[(d_{L,1} - d_{L,2}) \sqcup \phi_{d_1}(\eta)(1) + \\ &\quad d_{L,2} \sqcup (\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1))] + \\ &\quad x_0[d_R \sqcup (\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1))]) \\ &\geq 1 + \min(\text{ord}([d_{L,1} - d_{L,2}] \sqcup \phi_{d_1}(\eta)(1)), \\ &\quad \text{ord}(d_{L,2} \sqcup [\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1)]), \\ &\quad \text{ord}(d_R \sqcup [\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1)])) \\ &\geq 1 + \min(\text{ord}(d_{L,1} - d_{L,2}) + |\eta|, \text{ord}(\phi_{d_1}(\eta)(1) - \\ &\quad \phi_{d_2}(\eta)(1))) \\ &= |\eta| + 1 + \text{ord}(d_{L,1} - d_{L,2}). \end{aligned}$$

In which case, (2) holds for any $\eta \in X^*$. The inequality (1a) is now derived. Observe

$$\begin{aligned} &\text{dist}(c \tilde{\circ} (d_{L,1}, d_R), c \tilde{\circ} (d_{L,2}, d_R)) \\ &= \text{dist}(c' \tilde{\circ} (d_{L,1}, d_R), c' \tilde{\circ} (d_{L,2}, d_R)) \\ &= \sigma^{\text{ord}(\sum_\eta (c', \eta)[\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1)])} \\ &\leq \sigma^{\min_{\eta \in \text{supp}(c')} \text{ord}(\phi_{d_1}(\eta)(1) - \phi_{d_2}(\eta)(1))} \\ &\leq \sigma^{\min_{\eta \in \text{supp}(c')} |\eta| + \text{ord}(d_{L,1} - d_{L,2})} \\ &= \sigma^{\text{ord}(c')} \text{dist}(d_{L,1}, d_{L,2}). \end{aligned}$$

The proof for (1b) is completely analogous. The final step of the proof is to employ the ultrametric triangle inequality in conjunction with (1). Observe

$$\begin{aligned} &\text{dist}(c \tilde{\circ} d_{\delta,1}, c \tilde{\circ} d_{\delta,2}) \\ &= \text{dist}(c \tilde{\circ} (d_{L,1}, d_{R,1}), c \tilde{\circ} (d_{L,2}, d_{R,2})) \\ &\leq \max(\text{dist}(c \tilde{\circ} (d_{L,1}, d_{R,1}), c \tilde{\circ} (d_{L,2}, d_{R,1})), \\ &\quad \text{dist}(c \tilde{\circ} (d_{L,2}, d_{R,1}), c \tilde{\circ} (d_{L,2}, d_{R,2}))) \\ &\leq \sigma^{\text{ord}(c')} \max(\text{dist}(d_{L,1}, d_{L,2}), \text{dist}(d_{R,1}, d_{R,2})) \end{aligned}$$

$$= \sigma^{\text{ord}(c')} \text{dist}(d_{\delta,1}, d_{\delta,2}).$$

■

Analogous to the special case $d_L = 1$ in [11], where the modified composition product is used to define the group product, here the mixed composition product is used to define a group product on $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ as generalized in Definition 3.1. Its basic properties are given in the subsequent lemma.

Definition 3.2: The composition product on $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ is defined as

$$c_\delta \circ d_\delta = ((c_L \tilde{\circ} d_\delta) \sqcup d_L, (c_L \tilde{\circ} d_\delta) \sqcup d_R + c_R \tilde{\circ} d_\delta).$$

Lemma 3.3: The composition product on $\mathbb{R}\langle\langle X_\delta \rangle\rangle$

- (1) is left \mathbb{R} -linear;
- (2) satisfies $(c \tilde{\circ} d_\delta) \tilde{\circ} e_\delta = c \tilde{\circ} (d_\delta \circ e_\delta)$ (mixed associativity);
- (3) is associative.

Proof:

(1) This claim is a direct consequence of the left linearity of the mixed composition product.

