

The centroid as an estimate for the quadratic min-power centre

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Abstract—Given a set of nodes in the plane, the min-power centre is a point that minimises the cost of the star centred at this point and spanning all nodes. The cost of the star is defined as the sum of the costs of its nodes, where the cost of a node is an increasing function of the length of its longest incident edge. The min-power centre problem provides a model for optimally locating a cluster-head amongst a set of radio transmitters. We provide upper bounds for the performance of the centroid (centre of mass) of the given nodes as an approximation to the quadratic min-power centre.

Keywords: wireless ad hoc networks; cluster-head location; minimum power range assignment; single facility location.

I. INTRODUCTION

A number of important issues arise in the design of wireless ad hoc radio networks, including communication, robustness and power minimisation. All these problems must be considered in both the physical design and the routing design phases [3], [2], [8], [9]. In this paper we consider power minimisation modelled as the *power efficient range assignment problem*, where a communication range r_i is assigned to each transmitter x_i such that the resultant network is connected and the total power $\sum r_i^\alpha$ is minimised (see [1]). The exponent α is called the *path loss exponent* and usually takes a value between 2 and 4, with $\alpha = 2$ corresponding to transmission in free-space.

We consider the problem of optimally locating a *single* cluster-head amongst a given set of transmitters, where each transmitter can send and receive data *directly* to and from the cluster-head. To solve the problem one needs to find the position of a cluster-head and assign ranges to all nodes such that total power is minimised. The graph induced by the assignment of ranges will be an undirected star with the cluster head as its centre and the complete set of transmitters as its leaves.

In more formal terms we denote the given finite set of transmitters by $X \subset \mathbb{R}^2$ and the cluster-head by $s \in \mathbb{R}^2$.

The *power* of any $x \in X$ is $P_x = \|s - x\|^\alpha$ and the *power* of s is $P_s = \max\{\|s - x\|^\alpha : x \in X\}$, where $\|\cdot\|$ is the Euclidean norm. The *total power* of the system is denoted by $P(s) = P_s + \sum_{x \in X} P_x$, and a *min-power centre* of X is a point s^* which minimises $P(s)$. The *min-power problem* seeks to locate a min-power centre of a given set X in the plane.

Single-facility location problems similar to the min-power problem, but under very general norms and convex cost functions, have been studied by Durier [5] and by Durier and Michelot [6]. The principal contribution of the research presented by Durier and Michelot in these papers involves a description of the set of solutions for generalised Fermat-Weber problems, which is a feasible and interesting endeavor when the objective cost function is not strictly convex, unlike the min-power problem. A geometric characterisation of the min-power centre when $\alpha = 2$ has been obtained by Brazil et al. in [4]. Durocher and Kirkpatrick studied the feasibility of approximating the Fermat-Weber point by the centroid and the projection median [7].

In this paper we present various upper bounds for the performance of the centroid (centre of mass) as an approximation to the min-power centre when $\alpha = 2$. This provides a significant improvement over the analogous result in [4]. The centroid is easy to calculate (it can be constructed in constant time given the locations of the nodes in X), which justifies it as an approximation point.

II. PRELIMINARIES

Let $X = \{x_i : i \in J\}$ be a given finite set of points in the Euclidean plane, for some index-set J , and let $\alpha > 1$ be a given real number. The function $P(s)$ can be written as

$$P(s) = \sum_{i \in J} \|s - x_i\|^\alpha + \max_{i \in J} \{\|s - x_i\|^\alpha\}.$$

It follows that $P(s)$ is strictly convex, and therefore the min-power centre s^* is unique. Although $P(s)$ is not smooth, it can be expressed as the maximum of a set of smooth functions: for any $j \in J$ let $F_j(s) = \|s - x_j\|^\alpha + \sum_{i \in J} \|s - x_i\|^\alpha$. Then $P(s) = \max_{j \in J} \{F_j(s)\}$.

Note that F_j is continuously differentiable and strictly convex, since it is the sum of such functions.

Consider the following non-linear optimization program.

