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Design of Reduced-order Observers for a Class of Nonlinear Sampled-data Systems

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***Abstract*— The design of semiglobal and practical discrete-time reduced-order observers for nonlinear sampled-data systems of strict-feedback-like form is given. Robustness analysis of the designed observer and the comparison to the two-step deadbeat reduced-order observer are also discussed. A numerical example is given to show an efficiency of the designed observer.**

I. INTRODUCTION

Most practical and modern control systems use digital computers to control continuous-time plants, due to a rapid development of digital technologies. Since most practical systems are nonlinear, it is important to consider nonlinear sampled-data systems from both theoretical and practical points of view. For linear systems, the sampled-data control theory has been widely discussed and many design methods of controllers have been proposed [2]. Recently, a framework to design controllers based on discrete-time approximate models has been proposed for nonlinear sampled-data systems ([5], [12], [13]). Several design methods of state feedback controllers, such as an emulation and controller redesigns of continuous-time controllers [8], an integrator backstepping for discrete-time nonlinear systems [11], and controller designs by receding horizon methods [9] have been proposed to guarantee the stability of nonlinear sampled-data systems.

In most practical situations, all states cannot be available from measurement and a use of state feedback controllers is not realistic. In this case, we usually design observers to estimate unmeasured states and combine the designed state feedback controllers and observers to construct output feedback controllers. For nonlinear sampled-data systems, a framework to design observers has been also given based on discrete-time approximate models and two approaches of the observer design have been discussed in the proposed framework [1]. Then the two-step deadbeat reduced-order observer for nonlinear sampled-data strict-feedback systems has been given in [6] and by using an orthogonality-preserving algorithm, the design of discrete-time observers for sampled-data spacecraft attitude determination has been discussed in [7]. In [3], we have given a simple design of semiglobal and practical reduced-order observers for the exact (discrete-time) model of nonlinear sampled-data strict-feedback systems by using the design method of reduced-order observers for linear systems [14] and the property of the Euler model of strict-feedback form. We have applied the proposed design to straight-line trajectory tracking control of sampled-data underactuated ships. By simulation and experimental results, we have shown the efficiency of the designed observers.

In this paper we first extend the results in [3] to nonlinear sampled-data systems of strict-feedback-like form. Then we discuss the robustness of the designed reduced-order observers against sampled observation noise. We also show that the two-step deadbeat reduced-order observer given in [6] is a special case of the designed observers and the designed observers are superior to the two-step deadbeat observer in terms of robustness. A numerical example is given to show an efficiency of the proposed observers.

Notation: Let $\|\cdot\|$ denote the Euclidean norm of a vector and an induced norm of a matrix. Let $\mathbf{D} = \{\lambda = a + ib : \sqrt{a^2+b^2} < 1\}$ and $\mathbf{B}_r = \{x \in \mathbf{R}^n : \|x\| \le r\}$. Let $\sigma(M)$, $\lambda_{\min}(M)$, and $\lambda_{\max}(M)$ be a set of eigenvalues, the minimum eigenvalue, and the maximum eigenvalue of a real-valued matrix M, respectively. A function $\alpha : \mathbf{R}_{\ge 0} \to \mathbf{R}_{\ge 0}$ is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function β: $\mathbf{R}_{\ge 0} \times \mathbf{R}_{\ge 0} \to \mathbf{R}_{\ge 0}$ is of class $\mathcal{KL}$ if for any fixed $t \ge 0$, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \ge 0$, $\beta(s, \cdot)$ is decreasing to zero as its argument tends to infinity [4]. For simplicity of expression, we write $f(x(\cdot), y(\cdot)) = f(x, y)(\cdot)$.

II. PROBLEM FORMULATION

Consider the nonlinear sampled-data system of the form

$$\begin{aligned} \dot{x} &= f_1(x,u) + g(x,u)z, \\ \dot{z} &= f_2(x,z,u), \\ y(k) &= x(kT) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^{n_x}$, $z \in \mathbf{R}^{n_z}$ are the continuous-time states, $u \in \mathbf{R}^m$ is the control input realized through a zero-order hold, $y \in \mathbf{R}^{n_x}$ is the sampled observation, and $T > 0$ is a sampling period. We assume that for each initial condition and each constant control, there exists a unique solution of (1) defined on some bounded interval $[0, s)$. For the system (1) we also assume

A1 The mappings f_1, f_2, g are smooth over the domain of interest, $f_1(0,0) = 0$, and $f_2(0,0,0) = 0$.

A2 The $m \times m$ matrix $\Phi(\cdot,\cdot) = g(\cdot,\cdot)^T g(\cdot,\cdot)$ is nonsingular and its inverse is bounded over the domain of interest.

Let $u(t) = u(kT) =: u(k)$ for any $t \in [kT, (k+1)T)$. Then the difference equations corresponding to the exact

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model and the Euler model of (1) are given by

$$\begin{aligned} x(k+1) &= F_{1T}^e(x(k), z(k), u(k)), \\ z(k+1) &= F_{2T}^e(x(k), z(k), u(k)), \\ y(k) &= x(k) \end{aligned} \tag{2}$$

and

$$\begin{aligned} x(k+1) &= F_{1T}^a(x(k), z(k), u(k)), \\ z(k+1) &= F_{2T}^a(x(k), z(k), u(k)), \\ y(k) &= x(k), \end{aligned} \tag{3}$$

respectively, where $(F_{1T}^e(x,z,u), F_{2T}^e(x,z,u))$ is the solution of (1) at time T starting at (x,z) with the constant input u, $F_{1T}^a(x,z,u) = x + T[f_1(x,u) + g(x,u)z]$, and $F_{2T}^a(x,z,u) = z + Tf_2(x,z,u)$. The exact model (2) is not analytically computable in general and that is why the Euler model (3) is considered. Here we assume that the sampling period is a design parameter and can be assigned arbitrarily.

Let $F_T^i(x,z,u) = [\, F_{1T}^i(x,z,u)^T \quad F_{2T}^i(x,z,u)^T \,]^T$ for $i = a,\, e$. By [5] and [12], it is well-known that $F_T^a(x,z,u)$ is one-step consistent with $F_T^e(x,z,u)$, i.e., for each compact set $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$, there exist $\gamma \in \mathcal{K}$ and $T^* > 0$ such that

$$\|F_T^e(x,z,u) - F_T^a(x,z,u)\| \leq T\gamma(T)$$

for all $(x,z,u) \in \Omega$ and $T \in (0, T^*)$.

Since $y(k) = x(k)$, we want to design reduced-order observers of the form

$$\hat{z}(k+1) = O_T(\hat{z}(k), y(k), y(k+1), u(k)) \tag{4}$$

that estimate $z(k)$ of the exact model (2). Here we must use the Euler model (3) instead of the exact model (2) to design reduced-order observers (4). We now introduce the following property for reduced-order observers (4).

