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On positive completions of matrices over real C^*-algebras

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***Abstract*— Fill in.**

I. MAIN RESULT

Positive (in the sense of positive semidefiniteness) completions for real and complex matrices have been extensively studied. They appear in particular in maximal entropy problems and completions of covariance matrices, see for example [2], [10], [3] and the bibliography there. These studies have been extended to positive completions of operator matrices (see [1] and references therein), and more generally to positive completions of matrices with entries in a (complex) C^*-algebra [9], [11].

In this paper we study positive completions in the context of matrices with entries in a *real* C^*-algebra. We adopt here the following definition of a real C^*-algebra. A real Banach algebra $\mathcal{A}$ with continuous involution * and unity 1 is called a C^*-algebra if $\|aa^*\| = \|a\|^2$ for all $a \in \mathcal{A}$ and $1 + aa^*$ is invertible for all $a \in \mathcal{A}$. See, for example, [8] for information on real C^*-algebras. An element a of a real C^*-algebra $\mathcal{A}$ is called *positive* if $a = b^*b$ for some $b \in \mathcal{A}$ and *strictly positive* if $a = b^*b$ for some invertible $b \in \mathcal{A}$. If $\mathcal{A}$ is a real C^*-algebra, we denote by $M_n(\mathcal{A})$ the real C^*-algebra of all $n \times n$ matrices with entries in $\mathcal{A}$.

All graphs Γ in this paper are assumed to be undirected, with finite vertex set $V(\Gamma)$, without multiple edges, and with all loops (x, x), $x \in V(\Gamma)$ contained in the set of edges $E(\Gamma)$. As Γ is undirected, we have $(x, y) \in E(\Gamma)$ if and only if $(y, x) \in E(\Gamma)$; here $x, y \in V(\Gamma)$. We label the vertices $V(\Gamma) = \{1, 2, \ldots, n\}$.

A *clique* of a graph Γ is a set of vertices V_0 such that $(x, y) \in E(\Gamma)$ for all $x, y \in V_0$. A graph Γ is said to *chordal*, or triangulated, if for every loop

$$(i_1, i_2), (i_2, i_3), \ldots, (i_{p-1}, i_p), (i_p, i_1) \in E(\Gamma)$$

of distinct vertices $i_1, \ldots, i_p$, $p \geq 4$, there is a chord $(i_j, i_k) \in E(\Gamma)$ for some indices $j, k \in \{1, 2, \ldots, p\}$ such that $1 < k - j < p - 1$.

We now state our main result; it is restricted to finite dimensional real C^*-algebras.

Theorem 1.1: Let G be a chordal graph, and let $\mathcal{A}$ be a finite dimensional real C^*-algebra. Then, for a given $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A})$ there exists $B = [b_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A})$ such that $b_{i,j} = 0$ for every edge $(i, j) \in E(\Gamma)$ and $A + B$ is positive, resp. strictly positive, if and only if for every clique V_0 of Γ the submatrix $[a_{i,j}]_{i,j \in V_0}$ is positive, resp. strictly positive.

Results of this nature are well known for the algebras $M_n(\mathcal{C})$ and $M_n(\mathcal{R})$, going back to [7], where it was proved for the strictly positive matrices and complex numbers. Theorem 1.1 holds also for the (complex) algebra of linear bounded operators on a (complex) Hilbert space (see [1] and references there), and more generally for von Neumann algebras [9]. However, the result of Theorem 1.1 is generally false for (complex) infinite dimensional C^*-algebras, even for commutative ones (the first counterexample was given in [4], and see [11] for a discussion on this problem).

The important special case of Toeplitz matrices deserves to be stated separately. We denote by $T(a_0, a_1, \ldots, a_{n-1}) \in M_n(\mathcal{A})$ the Hermitian Toeplitz matrix having a_j in the (k, ℓ)-positions and having a_j^* in the (ℓ, k)-positions, where $\ell - k = j \geq 0$.

Theorem 1.2: Let $\mathcal{A}$ be a finite dimensional real C^*-algebra, and let there be given $a_0, a_1, \ldots, a_k \in \mathcal{A}$, for some integer k, $0 \leq k \leq n-2$. Then there exist $a_{k+1}, \ldots, a_{n-1} \in \mathcal{A}$ such that $T(a_0, a_1, \ldots, a_{n-1})$ is positive, resp. strictly positive, if and only if $T(a_0, \ldots, a_k)$ is positive, resp. strictly positive.

Theorem 1.2 is essentially a particular case of Theorem 1.1.

It is an open question whether or not Theorem 1.2 is valid for infinite dimensional complex, as well as real, C^*-algebras. It was proved in [11] that Theorem 1.2 holds true for commutative complex unital C^*-algebras.