Problem P1.

$$\begin{aligned} & \underset{(s,v) \in \mathbb{R}^2 \times \mathbb{R}}{\text{minimise}} && v \\ & \text{subject to} && \\ & && F_j(s) - v \leq 0, \quad \forall j \in J. \end{aligned}$$

Clearly Problem P1 is smooth and convex. It is easy to see that (s, v) is optimal for this problem if and only if $s = s^*$ and $v = P(s^*)$. From the Karush-Kuhn-Tucker (KKT) conditions it follows that the point $(s^*, v^*) \in \mathbb{R}^2 \times \mathbb{R}$ is optimal for Problem P1 if and only if there exist multipliers $\lambda_j^* \geq 0$, $j \in J$ such that $\nabla v + \sum_{j \in J} \lambda_j^* (\nabla F_j(s^*), -1) = 0$ and $\lambda_j^* (F_j(s^*) - v^*) = 0$ for all $j \in J$. Observe that $\lambda_j^* (F_j(s^*) - v^*) = 0 \Leftrightarrow \lambda_j^* (\|s^* - x_j\| - \max_{i \in J} \{\|s^* - x_i\|\}) = 0$.

Note: For the remainder of this paper we focus on the case $\alpha = 2$.

Definition: Recall that the *centroid* (or centre of mass) of X , which we denote as M , is defined by $M = \frac{1}{n} \sum_{j \in J} x_j$, where $n = |X|$. We define the set of *2-centroids* of X to be the set $\mathcal{M} = \{M_j : j \in J\}$ where each $M_j = \frac{1}{n+1} \left(x_j + \sum_{i \in J} x_i \right)$. See Figure 1.

Observe that \mathcal{M} is the image of an affine transformation on X . A consequence of this observation is the following lemma. Let $\text{conv}(\cdot)$ denote the the convex hull of a set of points. By two planar polygonal regions being *similar* we mean that they have corresponding sides, all of which are in proportion.

Lemma 1 ([4]): The region $\text{conv}(\mathcal{M})$ is similar to $\text{conv}(X)$, and M is the centroid of \mathcal{M} (as well as X). For any j the point M_j divides the line segment Mx_j into a ratio of $1 : n$. See Figure 1.

We now simplify the KKT conditions for the case $\alpha = 2$. Note that $\nabla F_j(s) = 2(n+1)s - 2x_j - 2 \sum_{i \in J} x_i = 2(n+1)(s - M_j)$.

Then s^* is the min-power centre of X if and only if there exists $\lambda_j^* \geq 0$ such that

$$s^* = \sum_{j \in J} \lambda_j^* M_j, \tag{1}$$

$$\sum_{j \in J} \lambda_j^* = 1, \tag{2}$$

$$\lambda_j^* (\|s^* - x_j\| - \max_{i \in J} \{\|s^* - x_i\|\}) = 0, \quad \forall j \in J. \tag{3}$$

Finally, by Conditions (1) and (2) we have the following important result.

Lemma 2 ([4]): $s^* \in \text{conv}(\mathcal{M})$.

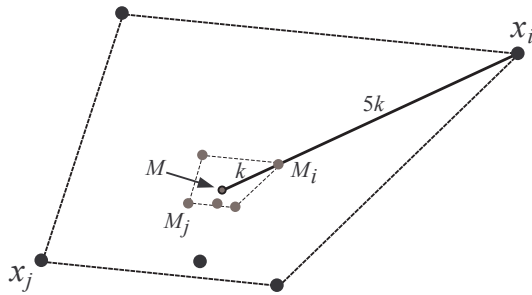


Fig. 1. Example illustrating the definitions

III. THE CENTROID AS AN APPROXIMATION TO s^*

In this section we give upper bounds for the performance of the centroid M as an approximation point for s^* . The next two results are from [4]; we include the proofs since we will subsequently extend these results and their method of proof. Let $x_r \in X$ be the farthest node from M .

Lemma 3 ([4]): $\|M_r - x_r\| \leq \|s^* - x_r\|$.