Definition 2.1: [1] We say that reduced-order observers (4) are semiglobal and practical in T if there exists $\beta \in \mathcal{KL}$ such that for any $D > d > 0$ and compact sets $\mathcal{X} \subset \mathbf{R}^{n_x}$, $\mathcal{Z} \subset \mathbf{R}^{n_z}$, $\mathcal{U} \subset \mathbf{R}^m$, we can find $T^* > 0$ with the property that $\|z(0) - \hat{z}(0)\| \leq D$ and $x(k) \in \mathcal{X}$, $z(k) \in \mathcal{Z}$, $u(k) \in \mathcal{U}$ for any $k \geq 0$ imply

$$\|z(k) - \hat{z}(k)\| \leq \beta(\|z(0) - \hat{z}(0)\|, kT) + d$$

for all $T \in (0, T^*)$.

III. Design of Reduced-order Observers

We first design reduced-order observers for the Euler model (3). Then we show that the designed observers are semiglobal and practical in T for the exact model (2). Following the design method of reduced-order observers for linear discrete-time systems [14], we introduce a change of state variables $\bar{z}(k) = J(k)x(k) + z(k)$ for the Euler model (3) where J is an $n_z \times n_x$ matrix that corresponds to an observer gain and will be assigned later. Then we obtain

$$\begin{aligned} \bar{z}(k+1) &= M_T(x,z,J,\rho J,u)(k) \\ &\quad + [I + TJ(k+1)g(x,u)(k)]\bar{z}(k), \\ z(k) &= \bar{z}(k) - J(k)x(k) \end{aligned} \tag{5}$$

where $M_T(x,z,J,\rho J,u) = (\rho J - J)x + T\{\rho J[f_1(x,u) - g(x,u)Jx] + f_2(x,z,u)\}$ and ρ denotes the shift, i.e., $(\rho J)(k) = J(k+1)$. At time $k+1$, we can obtain $z(k)$ exactly from (3) by

$$\begin{aligned} z(k) &= \Phi(y,u)^{-1} g(y,u)^T \left\{ \frac{\rho y - y}{T} - f_1(y,u) \right\}(k) \\ &=: \Psi_T(y, \rho y, u)(k). \end{aligned} \tag{6}$$

By (5) and (6) we consider

$$\begin{aligned} s(k+1) &= \bar{M}_T(y,\rho y,J,\rho J,u)(k) \\ &\quad + [I + TJ(k+1)g(y,u)(k)]s(k), \\ \hat{z}(k) &= s(k) - J(k)y(k) \end{aligned} \tag{7}$$

as a candidate of reduced-order observers where $\bar{M}_T(y,\rho y,J,\rho J,u) = (\rho J - J)y + T\{\rho J[f_1(y,u) - g(y,u)Jy] + f_2(y, \Psi_T(y,\rho y,u), u)\}$. Let $e = z - \hat{z}$. Then by direct calculation, we obtain

$$e(k+1) = [I + TJ(k+1)g(y,u)(k)]e(k).$$

For given $\hat{T} > 0$ and $k \geq 0$, let

$$J(k+1) = -H\Phi(y,u)^{-1}(k)g(y,u)^T(k)$$

where $H = \mathrm{diag}\{h_1, ..., h_{n_z}\}$. Then we have $I + TJ(k+1)g(y,u)(k) = I - TH$ for any $k \geq 0$ and the system (7) can be rewritten as

$$\hat{z}(k+1) = (I - TH)\hat{z}(k) + TN_T(y(k), y(k+1), u(k)) \tag{8}$$

where $N_T(y,\rho y,u) = H\Psi_T(y,\rho y,u) + f_2(y, \Psi_T(y,\rho y,u), u)$. Now we assume

A3 $|1 - Th_i| < 1$, $i = 1, .., n_z$ for any $T \in (0, \hat{T})$.

Note that $0 < h_i < 2/\hat{T}$, $i = 1, .., n_z$ satisfy the assumption **A3** and in this case $\sigma(I - TH) \subset \mathbf{D}$ for any $T \in (0, \hat{T})$. Summing up we have the following result.

Lemma 3.1: Let $\hat{T} > 0$ be given and assume **A1**-**A3**. Then the system (8) is the reduced-order observer of the Euler model (3) for any $T \in (0, \hat{T})$, i.e., the system (8) satisfies $z(k) - \hat{z}(k) \to 0$ as $k \to \infty$ for the Euler model (3).

Then we have the following result.

Theorem 3.1: Assume **A1**-**A3**. Then the reduced-order observer (8) is semiglobal and practical in T for the exact model (2).

Remark 3.1: In [6], the two-step deadbeat reduced-order observer for the system (1) with $f_1(x,u) = 0$, $g(x,u) = g_1(x)$, and $f_2(x,z,u) = \hat{f}_2(x,z) + g_2(x)u$ is given. The two-step deadbeat reduced-order observer for the system (1) is given by

$$\begin{aligned} \hat{z}(k+1) &= \Psi_T(y,\rho y,u)(k) \\ &\quad + Tf_2(y, \Psi_T(y,\rho y,u), u)(k) \\ &=: O_T^{2d}(y,\rho y,u)(k). \end{aligned} \tag{9}$$

Note that if we set $H = \frac{1}{T}I$, then the observer (8) coincides with the observer (9) and hence the observer (9) is a special case of the observer (8). Let $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$ be a compact set. Then we have

$$F_{2T}^a(y,z,u) - O_T^{2d}(y, F_{1T}^a(y,z,u), u) = 0$$

and

$$\begin{aligned}\|F^e_{2T}(y,z,u) - O^{2d}_T(y, F^e_{1T}(y,z,u),u)\| \\ \le [T + l_1 l_2(1+TL_2)]\gamma(T)\end{aligned}$$

for any $(y,z,u) \in \Omega$ where l_1, l_2, and etc are given in Section III-A. Hence the observer (9) is semiglobal and practical in T for the exact model (2).

A. Proof of Theorem 3.1

Let $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$ be a given compact set. By **A1** and **A2**, first note that there exist positive real numbers L_1, L_2, L_3, L_4, l_1, and l_2 such that $\|f_1(y,u) - f_1(\bar y, \bar u)\| \le L_1(\|y-\bar y\| + \|u-\bar u\|)$, $\|f_2(y,z,u) - f_2(\bar y,\bar z,\bar u)\| \le L_2(\|y-\bar y\| + \|z-\bar z\| + \|u-\bar u\|)$, $\|g(y,u) - g(\bar y,\bar u)\| \le L_3(\|y-\bar y\| + \|u-\bar u\|)$, $\|\Phi(y,u)^{-1} - \Phi(\bar y,\bar u)^{-1}\| \le L_4(\|y-\bar y\| + \|u-\bar u\|)$, $\|g(y,u)\| \le l_1$, and $\|\Phi(y,u)^{-1}\| \le l_2$ for any (y,z,u), $(\bar y,\bar z,\bar u) \in \Omega$.