(2) In light of the first item it is sufficient to prove the claim only for $c = \eta \in X^k$, $k \geq 0$. The cases $k = 0$ and $k = 1$ are trivial. Assume the claim holds up to some fixed $k \geq 0$. Then via Lemma 3.1 (4) and the induction hypothesis it follows that

$$\begin{aligned} ((x_0\eta) \tilde{\circ} d_\delta) \tilde{\circ} e_\delta &= (x_0(\eta \tilde{\circ} d_\delta)) \tilde{\circ} e_\delta = x_0((\eta \tilde{\circ} d_\delta) \tilde{\circ} e_\delta) \\ &= x_0(\eta \tilde{\circ} (d_\delta \tilde{\circ} e_\delta)) = (x_0\eta) \tilde{\circ} (d_\delta \tilde{\circ} e_\delta). \end{aligned}$$

In a similar fashion, apply the properties in Lemma 3.1 (1), (4), and (5) to get

$$\begin{aligned} ((x_1\eta) \tilde{\circ} d_\delta) \tilde{\circ} e_\delta &= [x_1(d_L \sqcup (\eta \tilde{\circ} d_\delta)) + x_0(d_R \sqcup (\eta \tilde{\circ} d_\delta))] \tilde{\circ} e_\delta \\ &= [x_1(d_L \sqcup (\eta \tilde{\circ} d_\delta))] \tilde{\circ} e_\delta + [x_0(d_R \sqcup (\eta \tilde{\circ} d_\delta))] \tilde{\circ} e_\delta \\ &= x_1[e_L \sqcup ((d_L \sqcup (\eta \tilde{\circ} d_\delta)) \tilde{\circ} e_\delta)] + \\ &\quad x_0[e_R \sqcup ((d_L \sqcup (\eta \tilde{\circ} d_\delta)) \tilde{\circ} e_\delta)] + \\ &\quad x_0[(d_R \sqcup (\eta \tilde{\circ} d_\delta)) \tilde{\circ} e_\delta] \\ &= x_1[e_L \sqcup \underbrace{(d_L \tilde{\circ} e_\delta)}_{(d_\delta \circ e_\delta)_L} \sqcup ((\eta \tilde{\circ} d_\delta) \tilde{\circ} e_\delta)] + \\ &\quad x_0[\underbrace{((d_L \tilde{\circ} e_\delta) \sqcup e_R + d_R \tilde{\circ} e_\delta)}_{(d_\delta \circ e_\delta)_R} \sqcup ((\eta \tilde{\circ} d_\delta) \tilde{\circ} e_\delta)]. \end{aligned}$$

Now employ the induction hypothesis so that

$$\begin{aligned} ((x_1\eta) \tilde{\circ} d_\delta) \tilde{\circ} e_\delta &= x_1[(d_\delta \circ e_\delta)_L \sqcup (\eta \tilde{\circ} (d_\delta \circ e_\delta))] + \\ &\quad x_0[(d_\delta \circ e_\delta)_R \sqcup (\eta \tilde{\circ} (d_\delta \circ e_\delta))] \\ &= (x_1\eta) \tilde{\circ} (d_\delta \circ e_\delta). \end{aligned}$$

Therefore, the claim holds for all $\eta \in X^*$, and the identity is proved.

(3) First apply Definition 3.2 twice, Lemma 3.1 (1) and (5) to get

$$\begin{aligned} (c_\delta \circ d_\delta) \circ e_\delta &= ((c_L \tilde{\circ} d_\delta) \sqcup d_L, (c_L \tilde{\circ} d_\delta) \sqcup d_R + c_R \tilde{\circ} d_\delta) \circ e_\delta \\ &= (((c_L \tilde{\circ} d_\delta) \sqcup d_L) \tilde{\circ} e_\delta) \sqcup e_L, [((c_L \tilde{\circ} d_\delta) \sqcup d_L) \tilde{\circ} e_\delta] \sqcup \\ &\quad e_R + [(c_L \tilde{\circ} d_\delta) \sqcup d_R + c_R \tilde{\circ} d_\delta] \tilde{\circ} e_\delta) \\ &= (((c_L \tilde{\circ} d_\delta) \tilde{\circ} e_\delta) \sqcup [d_L \tilde{\circ} e_\delta] \sqcup e_L, [(c_L \tilde{\circ} d_\delta) \tilde{\circ} e_\delta] \sqcup \end{aligned}$$

$$\begin{aligned} [d_L \tilde{\circ} e_\delta] \sqcup e_R + ((c_L \tilde{\circ} d_\delta) \tilde{\circ} e_\delta) \sqcup (d_R \tilde{\circ} e_\delta) + \\ (c_R \tilde{\circ} d_\delta) \tilde{\circ} e_\delta). \end{aligned}$$