Proof.

$$\begin{aligned} \|x_r - M\| &\leq \|s^* - x_r\| \\ &+ \|s^* - M\| \quad (\text{by triangle inequality}) \\ &\leq \|s^* - x_r\| \\ &+ \|M_r - M\| \quad (\text{since } M_r \text{ is farthest} \\ &\quad \text{point of } \text{conv}(\mathcal{M}) \text{ from} \\ &\quad M). \end{aligned}$$

Therefore

$$\begin{aligned}\|s^* - x_r\| &\geq \|x_r - M\| - \|M_r - M\| \\ &= \|M_r - x_r\|.\end{aligned}$$

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For point set X , define $\rho(X) := \frac{P(M)}{P(s^*)}$. Let k be the number of farthest elements of X from s^* .

Theorem 4 ([4]): Then

$$\rho(X) \leq \frac{1}{k+1} \left(\frac{n+1}{n} \right)^2 + \frac{k}{k+1}.$$

Proof. For any point y we denote by $F(y)$ the sum $F(y) = \sum_{j \in J} \|y - x_j\|^2$. Let $x_p \in X$ be a farthest node from s^* . By the triangle inequality $\|M - x_r\| \leq \|s^* - M\| + \|s^* - x_r\| \leq \|s^* - M\| + \|s^* - x_p\|$.

Therefore

$$\begin{aligned}\frac{\|M - x_r\|}{\|s^* - x_p\|} &\leq \frac{\|s^* - M\|}{\|s^* - x_p\|} + 1 \\ &\leq \frac{\|M_r - M\|}{\|s^* - x_r\|} + 1 \quad (\text{by Lemma 2}) \\ &\leq \frac{\|M_r - M\|}{\|M_r - x_r\|} + 1 \quad (\text{by Lemma 3}) \\ &= \frac{1}{n} + 1 = \frac{n+1}{n} \quad (\text{by Lemma 1}).\end{aligned}$$

We now have

$$\begin{aligned}\frac{P(M)}{P(s^*)} &\leq \frac{\|M - x_r\|^2 + F(s^*)}{\|s^* - x_p\|^2 + F(s^*)} \\ &= \frac{\|M - x_r\|^2 - \|s^* - x_p\|^2}{\|s^* - x_p\|^2 + F(s^*)} + 1 \\ &\leq \frac{\|M - x_r\|^2 - \|s^* - x_p\|^2}{(k+1)\|s^* - x_p\|^2} + 1 \\ &= \frac{\|M - x_r\|^2}{(k+1)\|s^* - x_p\|^2} + \frac{k}{k+1} \\ &\leq \frac{1}{k+1} \left(\frac{n+1}{n} \right)^2 + \frac{k}{k+1},\end{aligned}$$

where the last inequality follows from the fact that $\frac{\|M - x_r\|}{\|s^* - x_p\|} \leq \frac{n+1}{n}$. ■

Since $k \geq 1$ we conclude that $\rho(X) \leq \frac{1}{2} \left(\frac{n+1}{n} \right)^2 + \frac{1}{2}$ for any X . The above approximation becomes exact as n increases indefinitely. Next we show that the above bound can be improved in the case $k = 1$.

Lemma 5:

$$\frac{\sum_{i=1}^n \|M - x_i\|^2}{\max_{i=1, \dots, n} \|M - x_i\|^2} \geq \frac{n}{n-1}.$$

Proof. It is sufficient to prove that

$$(n-1) \sum_{i=1}^n \|M - x_i\|^2 \geq n \|M - x_1\|^2. \quad (4)$$

We start off as follows:

$$\begin{aligned}(n-1) \sum_{i=2}^n \|M - x_i\|^2 &\geq \left(\sum_{i=2}^n \|M - x_i\| \right)^2 \\ &\geq \left\| \sum_{i=2}^n (M - x_i) \right\|^2 \\ &= \|M - x_1\|^2.\end{aligned}$$

Now add $(n-1)\|M - x_1\|^2$ to both sides to obtain (4).