By Remark 3.1, if $H = \frac{1}{T}I$, then the proof of Theorem 3.1 is obvious. Hence we assume $h_i \neq \frac{1}{T}$, $i = 1,..,n_z$ in the rest of this section.

Let $\hat T > 0$ be given and assume $\sigma(I - TH) \subset \mathbf{D}$ for any $T \in (0, \hat T)$. Then there exists $P_T > 0$ satisfying

$$(I-TH)^T P_T (I - TH) - P_T = -TcI \tag{10}$$

for any $c > 0$ and a fixed $T \in (0, \hat T)$. Then by direct calculation we have

$$P_T = \text{diag}\left\{\frac{c}{h_1(2-Th_1)}, ..., \frac{c}{h_{n_z}(2-Th_{n_z})}\right\}$$

and

$$\begin{aligned}\lambda_{min}(P_T) &= \min_{i=1,..,n_z} \frac{c}{h_i(2-Th_i)} \ge cq_1, \\ \lambda_{max}(P_T) &= \max_{i=1,..,n_z} \frac{c}{h_i(2-Th_i)} \le cq_2\end{aligned}$$

where $q_1 = (2h_{max})^{-1}$, $q_2 = (h_{min}(2 - \hat T h_{max}))^{-1}$, $h_{min} = \min_{i=1,..,n_z} h_i$, and $h_{max} = \max_{i=1,..,n_z} h_i$. Let $O_T(\hat z, y, \rho y, u) := (I - TH)\hat z + TN_T(y, \rho y, u)$ and $V_T(z,\hat z) = e^T P_T e$ where $e = z - \hat z$. Then we have

$$\alpha_1(\|e\|) \le V_T(z,\hat z) \le \alpha_2(\|e\|), \tag{11}$$

$$\begin{aligned}V_T(F^a_{2T}(y,z,u), O_T(\hat z, y, F^a_{1T}(y,z,u),u)) \\ -V_T(z,\hat z) = -Tc\alpha_3(\|e\|)\end{aligned} \tag{12}$$

where $\alpha_1(s) = cq_1 s^2$, $\alpha_2(s) = cq_2 s^2$, and $\alpha_3(s) = s^2$. Let $\mathcal{Z}$, $\hat{\mathcal{Z}} \subset \mathbf{R}^{n_z}$ be compact sets. Then we also have

$$|V_T(z_1,\hat z_1) - V_T(z_2,\hat z_2)| \le M_1(\|z_1 - z_2\| + \|\hat z_1 - \hat z_2\|) \tag{13}$$

for any z_1, $z_2 \in \mathcal{Z}$ and $\hat z_1$, $\hat z_2 \in \hat{\mathcal{Z}}$ where $M_1 = 2cq_2\hat M_1$ and $\hat M_1 = \sup_{z\in\mathcal{Z}} \|z\| + \sup_{z\in\hat{\mathcal{Z}}}\|z\|$.

We first introduce the following result. A proof is given in Appendix A.

Lemma 3.2: Assume **A1**-**A3**. For any positive real numbers $(\Delta_x, \Delta_z, \Delta_{\hat z}, \Delta_u, \nu)$, there exists $T^* > 0$ such that

$$\begin{aligned}V_T(F^e_{2T}(y,z,u), O_T(\hat z, y, F^e_{1T}(y,z,u),u)) \\ -V_T(z,\hat z) \le -T\alpha_3(\|e\|) + T\nu\end{aligned} \tag{14}$$

for all y, z, $\hat z$, u, and T satisfying $\|y\| \le \Delta_x$, $\|z\| \le \Delta_z$, $\|\hat z\| \le \Delta_{\hat z}$, $\|u\| \le \Delta_u$, and $T \in (0, T^*)$.

Proof of Theorem 3.1: Let x and z be the states of the exact model (2) and let $\mathcal{X}$, $\mathcal{Z}$, and $\mathcal{U}$ give a compact set $\Omega = \mathcal{X} \times \mathcal{Z} \times \mathcal{U}$ that guarantees the one-step consistency between F^e_T and F^a_T. Let $0 < r < R$ be given. Then we first show that if $r \le V_T(z,\hat z)(k) \le R$, then there exists $T^*_1 > 0$ such that

$$\Delta V_T(k) \le -\frac{1}{2}T\alpha_3(\|e(k)\|) \tag{15}$$

for any $T \in (0, T^*_1)$ where $\Delta V_T(k) = V_T(z,\hat z)(k+1) - V_T(z,\hat z)(k)$. To see this, let $(\Delta_x, \Delta_z, \Delta_{\hat z}, \Delta_u, \nu)$ be real numbers satisfying

$$\begin{aligned}&\Delta_x \ge \sup_{x\in\mathcal{X}}\|x\|,\ \Delta_u \ge \sup_{u\in\mathcal{U}}\|u\|,\ \Delta_z \ge \sup_{z\in\mathcal{Z}}\|z\|, \\ &\Delta_{\hat z} \ge \sup_{z\in\mathcal{Z}}\|z\| + \alpha_1^{-1}(R),\ \nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)).\end{aligned} \tag{16}$$

By (11) and $V_T(z,\hat z)(k) \le R$, we have

$$\begin{aligned}\|\hat z(k)\| &\le \|z(k)\| + \alpha_1^{-1}(V_T(z,\hat z)(k)) \\ &\le \|z(k)\| + \alpha_1^{-1}(R) \le \Delta_{\hat z}.\end{aligned}$$

Hence if we choose $T^*_1 > 0$ from Lemma 3.2, we have $\Delta V_T(k) \le -T\alpha_3(\|e(k)\|) + T\nu$ for any $T \in (0, T^*_1)$. Moreover $r \le V_T(z,\hat z)(k)$ and (11) imply

$$\|e(k)\| \ge \alpha_2^{-1}(V_T(z,\hat z)(k)) \ge \alpha_2^{-1}(r).$$

By the choice of ν in (16), we have

$$\nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)) \le \frac{1}{2}\alpha_3(\|e(k)\|)$$

and hence (15).

Note that if $V_T(z,\hat z)(k) \le r$, then by (14), we have $V_T(z,\hat z)(k+1) \le r + T\nu$. Let $T > 0$ be such that $r + T\nu \le R$. Then from the proof of Theorem 1 in [1], we can show that $V_T(z,\hat z)(0) \le R$ implies

$$V_T(z,\hat z)(k) \le \max\{\beta_1(V_T(z,\hat z)(0), kT), r + T\nu\} \tag{17}$$

for some $\beta_1 \in \mathcal{KL}$.