Now apply the mixed associativity property from the previous item and then recombine terms according to Definition 3.2 so that

$$\begin{aligned} (c_\delta \circ d_\delta) \circ e_\delta &= ((c_L \tilde{\circ} (d_\delta \circ e_\delta)) \sqcup \underbrace{[d_L \tilde{\circ} e_\delta] \sqcup e_L}_{(d_\delta \circ e_\delta)_L}, [c_L \tilde{\circ} (d_\delta \circ e_\delta)] \sqcup \\ &\quad [d_L \tilde{\circ} e_\delta] \sqcup e_R + (c_L \tilde{\circ} (d_\delta \circ e_\delta)) \sqcup (d_R \tilde{\circ} e_\delta) + \\ &\quad c_R \tilde{\circ} (d_\delta \circ e_\delta)) \\ &= ((c_L \tilde{\circ} (d_\delta \circ e_\delta)) \sqcup (d_\delta \circ e_\delta)_L, [c_L \tilde{\circ} (d_\delta \circ e_\delta)] \sqcup \\ &\quad \underbrace{[d_L \tilde{\circ} e_\delta] \sqcup e_R + d_R \tilde{\circ} e_\delta}_{(d_\delta \circ e_\delta)_R} + c_R \tilde{\circ} (d_\delta \circ e_\delta)) \\ &= ((c_\delta \circ (d_\delta \circ e_\delta))_L, (c_\delta \circ (d_\delta \circ e_\delta))_R) \\ &= c_\delta \circ (d_\delta \circ e_\delta), \end{aligned}$$

and the lemma is proved. ■

For any $c_\delta \in \mathbb{R}\langle\langle X_\delta \rangle\rangle$ associate the functional $F_{c_\delta}[u] := uF_{c_L}[u] + F_{c_R}[u]$. The primary motivation behind the two series products defined above is given in the following theorem.

Theorem 3.1: For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ and $c_\delta, d_\delta \in \mathbb{R}\langle\langle X_\delta \rangle\rangle$ the following identities hold:

- (1) $F_c \circ F_{d_\delta} = F_{c \tilde{\circ} d_\delta}$
- (2) $F_{c_\delta} \circ F_{d_\delta} = F_{c_\delta \circ d_\delta}$.

Proof:

(1) It is sufficient to prove the claim for $c = \eta \in X^*$. This is done by induction on the length of η . The case for the empty word is trivial. Assume the identity holds for words $\eta \in X^k$ up to some fixed length $k \geq 0$. Then

$$\begin{aligned} E_{x_0\eta} \circ F_{d_\delta}[u](t) &= \int_{t_0}^t E_\eta[F_{d_\delta}[u]](\tau, t_0) d\tau \\ &= \int_{t_0}^t F_{\eta \tilde{\circ} d_\delta}[u](\tau) d\tau = F_{x_0(\eta \tilde{\circ} d_\delta)}[u](t) \\ &= F_{(x_0\eta) \tilde{\circ} d_\delta}[u](t). \end{aligned}$$

Similarly,

$$\begin{aligned} E_{x_1\eta} \circ F_{d_\delta}[u](t) &= \int_{t_0}^t [u(\tau)F_{d_L}[u](\tau) + F_{d_R}[u](\tau)]F_{\eta \tilde{\circ} d_\delta}[u](\tau) d\tau \\ &= \int_{t_0}^t u(\tau)F_{d_L \sqcup (\eta \tilde{\circ} d_\delta)}[u](\tau) + F_{d_R \sqcup (\eta \tilde{\circ} d_\delta)}[u](\tau) d\tau \\ &= F_{x_1(d_L \sqcup (\eta \tilde{\circ} d_\delta)) + x_0(d_R \sqcup (\eta \tilde{\circ} d_\delta))}[u](t) \\ &= F_{(x_1\eta) \circ d_\delta}[u](t). \end{aligned}$$

Hence, the claim holds for all $\eta \in X^*$.