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A straight-forward calculation shows that for any x ,

$$\sum_{i=1}^n \|x - x_i\|^2 = \sum_{i=1}^n \|M - x_i\|^2 + n \|x - M\|^2.$$

Once again, let x_p be a point of X farthest from s^* .

$$\begin{aligned}\rho(X) &= \frac{P(M)}{P(s^*)} = \frac{\sum_{i=1}^n \|M - x_i\|^2 + \|M - x_r\|^2}{\sum_{i=1}^n \|s^* - x_i\|^2 + \|s^* - x_p\|^2} \\ &= 1 + \frac{\|M - x_r\|^2 - n \|s^* - M\|^2 - \|s^* - x_p\|^2}{\sum_{i=1}^n \|M - x_i\|^2 + n \|s^* - M\|^2 + \|s^* - x_p\|^2}.\end{aligned} \quad (5)$$

Theorem 6: If the number of farthest elements of X from s^* is $k = 1$ then

$$\rho(X) \leq 1 + \frac{1}{2n} - \frac{1}{2n^2}$$

Proof.

Since $k = 1$ it follows that s^* minimises $F_p(s)$ (as defined in Section II), and therefore $s^* = M_p = \frac{1}{n+1} (\sum_{i=1}^n x_i + x_p) = \frac{1}{n+1} x_p + \frac{n}{n+1} M$. Since there is only one farthest point x_p from s^* and, from the previous equation, M is on the line through x_p and s^* , it follows that x_p is the farthest point from M , that is, $x_r = x_p$. It follows that $s^* - M = M_r - M = \frac{1}{n+1} (x_r - M)$ and $s^* - x_p = M_r - x_r = -\frac{n}{n+1} (x_r - M)$.

Finally, by Lemma 5 we have $\sum_{i=1}^n \|M - x_i\|^2 \geq \frac{n}{n-1} \|M - x_r\|^2$. Substituting back into (5) we obtain

$$\begin{aligned} \rho(X) &\leq 1 + \frac{\|M - x_r\|^2 - A}{\frac{n}{n-1} \|M - x_r\|^2 + A} \\ &= 1 + \frac{1}{2n} - \frac{1}{2n^2}, \end{aligned}$$

where $A = \frac{n}{(n+1)^2} \|M - x_r\|^2 + \frac{n^2}{(n+1)^2} \|M - x_r\|^2$

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This bound is tight: it is attained when all but one of the x_i 's coincide. Note also that it is a slight improvement over the previous bound when $k = 1$. Finally, since k is not necessarily known at the outset, we construct a bound in terms of k' , the number of farthest elements of X from the centroid M . Observe that k' can be easily constructed given X .

Theorem 7:

$$\rho(X) \leq 1 + \frac{2n + 1}{k'(n + 1)^2 + n^2}$$

Proof. Observe that $\sum_{i=1}^n \|M - x_i\|^2 \geq k' \|M - x_r\|^2$. But, by the proof of Theorem 4, we have $\frac{\|M - x_r\|}{\|s^* - x_p\|} \leq \frac{n+1}{n}$.

Substituting into (5) we obtain

$$\begin{aligned} \rho(X) &\leq 1 + \frac{\|M - x_r\|^2 - n \|s^* - M\|^2 - \|s^* - x_p\|^2}{\sum_{i=1}^n \|M - x_i\|^2 + n \|s^* - M\|^2 + \|s^* - x_p\|^2} \\ &\leq 1 + \frac{\|M - x_r\|^2 - n \|s^* - M\|^2 - \|s^* - x_p\|^2}{k' \|M - x_r\|^2 + n \|s^* - M\|^2 + \|s^* - x_p\|^2} \\ &\leq \frac{(k' + 1) \|M - x_r\|^2}{k' \|M - x_r\|^2 + 0 + \left(\frac{n}{n+1}\right)^2 \|M - x_r\|^2} \\ &= \frac{(k' + 1)(n + 1)^2}{k'(n + 1)^2 + n^2} = 1 + \frac{2n + 1}{k'(n + 1)^2 + n^2}. \end{aligned}$$

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IV. CONCLUSION

In this paper we improve upon the results of [4] by providing new upper-bounds for the performance of the centroid as an approximation to the quadratic min-power centre. When the longest edge incident to the min-power centre is unique we provide a tight bound.

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