If $\|e(0)\| \le \alpha_2^{-1}(R)$, then $V_T(z,\hat z)(0) \le R$. By (11) and (17), we have

$$\begin{aligned}\|e(k)\| &\le \alpha_1^{-1}(\beta_1(V_T(z,\hat z)(0), kT) + r + T\nu) \\ &\le \beta(\|e(0)\|, kT) + \alpha_1^{-1}(2(r + T\nu))\end{aligned}$$

where $\beta(s,t) = \alpha_1^{-1}(2\beta_1(\alpha_2(s), t)) \in \mathcal{KL}$. For any $D > d > 0$, let $R = \alpha_2(D)$, $r = \frac{1}{4}\alpha_1(d)$ and choose $0 < T^* \le T^*_1$ such that $\nu T^* \le \frac{1}{4}\alpha_1(d)$. Then we have

$$\begin{aligned}\|e(0)\| &\le \alpha_2^{-1}(R) = D, \\ \alpha_1^{-1}(2(r + T\nu)) &\le \alpha_1^{-1}(2(r + T^*\nu)) \le d.\end{aligned}$$

Hence we have the assertion. ∎

Remark 3.2: For the case of $H = \frac{1}{T}I$, we can show that $V_T(z,\hat z) = c\hat T\|e\|^2$ satisfies (11)-(13) with $\alpha_1(s) = \alpha_2(s) = c\hat T s^2$ and $M_1 = 2c\hat T\hat M_1$. Hence Theorem 3.1 for $H = \frac{1}{T}I$ can be also proved in a same manner.

IV. Robustness Analysis

For the sampled-data system (1) and the models (2) and (3), we replace $y(k)$ by the sampled observation with noise

$$y_n(k) = y(k) + n(k)$$

and we consider robustness of the observer (8) against sampled observation noise. We introduce the following assumption.

A4 For given positive real numbers (Δ_n, Δ_w), n belongs to $\Xi(\Delta_n, \Delta_w) = \{n \in l_\infty | \|n\|_\infty \le \Delta_n$ and $\|w_T\|_\infty \le \Delta_w$ for any $T > 0\}$ where $w_T(k+1) = \frac{1}{T}[n(k+1) - n(k)]$ for any $k \ge 0$.

If $n(k)$ is a sampled signal of a bounded differentiable continuous-time signal $n(t)$, i.e., $n(k) := n(kT)$, then **A4** is satisfied. Let x and z be the states of the Euler model (3) and let $\hat{z}_n$ be the state of the observer (8) with y replaced by y_n. Then we have

$$e(k+1) = (I - TH)e(k) - Tv(y, \rho y, n, \rho n, u)(k)$$

where $e = z - \hat{z}_n$ and $v(y, \rho y, n, \rho n, u) = N_T(y_n, \rho y_n, u) - N_T(y, \rho y, u)$. Then we have the following result.

Theorem 4.1: Assume **A1-A4** with $h_i \neq \frac{1}{T}$, $i = 1, .., n_z$. Then the reduced-order observer (8) achieves semiglobal practical input-to-state stable convergence for the exact model (2) with y replaced by y_n, i.e., there exist $\beta \in \mathcal{KL}$ and γ_n, $\gamma_w \in \mathcal{K}$ such that for any $D > d > 0$, positive real numbers (Δ_n, Δ_w), and compact sets $\mathcal{X} \subset \mathbf{R}^{n_x}$, $\mathcal{Z} \subset \mathbf{R}^{n_z}$, $\mathcal{U} \subset \mathbf{R}^m$, we can find $T^* > 0$ such that $\|z(0) - \hat{z}_n(0)\| \le D$, $x(k) \in \mathcal{X}$, $z(k) \in \mathcal{Z}$, $u(k) \in \mathcal{U}$ for any $k \ge 0$, and $n \in \Xi(\Delta_n, \Delta_w)$ imply

$$\begin{aligned}\|z(k) - \hat{z}_n(k)\| \;\; &\le \;\; \beta(\|z(0) - \hat{z}_n(0)\|, kT) + d \\ &\quad + \gamma_n(\|n(k)\|) + \gamma_w(\|\rho w_T(k)\|)\end{aligned}$$

for all $T \in (0, T^*)$ where z is the state of the exact model (2).

Remark 4.1: 1) In [15], the design of quasi-ISS observers for general nonlinear continuous-time systems with the measurement corrupted by a disturbance is considered. Classes of systems that admit quasi-ISS reduced-order observers and an application to quantized output feedback control are also given.

2) SP-ISS property of nonlinear sampled-data systems and parameterized discrete-time systems and its related topics are given in [10].

A. Proof of Theorem 4.1

For given positive real numbers $(\Delta_x, \Delta_z, \Delta_{\hat{z}}, \Delta_u, \Delta_n, \Delta_w)$, let $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$. Let x and z be the states of the Euler model (3) with y replaced by y_n and let $V_T(z, \hat{z}_n) = e^T P_T e$ where $e = z - \hat{z}_n$ and $P_T > 0$ is the solution of (10) with $c = 2$. Then $V_T(z, \hat{z}_n)$ satisfies (11), (13), and

$$\begin{aligned}& V_T(F^a_{2T}, O^a_T) - V_T(z, \hat{z}_n) \\ \le \;\; & -T\alpha_3(\|e\|) + Tv^T P_T(TI + P_T)v \\ & -Te^T[I - (I - T\Lambda)^T(I - T\Lambda)]e \\ \le \;\; & -T\alpha_3(\|e\|) + Tb_0(\hat{T})|v|^2\end{aligned}$$

for any $(y, z, u) \in \Omega$, $\hat{z}_n \in \mathbf{B}_{\Delta_{\hat{z}}}$, $n \in \Xi(\Delta_n, \Delta_w)$, and $T \in (0, \hat{T})$ where $O^i_T = O_T(\hat{z}_n, y_n, F^i_{1T}(y, z, u) + \rho n, u)$ for $i = e, a$, $b_0(T) = 2q_2(2q_2 + T)$, and $v = v(y, \rho y, n, \rho n, u)$. By **A1** and **A2**, we have $\|f_1(y_n, u) - f_1(y, u)\| \le L_1\|n\|$, $\|f_2(y_n, \Psi_T(y_n, \rho y_n, u), u) - f_2(y, \Psi_T(y, \rho y, u), u)\| \le L_2(\|n\| + \|\Psi_T(y_n, \rho y_n, u) - \Psi_T(y, \rho y, u)\|)$, $\|g(y_n, u) - g(y, u)\| \le L_3\|n\|$, $\|\Phi(y_n, u)^{-1} - \Phi(y, u)^{-1}\| \le L_4\|n\|$, $\|g(y_n, u)\|$, $\|g(y, u)\| \le l_1$, and $\|\Phi(y_n, u)^{-1}\|$, $\|\Phi(y, u)^{-1}\| \le l_2$ for any $(y, z, u) \in \Omega$ and $n \in \Xi(\Delta_n, \Delta_w)$. Then by direct calculation, we have