(2) Observe

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta}[u] &= (uF_{c_L}[u] + F_{c_R}[u]) \circ (uF_{d_L}[u] + F_{d_R}[u]) \\ &= uF_{d_L}[u]F_{c_L}[F_{d_\delta}[u]] + F_{d_R}[u]F_{c_L}[F_{d_\delta}[u]] + \\ &\quad F_{c_R}[F_{d_\delta}[u]] \\ &= uF_{(c_L \tilde{\circ} d_\delta) \sqcup d_L}[u] + F_{(c_L \tilde{\circ} d_\delta) \sqcup d_R}[u] + \end{aligned}$$

$$\begin{aligned} & F_{c_R \tilde{\circ} d_\delta}[u] \\ &= uF_{(c_\delta \circ d_\delta)_L}[u] + F_{(c_\delta \circ d_\delta)_R}[u] \\ &= F_{c_\delta \circ d_\delta}[u]. \end{aligned}$$

Let $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$ denote the subset of $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ with the defining property that the left series are *not* proper. A main result of the paper is given below.

Theorem 3.2: The set $(\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle, \circ, (1, 0))$ is a group.

Proof: Using the identities $1 \tilde{\circ} c_\delta = 1$ and $0 \tilde{\circ} c_\delta = 0$, it is straightforward to show that $c_\delta \circ (1, 0) = (1, 0) \circ c_\delta = (1, 0)$. Associativity was established in Lemma 3.3 (3). So the only open issue is the existence of inverses. Suppose c_δ is fixed and one seeks a right inverse $c_\delta^{-1} = (c_L^{\circ-1}, c_R^{\circ-1})$, that is, $c_\delta \circ c_\delta^{-1} = (1, 0)$. Then it follows directly from Theorem 2.2 and Definition 3.2 that

$$c_L^{\circ-1} = (c_L \tilde{\circ} (c_L^{\circ-1}, c_R^{\circ-1})) \sqcup^{-1} \quad (3a)$$

$$c_R^{\circ-1} = -c_L^{\circ-1} \sqcup (c_R \tilde{\circ} (c_L^{\circ-1}, c_R^{\circ-1})). \quad (3b)$$

It is first shown that the mapping

$$S_R : (e_L, e_R) \mapsto ((c_L \tilde{\circ} (e_L, e_R)) \sqcup^{-1}, -e_L \sqcup (c_R \tilde{\circ} (e_L, e_R)))$$

is an ultrametric contraction, and therefore, has a unique fixed point which by design is a right inverse, c_δ^{-1} . Note that for any e_δ it follows that $(S_R(e_L, e_R)_L, \emptyset) = (c_L, \emptyset)^{-1} \neq 0$. Thus, the fixed point will always be in the group. Then it is shown that this same series is also a left inverse, that is, $c_\delta^{-1} \circ c_\delta = (1, 0)$, or equivalently,

$$c_L = (c_L^{\circ-1} \tilde{\circ} (c_L, c_R)) \sqcup^{-1} \quad (4a)$$

$$c_R = -c_L \sqcup (c_R^{\circ-1} \tilde{\circ} (c_L, c_R)). \quad (4b)$$

To establish the first claim, observe via Corollary 2.1 and Lemma 2.2 that for arbitrary e_δ, \bar{e}_δ

$$\begin{aligned} & \text{dist}(S_R(e_\delta), S_R(\bar{e}_\delta)) \\ &= \max(\text{dist}((c_L \tilde{\circ} (e_L, e_R)) \sqcup^{-1}, (c_L \tilde{\circ} (\bar{e}_L, \bar{e}_R)) \sqcup^{-1}), \\ & \quad \text{dist}(-e_L \sqcup (c_R \tilde{\circ} (e_L, e_R)), -\bar{e}_L \sqcup (c_R \tilde{\circ} (\bar{e}_L, \bar{e}_R)))) \\ &\leq \max(\text{dist}(c_L \tilde{\circ} (e_L, e_R), c_L \tilde{\circ} (\bar{e}_L, \bar{e}_R)), \\ & \quad \text{dist}(c_R \tilde{\circ} (e_L, e_R), c_R \tilde{\circ} (\bar{e}_L, \bar{e}_R))). \end{aligned}$$

In which case, from Lemma 3.2 it follows that

$$\begin{aligned} & \text{dist}(S_R(e_\delta), S_R(\bar{e}_\delta)) \\ &\leq \max(\sigma^{\text{ord}(c'_L)} \text{dist}((e_L, e_R), (\bar{e}_L, \bar{e}_R)), \\ & \quad \sigma^{\text{ord}(c'_R)} \text{dist}((e_L, e_R), (\bar{e}_L, \bar{e}_R))) \\ &\leq \sigma \text{dist}(e_\delta, \bar{e}_\delta). \end{aligned}$$