II. PROOF OF THEOREM 1.1

For the proof of the "if" part note that by the Wedderburn type theorem for finite dimensional real C^*-algebras, every such algebra is isometrically *-isomorphic to direct sum of full matrix algebras over reals, complexes, and quaternions $\mathcal{H}$ (Theorem 5.22 in [5]). We may assume therefore that $\mathcal{A}$ is equal to one of $M_n(\mathcal{R})$, $M_n(\mathcal{C})$, or $M_n(\mathcal{H})$. As the result is known for $M_n(\mathcal{R})$ and $M_n(\mathcal{C})$, it remains to consider the case $\mathcal{A} = M_n(\mathcal{H})$. A standard argument using the perfect elimination scheme property of chordal graphs (see, e.g. [6]) reduces the proof of the "if" part to the following statement.

Lemma 2.1: Let there be given $A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{H})$, and assume that the submatrices $[a_{i,j}]_{i,j=1}^{n-1}, [a_{i,j}]_{i,j=2}^n \in M_{n-1}(\mathcal{H})$ are positive, resp. strictly positive. Then there exists $x \in \mathcal{H}$ such that the matrix $A + X$ is positive, resp. positive definite; here $X \in M_n(\mathcal{H})$ has x in the $(1, n)$-position, x^* in the $(n, 1)$-position, and zeros everywhere else.

In turn, for the proof of Lemma 2.1 the following characterizations of positive and strictly positive matrices over quaternions will be useful. We use in the next lemma the

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inner product in $\mathcal{H}^n$ given by $\langle x,y\rangle = y^*x \in \mathcal{H}$, for $x,y\in\mathcal{H}^n$.

Lemma 2.2: The following statements are equivalent for a given $A\in M_n(\mathcal{H})$:

(1) $A=BB^*$ for some $B\in M_n(\mathcal{H})$;
(2) $A=BB^*$ for some $n\times m$ matrix B over $\mathcal{H}$, and some positive integer m;
(3) $\langle x,Ax\rangle = \langle Ax,x\rangle \geq 0$ for all $x\in\mathcal{H}^n$;
(4) the matrix A is Hermitian and all its eigenvalues are nonnegative (recall that $\lambda\in\mathcal{H}$ is an eigenvalue of A if $Ax=x\lambda$ holds for some nonzero $x\in\mathcal{H}^n$).
(5) there exists an upper triangular matrix $B\in M_n(\mathcal{H})$ with real and positive diagonal such that B^*AB is a diagonal matrix whose diagonal consists of zeros and ones;
(6) there exists a lower triangular matrix $B\in M_n(\mathcal{H})$ with real and positive diagonal such that B^*AB is a diagonal matrix whose diagonal consists of zeros and ones.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), as well as (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1), are obvious.

Assume (3) holds. Then elementary calculations (for each 2×2 principal submatrix of A) show that A is Hermitian. It is easy to see that all eigenvalues of A are nonnegative. Thus (4) follows. The implication (4) $\Rightarrow$ (1) is obtained by using the fact that every quaternion Hermitian matrix is unitarily similar to a real diagonal matrix with its eigenvalues on the diagonal.

We omit the proofs of (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6), and only remark that the following fact is used in the proofs.

Fact 2.3: If $A=[a_{i,j}]_{i,j=1}^n\in M_n(\mathcal{H})$ satisfies (4) (or equivalently (1), (2), or (3)) and $a_{1,1}=0$, then $a_{1,j}=a_{j,1}=0$ for $j=2,3,\ldots,n$.

Proof of the "if" part of Lemma 2.1. Assume first that the matrices $[a_{i,j}]_{i,j=1}^{n-1}, [a_{i,j}]_{i,j=2}^{n}\in M_{n-1}(\mathcal{H})$ are positive. Using statement (5) of Lemma 2.2, we may assume that $[a_{i,j}]_{i,j=1}^{n-1}$ is a diagonal matrix with zeros and ones in the main diagonal. If $a_{1,1}=0$, then take $x=0$. Assume therefore that $a_{1,1}=1$. For notational convenience suppose that $a_{j,j}=1$, $j=1,2,\ldots,r$, and $a_{j,j}=0$, $j=r+1,\ldots,n-1$, for some integer r, $1\leq r\leq n-1$. In view of Fact 2.3 the matrix $A+X$ has the following form:

$$A+X=\begin{bmatrix} 1 & 0 & 0 & x\\ 0 & I_{r-1} & 0 & A_0\\ 0 & 0 & 0 & 0\\ x^* & A_0^* & 0 & a_{n,n}\end{bmatrix},$$

for some $A_0\in\mathcal{H}^{r-1}$. Then $x=0$ will do.

If the matrices $[a_{i,j}]_{i,j=1}^{n-1}, [a_{i,j}]_{i,j=2}^{n}$ are strictly positive, then we have $r=n-1$, and a similar argument shows that we can take $x=0$ as well.

As for the "only if" part of Lemma 2.1, note that it is obtained immediately from Lemma 2.2.

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