$$\begin{aligned}&\|\Psi_T(y_n, F^a_{1T} + \rho n, u) - \Psi_T(y, F^a_{1T}, u)\| \\ &\quad \le l_1 l_2\|\rho w_T\| + l_1[l_2 L_1 + (l_2 L_3 + l_1 L_4)\|z\|]\|n\|,\end{aligned}$$

$$\begin{aligned}&\|f_2(y_n, \Psi_T(y_n, F^a_{1T} + \rho n, u), u) - f_2(y, \Psi_T(y, F^a_{1T}, u), u)\| \\ &\le L_2\{l_1 l_2\|\rho w_T\| + [1 + l_1(l_2 L_1 + (l_2 L_3 + l_1 L_4))\|z\|]\|n\|\},\end{aligned}$$

and $\|v\| \le a_1\|\rho w_T\| + a_2(\|z\|)\|n\|$ where $a_1 = l_1 l_2(L_2 + h_{max})$ and $a_2(\|z\|) = L_2 + a_1 L_1 + l_1(L_2 + h_{max})(l_2 L_3 + l_1 L_4)\|z\|$. Hence we obtain

$$\begin{aligned}V_T(F^a_{2T}, O^a_T) - V_T(z, \hat{z}_n) &\le -T\alpha_3(\|e\|) \\ &\quad + T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)] \qquad (18)\end{aligned}$$

where $\gamma_n(s) = 2b_0(\hat{T})a_2^2 s^2$, $a_2 := a_2(\Delta_z)$, and $\gamma_w(s) = 2b_0(\hat{T})a_1^2 s^2$. To prove Theorem 4.1, we introduce the following result. A proof is given in Appendix.

Lemma 4.1: Assume **A1-A4**. For any positive real numbers $(\Delta_x, \Delta_z, \Delta_{\hat{z}}, \Delta_u, \Delta_n, \Delta_w, \nu)$, there exists $T^* > 0$ such that

$$\begin{aligned}& V_T(F^e_{2T}(y, z, u), O_T(\hat{z}_n, y_n, F^e_{1T}(y, z, u) + \rho n, u)) \\ &\qquad\qquad - V_T(z, \hat{z}_n) \\ \le \;\; & -T\alpha_3(\|e\|) + T\nu + T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)] \quad (19)\end{aligned}$$

for all y, z, $\hat{z}_n$, u, n, and T satisfying $\|y\| \le \Delta_x$, $\|z\| \le \Delta_z$, $\|\hat{z}_n\| \le \Delta_{\hat{z}}$, $\|u\| \le \Delta_u$, $n \in \Xi(\Delta_n, \Delta_w)$, and $T \in (0, T^*)$.

Proof of Theorem 4.1: Let x and z be the states of the exact model (2) with y replaced by y_n and let $\mathcal{X}$, $\mathcal{Z}$, and $\mathcal{U}$ give a compact set $\Omega = \mathcal{X} \times \mathcal{Z} \times \mathcal{U}$ that guarantees the one-step consistency between F^e_T and F^a_T. Let $\tilde{T} > 0$, Δ_n, $\Delta_w > 0$, and $0 < r < R$ be given. Then we first show that if $r \le V_T(z, \hat{z})(k) \le R + \tilde{T}[\gamma_n(\Delta_n) + \gamma_w(\Delta_w)] =: \tilde{R}$, then there exists $0 < T^*_1 \le \tilde{T}$ such that

$$\begin{aligned}\Delta V_T(k) \;\; &\le \;\; -\frac{1}{2}T\alpha_3(\|e(k)\|) \\ &\quad + T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)](k) \qquad (20)\end{aligned}$$

for any $T \in (0, T^*_1)$ and $n \in \Xi(\Delta_n, \Delta_w)$ where $\Delta V_T(k) = V_T(z, \hat{z}_n)(k+1) - V_T(z, \hat{z}_n)(k)$. To see this, let $(\Delta_x, \Delta_z, \Delta_{\hat{z}}, \Delta_u, \nu)$ be real numbers satisfying

$$\begin{aligned}&\Delta_x \ge \sup_{x \in \mathcal{X}} \|x\|, \;\; \Delta_u \ge \sup_{u \in \mathcal{U}} \|u\|, \;\; \Delta_z \ge \sup_{z \in \mathcal{Z}} \|z\|, \\ &\Delta_{\hat{z}} \ge \sup_{z \in \mathcal{Z}} \|z\| + \alpha_1^{-1}(\tilde{R}), \;\; \nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)). \qquad (21)\end{aligned}$$

By (11) and $V_T(z, \hat{z}_n)(k) \le \tilde{R}$, we have

$$\begin{aligned}\|\hat{z}(k)\| \;\; &\le \;\; \|z(k)\| + \alpha_1^{-1}(V_T(z, \hat{z}_n)(k)) \\ &\le \;\; \|z(k)\| + \alpha_1^{-1}(\tilde{R}) \le \Delta_{\hat{z}}.\end{aligned}$$

Hence if we choose $T_1^* > 0$ from Lemma 4.1, we have

$$\begin{aligned}\Delta V_T(k) &\le -T\alpha_3(\|e(k)\|) + T\nu \\ &\quad +T[\gamma_n(\|n(k)\|) + \gamma_w(\|\rho w_T(k)\|)]\end{aligned}$$

for any $T \in (0, T_1^*)$ and $n \in \Xi(\Delta_n, \Delta_w)$. Moreover $r \le V_T(z, \hat{z})(k)$ and (11) imply

$$\|e(k)\| \ge \alpha_2^{-1}(V_T(z, \hat{z}_n)(k)) \ge \alpha_2^{-1}(r).$$

Hence by the choice of ν in (21), we have

$$\nu \le \frac{1}{2}\alpha_3(\alpha_2^{-1}(r)) \le \frac{1}{2}\alpha_3(\|e(k)\|)$$

and hence (20).