To address the second claim, suppose c_δ^{-1} satisfies (3a). In which case,

$$\begin{aligned} & (c_L \tilde{\circ} c_\delta^{-1}) \sqcup c_L^{\circ-1} = 1 \\ & (c_L \tilde{\circ} c_\delta^{-1}) \sqcup (c_L^{\circ-1} \tilde{\circ} (c_\delta \circ c_\delta^{-1})) = 1 \\ & (c_L \tilde{\circ} c_\delta^{-1}) \sqcup ((c_L^{\circ-1} \tilde{\circ} c_\delta) \tilde{\circ} c_\delta^{-1}) = 1 \\ & (c_L \sqcup (c_L^{\circ-1} \tilde{\circ} c_\delta)) \tilde{\circ} c_\delta^{-1} = 1. \end{aligned}$$

Applying Lemma 3.1 (3) then gives

$$c_L \sqcup (c_L^{\circ-1} \tilde{\circ} c_\delta) = 1$$

$$c_L = (c_L^{\circ-1} \tilde{\circ} c_\delta) \sqcup^{-1},$$

which is (4a). If, in addition, c_δ^{-1} also satisfies (3b), then substituting (3a) into (3b) gives

$$c_R^{\circ-1} = -(c_L \tilde{\circ} c_\delta^{-1}) \sqcup^{-1} \sqcup (c_R \tilde{\circ} c_\delta^{-1}).$$

Therefore,

$$\begin{aligned} & -(c_L \tilde{\circ} c_\delta^{-1}) \sqcup c_R^{\circ-1} = c_R \tilde{\circ} c_\delta^{-1} \\ & (-c_L \sqcup (c_R^{\circ-1} \tilde{\circ} c_\delta)) \tilde{\circ} c_\delta^{-1} = c_R \tilde{\circ} c_\delta^{-1} \\ & (c_R + c_L \sqcup (c_R^{\circ-1} \tilde{\circ} c_\delta)) \tilde{\circ} c_\delta^{-1} = 0. \end{aligned}$$

Once again applying Lemma 3.1 (3) gives

$$c_R + c_L \sqcup (c_R^{\circ-1} \tilde{\circ} c_\delta) = 0,$$

which is equivalent to (4b). \blacksquare

Example 3.1: The subgroup with $c_L = 1$ was the main object of study in [11]–[13]. In this case, $F_{c_\delta}[u] = u + F_{c_R}[u]$ and (3)–(4) reduce to

$$\begin{aligned} (1, c_R^{\circ-1}) &= (1, -c_R \tilde{\circ} (1, c_R^{\circ-1})) \\ (1, c_R) &= (1, -c_R^{\circ-1} \tilde{\circ} (1, c_R)), \end{aligned}$$

respectively. \square

Corollary 3.1: The group $(\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle, \circ, (1, 0))$ acts as a right transformation group on $\mathbb{R}\langle\langle X \rangle\rangle$ via the action $c \tilde{\circ} d_\delta$.

Proof: See Lemma 3.1 (2) and Lemma 3.3 (2). \blacksquare

IV. RELATIVE DEGREE AND GROUP INVARIANTS

The relationship between relative degree and invariants under the transformation group $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$ is described in this section. The following definition describes relative degree from a generating series point of view. It reduces to the usual definition in a state space setting [18]. It uses the notion of a *linear word*, that is, any word in the language

$$L = \{\eta \in X^* : \eta = x_0^{n_1} x_1 x_0^{n_0}, n_1, n_0 \geq 0\}.$$

Furthermore, note that every $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$.

Definition 4.1: [17] Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, let $r \geq 1$ be the largest integer such that $\text{supp}(c_F) \subseteq x_0^{r-1} X^*$. Then c has *relative degree* r if the linear word $x_0^{r-1} x_1 \in \text{supp}(c)$, otherwise it is not well defined.

Observe that c having relative degree r is equivalent to saying that

$$c = c_N + c_F = c_N + K x_0^{r-1} x_1 + x_0^{r-1} e \quad (5)$$

for some $K \neq 0$ and some $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \text{supp}(e)$.

The main result of this section is given next.

Theorem 4.1: A series c has relative degree r if and only if it is on the orbit of $c_N + x_0^{r-1} x_1$ under $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$.