Note that if $V_T(z, \hat{z})(k) \le r$, then by (19), we have

$$\begin{aligned}& V_T(z, \hat{z}_n)(k+1) \\ \le\ & r + T\nu + T[\gamma_n(\|n(k)\|) + \gamma_w(\|\rho w_T(k)\|)] \\ \le\ & r + T\nu + \tilde{T}[\gamma_n(\Delta_n) + \gamma_w(\Delta_w)].\end{aligned}$$

Let $T > 0$ be such that $r + T\nu \le R$. Now we shall show that $V_T(z, \hat{z})(0) \le \tilde{R}$, then

$$\begin{aligned}V_T(z, \hat{z}_n)(k) \le \max\{&\beta_1(V_T(z, \hat{z}_n)(0), kT), \\ &r + T\nu + T[\gamma_n(\|n(k)\|) + \gamma_w(\|\rho w_T(k)\|)]\} \quad (22)\end{aligned}$$

for some $\beta_1 \in \mathcal{KL}$. Let

$$\begin{aligned}s_T(t) = V_T(z, \hat{z}_n)(k) + (\frac{t}{T} - k)\{&V_T(\rho z, \rho\hat{z}_n) - V_T(z, \hat{z}_n) \\ &-T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)]\}(k)\end{aligned}$$

for any $t \in [kT, (k+1)T)$ and $k \ge 0$. Then we have

$$\begin{aligned}\dot{s}_T(t) &= \frac{V_T(\rho z, \rho\hat{z}_n)(k) - V_T(z, \hat{z}_n)(k)}{T} \\ &\quad -[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)](k)\end{aligned}$$

for any $t \in [kT, (k+1)T)$ and

$$s_T(kT) = V_T(z, \hat{z}_n)(k).$$

If $V_T(z, \hat{z}_n)(k) < r$, we have

$$\begin{aligned}V_T(\rho z, \rho\hat{z}_n)(k) &\le r + T\nu \\ &+T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)](k) \le \tilde{R}\end{aligned}$$

and if $V_T(z, \hat{z}_n)(k) \ge r$, we have

$$\begin{aligned}\dot{s}_T(t) &\le -\frac{1}{2}\alpha_3(\|e(k)\|) \le 0, \\ s_T(t) &\le V_T(z, \hat{z}_n)(k) = s_T(kT)\end{aligned}$$

for any $t \in [kT, (k+1)T)$. Then by (11), we have

$$\|e(k)\| \ge \alpha_2^{-1}(s_T(kT)) \ge \alpha_2^{-1}(s_T(t))$$

and hence

$$\dot{s}_T(t) \le -\frac{1}{2}\alpha_3(\alpha_2^{-1}(s_T(t)))$$

for any $t \in [kT, (k+1)T)$. Hence if $V_T(z, \hat{z}_n)(0) \le \tilde{R}$, we obtain (22).

If $\|e(0)\| \le \alpha_2^{-1}(R)$, then $V_T(z, \hat{z}_n)(0) \le R \le \tilde{R}$. By (11) and (22), we have

$$\begin{aligned}\|e(k)\| &\le \alpha_1^{-1}(\beta_1(V_T(z, \hat{z}_n)(0), kT) + r + T\nu \\ &\qquad +T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)](k)) \\ &\le \beta(\|e(0)\|, kT) + \alpha_1^{-1}(3(r + T\nu)) \\ &\qquad +\alpha_1^{-1}(3T[\gamma_n(\|n\|) + \gamma_w(\|\rho w_T\|)](k)) \\ &\le \beta(\|e(0)\|, kT) + \alpha_1^{-1}(3(r + T\nu)) \\ &\qquad +\alpha_1^{-1}(6T\gamma_n(\|n(k)\|)) \\ &\qquad +\alpha_1^{-1}(6T\gamma_w(\|\rho w_T(k)\|))\end{aligned}$$

where $\beta(s, t) = \alpha_1^{-1}(3\beta_1(\alpha_2(s), t)) \in \mathcal{KL}$.

For any $D > d > 0$, let $R = \alpha_2(D)$, $r = \frac{1}{6}\alpha_1(d)$ and choose $0 < T^* \le T_1^*$ such that $\nu T^* \le \frac{1}{6}\alpha_1(d)$. Then we have

$$\begin{aligned}\|e(0)\| &\le \alpha_2^{-1}(R) = D, \\ \alpha_1^{-1}(3(r + T\nu)) &\le \alpha_1^{-1}(3(r + T^*\nu)) \le d.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|e(k)\| &\le \beta(\|e(0)\|, kT) + d \\ &\quad +\tilde{\gamma}_n(\|n(k)\|)) + \tilde{\gamma}_w(\|\rho w_T(k)\|))\end{aligned}$$

where $\tilde{\gamma}_n(s) = \alpha_1^{-1}(6T^*\gamma_n(s))$ and $\tilde{\gamma}_w(s) = \alpha_1^{-1}(6T^*\gamma_w(s))$. ∎

Remark 4.2: From the proof of Theorem 4.1, if we can choose sufficiently small $T^* > 0$, then an influence of the sampled observation noise $n \in \Xi(\Delta_n, \Delta_w)$ to the estimation error becomes small. We show it numerically in Section V.

Remark 4.3: Consider the robustness of the observer (9) against sampled observation noises. Then the observer (9) is rewritten as

$$\hat{z}_n(k+1) = O_T^{2d}(y, \rho y, u)(k) + r(y, \rho y, n, \rho n, u)(k)$$

where $r(y, \rho y, n, \rho n, u) = \Psi_T(y_n, \rho y_n, u) - \Psi_T(y, \rho y, u) + T[f_2(y_n, \Psi_T(y_n, \rho y_n, u), u) - f_2(y, \Psi_T(y, \rho y, u), u)]$. Let x and z be the states of the exact model (2) and $e = z - \hat{z}_n$. Then by **A1**-**A4**, we have

$$\begin{aligned}\|e(k+1)\| \le\ & [T + l_1 l_2(1 + TL_2)]\gamma(T) + TL_2\|n\| \\ &+(1 + TL_2)\|\Psi_T(y_n, F_{1T}^e + \rho n, u) - \Psi_T(y, F_{1T}^e, u)\| \quad (23)\end{aligned}$$

where $F_{iT}^j = F_{iT}^j(y, z, u)$ for $i = 1, 2$ and $j = e, a$. By the Taylor series expansion, we have

$$\begin{aligned}F_{1T}^e &= F_{1T}^a + O(T^2), \\ \frac{1}{T}(F_{1T}^e - y) - f_1(y, u) &= g(y, u)z + O(T)\end{aligned}$$

and we obtain

$$\begin{aligned}\|\Psi_T(y_n, F_{1T}^e + \rho n, u) - \Psi_T(y, F_{1T}^e, u)\| &\le l_1 l_2\|\rho w_T\| \\ +l_1[l_2 L_1 + (l_2 L_3 + l_1 L_4)\|z\|)]\|n\| &+ O(T). \quad (24)\end{aligned}$$

Hence by (23) and (24), we have

$$\begin{aligned}\|e(k+1)\| \le\ & [T + l_1 l_2(1 + TL_2)]\gamma(T) \\ &+\{TL_2 + l_1(1 + TL_2) \\ &\quad \times[l_2 L_1 + (l_2 L_3 + l_1 L_4)\|z\|)]\}\|n(k)\| \\ &\quad +l_1 l_2(1 + TL_2)\|\rho w_T(k)\| + O(T) \quad (25)\end{aligned}$$

for any $(x(k), z(k), u(k)) \in \Omega$ and $n \in \Xi(\Delta_n, \Delta_w)$ where $\Omega \subset \mathbf{R}^{n_x} \times \mathbf{R}^{n_z} \times \mathbf{R}^m$ is a given compact set. The inequality (25) implies that a large estimation error may remain even for sufficiently small sampling periods. Hence the observer (8) would be more useful than the observer (9) from robustness against sampled observation noise. We show it numerically in Section V.

V. Numerical Examples

Consider the Van der Pol equation with sampled observation

$$\dot{x} = z, \quad \dot{z} = x + 0.5(1 - x^2)z, \quad y(k) = x(kT). \qquad (26)$$

Note that the above system is of strict feedback form and it satisfies **A1** and **A2**. Moreover, the trajectories of the Van der Pole equation are bounded. Then by Theorem 3.1, the observer (8) with $N_T(y, \rho y) = H\Psi_T(y, \rho y) + f_2(y, \Psi_T(y, \rho y))$ and $\Psi_T(y, \rho y) = (\rho y - y)/T$ is semiglobal and practical in T for the exact model of the system (26). By Remark 4.3, the two-step deadbeat observer (9) is also semiglobal and practical in T for the exact model. Next we consider the system (26) with $y(k)$ replaced by

$$y_n(k) = x(kT) + 0.2\sin(10kT).$$

Note that $0.2\sin(10kT) \in \Xi(0.2, 2)$ and by Theorem 4.1, the observer (8) is still semiglobal and practical in T for the exact model of the system (26) with y replaced by y_n. Let $\hat{z}$ and $\hat{z}_n$ be the states of the observer (8) and the observer (8) with y replaced by y_n, respectively, and let $\hat{z}_{2d}$ and $\hat{z}_{2dn}$ be the states of the observer (9) and the observer (9) with y replaced by y_n, respectively. Let $x(0) = 1$, $z(0) = 0.5$, $\hat{z}(0) = \hat{z}_n(0) = \hat{z}_{2d}(0) = \hat{z}_{2dn}(0) = 0$, and $H = 0.5$. Then Figures 1 and 2 show the simulation results of the time responses of $z(t)$, $\hat{z}(k)$, $\hat{z}_n(k)$, $\hat{z}_{2d}(k)$, and $\hat{z}_{2dn}(k)$ for $T = 0.2,\ 0.05$. As we see Figures 1 and 2, the estimation errors $z(k) - \hat{z}(k)$, $z(k) - \hat{z}_n(k)$, and $z(k) - \hat{z}_{2d}(k)$ become smaller as the sampling period T is reduced. But the estimation error $z(k) - \hat{z}_{2dn}(k)$ does not become small for sufficiently small sampling periods.

VI. Conclusion

In this paper we have discussed the simple design of semiglobal and practical discrete-time reduced-order observers for the exact model of the nonlinear sampled-data systems of strict-feedback-like form. Then we have considered the robustness against sampled observation noise. We have also shown that the two-step deadbeat reduced-order observer given in [6] is a special case of the designed observer and the designed observer is superior to the two-step deadbeat observer in terms of robustness. A numerical example has been given to show an efficiency of the proposed observer.

Appendix

Proof of Lemma 3.2: Let $(\Delta_x, \Delta_z, \Delta_{\hat{z}}, \Delta_u, \nu)$ be given and $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$. Let $T_1^* > 0$ be such that $F_T^e(x, z, u)$ is well-defined for all $(x, z, u) \in \Omega$ and $T \in$

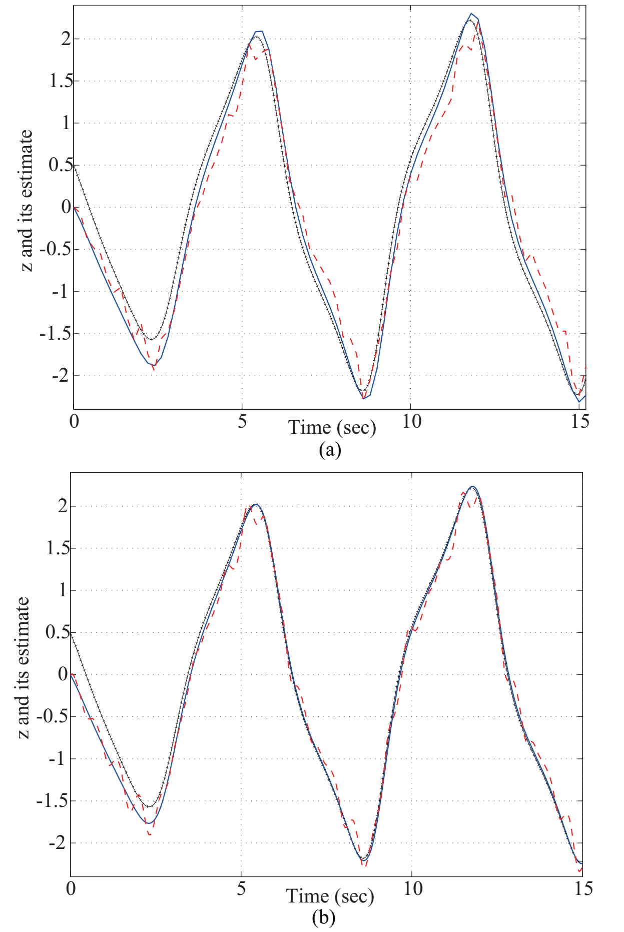

Fig. 1. Time responses of $z(t)$, $\hat{z}(k)$, and $\hat{z}_n(k)$. Blue lines: $\hat{z}$. Red broken lines: $\hat{z}_n$. Black dotted lines: z. (a) The observer (8) and $T = 0.2$ (sec). (b) The observer (8) and $T = 0.05$ (sec).