Proof: If c has well defined relative degree r then it can be decomposed as in (5), where without loss of generality $e = x_0 e_0 + x_1 e_1$ with e_1 proper. Then

$$\begin{aligned} c &= c_N + x_0^{r-1} x_1 (K + e_1) + x_0^r e_0 = c_N + \phi_e(x_0^{r-1} x_1) (1) \\ &= (c_N + x_0^{r-1} x_1) \tilde{\circ} e_\delta, \end{aligned}$$

where $e_\delta := (K + e_1, e_0) \in \mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$ since $K + e_1$ is not proper. In which case, $c \tilde{\circ} e_\delta^{-1} = c_N + x_0^{r-1}x_1$, or equivalently, c is on the orbit of $c_N + x_0^{r-1}x_1$ under $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$. The converse holds since all the steps above are reversible. ■

Another consequence of relative degree is given below.

Theorem 4.2: The transformation group $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$ acts freely on the subset of $\mathbb{R}\langle\langle X \rangle\rangle$ having well defined relative degree.

Proof: Assume c has relative degree r . Without loss of generality let $c_N = 0$. Then there exists an $e_\delta \in \mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$ such that $c \tilde{\circ} e_\delta^{-1} = x_0^{r-1}x_1$. So if $c \tilde{\circ} d_\delta = c$ for some $d_\delta \in \mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$, then it follows immediately that

$$\begin{aligned} (c \tilde{\circ} d_\delta) \tilde{\circ} e_\delta^{-1} &= c \tilde{\circ} e_\delta^{-1} \\ (c \tilde{\circ} e_\delta^{-1}) \tilde{\circ} d_\delta^e &= c \tilde{\circ} e_\delta^{-1}, \end{aligned}$$

where d_δ^e corresponds to the conjugate action $e_\delta \circ d_\delta \circ e_\delta^{-1}$. In which case,

$$\begin{aligned} x_0^{r-1}x_1 \tilde{\circ} d_\delta^e &= x_0^{r-1}x_1 \\ x_0^{r-1}x_1 d_L^e + x_0^r d_R^e &= x_0^{r-1}x_1, \end{aligned}$$

and therefore, $d_\delta^e := (d_L^e, d_R^e) = (1, 0)$, the identity element of $\mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$. Thus, $e_\delta \circ d_\delta \circ e_\delta^{-1} = (1, 0)$, which gives the desired conclusion that $d_\delta = (1, 0)$. ■

Theorem 4.1 is a generalization of the well known result stating that the relative degree of a finite dimensional control-affine state space realization is invariant under static state feedback [18]. To see the exact connection, recall that if $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has finite Lie rank, then $F_c : u \mapsto y$ is realizable by a smooth state space realization

$$\dot{z} = g_0(z) + g_1(z)u, \quad z(0) = z_0, \quad y = h(z),$$

where $(c, \eta) = L_{g_\eta}h(z_0)$, and

$$L_{g_\eta}h := L_{g_{i_1}} \cdots L_{g_{i_k}}h, \quad \eta = x_{i_k} \cdots x_{i_1},$$

with $L_{g_i}h$ denoting the Lie derivative of h with respect to g_i [8], [18]. If (f, g, h, z_0) has relative degree r in the classical sense, then the input-output system is put into the form $y^{(r)} = v$ by the state feedback law

$$u = \frac{v - L_f^r h(z)}{L_g L_f^{r-1} h(z)}.$$

If the solution to the state equation is written in the form $z = F_{c_z}[u]$ for some $c_z \in \mathbb{R}^n\langle\langle X \rangle\rangle$, then this feedback law is equivalent to

$$\begin{aligned} v &= u L_g L_f^{r-1} h(F_{c_z}[u]) + L_f^r h(F_{c_z}[u]) \\ &=: u F_{e_L}[u] + F_{e_R}[u] = F_{e_\delta}[u]. \end{aligned}$$

The relative degree assumption here, as above, ensures that e_L is not proper, thus $e_\delta \in \mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle$.

It follows directly from the proof of Theorem 4.1 that $u = F_{e_\delta^{-1}}[v]$ has the property

$$y = F_c[u] = F_c[F_{e_\delta^{-1}}[v]] = F_{c \tilde{\circ} e_\delta^{-1}}[v] = F_{c_N + x_0^{r-1}x_1}[v],$$

as expected.

V. CONCLUSIONS

A feedback transformation group for the class of nonlinear input-output systems that can be represented in terms of Chen-Fliess functional expansions was presented. It was then shown that this group acts freely on generating series having well defined relative degree, and, in particular, has invariants corresponding exactly to input-output maps of Brunovsky forms.

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