$(0, T_1^*)$. For simplicity of notation, let $F_T^i = F_T^i(y, z, u)$, $F_{jT}^i = F_{jT}^i(y, z, u)$, and $O_T^i = O_T(\hat{z}, y, F_{1T}^i, u)$ for $i = e, a$ and $j = 1, 2$. Then we can find $T_2^* > 0$ and $\gamma \in \mathcal{K}$ by the consistency between F_T^e and F_T^a. Let

$$\Delta_L = \sup_{(y,z,u)\in\Omega, \|\hat{z}\|\le\Delta_{\hat{z}}} \max\{\|F_T^e\|, \|O_T^e\|, \Delta_z, \Delta_{\hat{z}}\},$$

$\mathcal{Z} = \hat{\mathcal{Z}} = \mathbf{B}_{\Delta_L}$, and $\mathcal{U} = \mathbf{B}_{\Delta_u}$. Then there exist $M_1 > 0$ and $T_3^* > 0$ satisfying (11)-(13) with $c = 1$. Let $T_4^* > 0$ be such that

$$M_1[1 + l_1 l_2(h_{max} + L_2)]\gamma(T_4^*) \le \nu \qquad (27)$$

and $T^* = \min\{T_1^*, T_2^*, T_3^*, T_4^*\}$. Note that

$$\begin{aligned} V_T(F_{2T}^e, O_T^e) - V_T(z, \hat{z}) = {} & V_T(F_{2T}^a, O_T^a) - V_T(z, \hat{z}) \\ & + V_T(F_{2T}^e, O_T^e) - V_T(F_{2T}^a, O_T^a). \end{aligned}$$

By (11)-(13) and the consistency between F_T^e and F_T^a, we have

$$\begin{aligned} V_T(F_{2T}^a, O_T^a) - V_T(z, \hat{z}) &\le -T\alpha_3(\|e\|), \\ |V_T(F_{2T}^e, O_T^e) - V_T(F_{2T}^a, O_T^a)| &\le M_1[T\gamma(T) \\ &\qquad + \|O_T^e - O_T^a\|] \end{aligned}$$

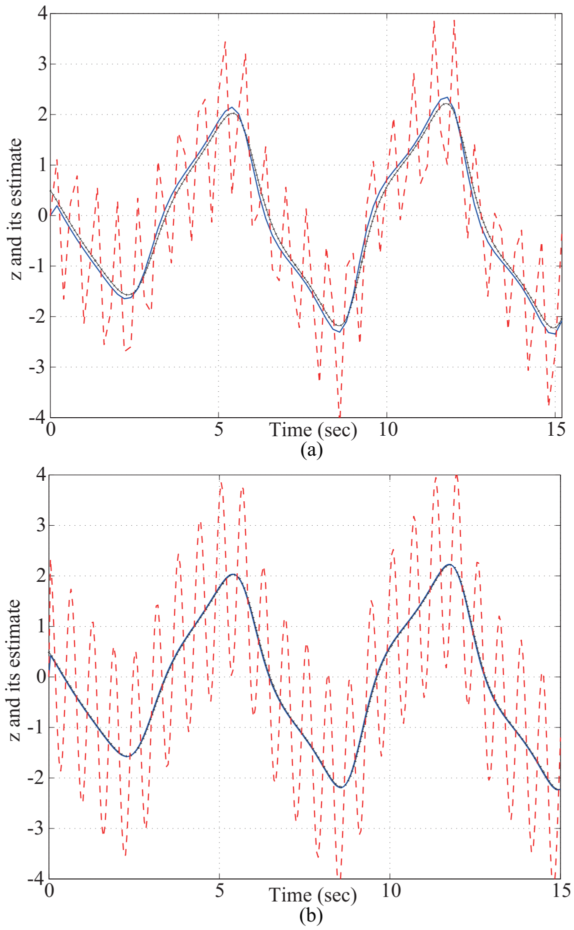

Fig. 2. Time responses of $z(t)$, $\hat{z}_{2d}(k)$, and $\hat{z}_{2dn}(k)$. Blue lines: $\hat{z}_{2d}$. Red broken lines: $\hat{z}_{2dn}$. Black dotted lines: z. (a) The observer (9) and $T = 0.2$ (sec). (b) The observer (9) and $T = 0.05$ (sec).

for any $(x, z, u) \in \Omega$ and $T \in (0, T^*)$. By direct calculation, we have

$$\begin{aligned} O_T^e - O_T^a &= H\Phi(y,u)^{-1} g(y,u)^T (F_{1T}^e - F_{1T}^a) \\ &\quad + T[f_2(y, \Psi_T(y, F_{1T}^e, u), u) \\ &\qquad - f_2(y, \Psi_T(y, F_{1T}^a, u), u)] \end{aligned}$$

and by the one-step consistency between F_T^e and F_T^a, we obtain

$$\begin{aligned} \|O_T^e - O_T^a\| &\leq (\|H\| + L_2)\|\Phi(y,u)^{-1}\|\|g(y,u)\| \\ &\quad \times \|F_{1T}^e - F_{1T}^a\| \\ &\leq T l_1 l_2 (h_{max} + L_2)\gamma(T^*). \end{aligned}$$

Hence we have

$$\begin{aligned} V_T(F_{2T}^e, O_T^e) - V_T(z, \hat{z}) &\leq -T\alpha_3(\|e\|) \\ &\quad + TM_1[1 + l_1 l_2 (h_{max} + L_2)]\gamma(T^*) \end{aligned}$$

and by (27), we obtain (14). ∎

Proof of Lemma 4.1: Let $(\Delta_x, \Delta_z, \Delta_{\hat{z}}, \Delta_u, \Delta_n, \Delta_w, \nu)$ be given and $\Omega = \mathbf{B}_{\Delta_x} \times \mathbf{B}_{\Delta_z} \times \mathbf{B}_{\Delta_u}$. Let T_1^*, $T_2^* > 0$, and $\gamma \in \mathcal{K}$ be given as in the proof of Lemma 3.2. Let

$$\Delta_L = \sup_{\substack{(y,z,u) \in \Omega, \\ \|\hat{z}_n\| \leq \Delta_{\hat{z}}, \\ n \in \Xi(\Delta_n, \Delta_w)}} \max\{\|F_T^e\|, \|O_T^e\|, \Delta_z, \Delta_{\hat{z}}\},$$

$\mathcal{Z} = \hat{\mathcal{Z}} = \mathbf{B}_{\Delta_L}$, and $\mathcal{U} = \mathbf{B}_{\Delta_u}$. Then there exist $M_1 > 0$ and $T_3^* > 0$ satisfying (11), (13) with $c = 2$, and (18). Let $T_4^* > 0$ be such that (27) is satisfied and $T^* = \min\{T_1^*, T_2^*, T_3^*, T_4^*\}$. Then similar to the proof of Lemma 3.2, we have

$$\begin{aligned} & V_T(F_{2T}^e, O_T^e) - V_T(z, \hat{z}_n) \\ \leq\ & -T\alpha_3(\|e\|) + T[\gamma_n(\|n\|) + \gamma_w(\|\rho w\|)] \\ & + TM_1[1 + l_1 l_2 (h_{max} + L_2)]\gamma(T^*) \end{aligned}$$

for any $(y, z, u) \in \Omega$, $\hat{z}_n \in \hat{\mathcal{Z}}$, $n \in \Xi(\Delta_n, \Delta_w)$, and $T \in (0, T^*)$. Hence by (27), we obtain (19). ∎

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