

Optimization schemes for synchronization of networked parabolic partial differential equations

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Abstract—This work considers the optimization of the synchronization gains for a class of networked systems governed by parabolic partial differential equations. The control and optimization objective is to ensure that (i) the state of each of the networked systems follows a reference model (leader) and (ii) all pairwise state differences of the networked systems converge to zero in an appropriate norm. A matching condition is assumed in order for a static output feedback controller to enable each of the networked systems to follow the state of the leader. A consensus protocol utilizing only the output signals from each of the networked systems is proposed to ensure that synchronization is attained. The synchronization gains utilized in the consensus protocol can be viewed as the strengths of the interconnections among the network nodes. Using the resulting aggregate systems, the choice of the synchronization gains is recast as an optimization problem in which the total energy of the aggregate dynamics is minimized. In this setting, the weighted version of the graph Laplacian matrix associated with an undirected connected graph, is used as an alternate parameterization of the synchronization gains. The resulting optimal value of the synchronization gains is thus found as the solution to a parameterized operator Lyapunov equation associated with the total energy of the aggregate system. The proposed results are demonstrated by numerical example for a 1D partial differential equation.

Index Terms—Distributed parameter systems; synchronization controllers; networked systems; gain optimization.

I. INTRODUCTION

This work considers the optimization of the synchronization gains, as used to enhance synchronization for a class of distributed parameter systems. The networked systems are assumed to have identical dynamics and the control objective is to choose a static output feedback controller so that each system follows a reference model (leader). In addition to the tracking objective, it is desired to augment the controllers with a term that enhances synchronization. The synchronization, or agreement, is defined as the convergence of the pairwise state differences of the networked systems.

The term utilized to enforce synchronization is based on a consensus protocol that uses only the output signal of each of the networked systems. In order to find the optimal choice of the synchronization gains, the networked systems are brought in an aggregate form and the choice of the optimal synchronization gains is recast as the minimization of an energy functional of the parameterized closed loop aggregate system. Using established results for a similar problem on optimization of second order distributed systems [1], [2],

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the optimal value of the synchronization gains is found by minimizing the trace of a parameterized operator Lyapunov equation.

The motivation and the problem formulation are presented in the next section. The controller architecture for enhancing synchronization is presented in Section III and the optimization scheme is summarized in Section IV. Numerical studies of networked systems modeled by 1D diffusion PDEs is presented in Section V. Conclusions follow in Section VI.

II. MATHEMATICAL FRAMEWORK

Typical systems of coupled parabolic partial differential equations, including reaction-diffusion systems of the activator-inhibitor type [3] and the FitzHugh-Nagumo type [4], are described by the following coupled equations

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= a_1 \Delta x_1 - f_1(x_1, \dots, x_N) \\ \frac{\partial x_2}{\partial t} &= a_2 \Delta x_2 - f_2(x_1, \dots, x_N) \\ &\vdots \\ \frac{\partial x_N}{\partial t} &= a_n \Delta x_N - f_N(x_1, \dots, x_N). \end{aligned}$$

The functions x_1, \dots, x_N denote the states of the N coupled PDEs, Δ the Laplacian operator and the nonlinear functions f_1, \dots, f_N describe the coupling of the N PDEs. When these functions are described a priori, then one may consider the asymptotic behavior of the N states or perform sensitivity analysis to assess their quantitative behavior as a function of system parameters. Finally, one may be able to choose the structure of the nonlinear functions so that a desired collective behavior of the system of PDEs is achieved.

In this work, we consider a class of networked controlled partial differential equations with identical dynamics. The goal is to choose the control signals and the coupling terms so that (i) the states of the individual systems have a desired behavior (model following) and (ii) the states of the entire network of these coupled systems come to an agreement (synchronization).

We consider the following class of networked systems with identical dynamics, written in abstract form

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_2 u_i(t), \quad x_i(0) = x_{i0} \in D(A), \\ y_i(t) &= C_2 x_i(t) \end{aligned} \quad (1)$$

for $i = 1 \dots, N$, where the state space is the Hilbert space $\{\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}, | \cdot |_{\mathcal{H}}\}$. Before providing details on the state

operator A , the input operator B_2 and the output operator C_2 , we define the spaces associated with these operators.

The control and synchronization problem is presented in a space setting associated with the Gelfand triple [5]. The space $\{\mathcal{V}, \|\cdot\|_{\mathcal{V}}\}$ is a reflexive Banach space that is densely and continuously embedded in \mathcal{H} with $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$ with these embeddings dense and continuous where the space \mathcal{V}^* denotes the continuous dual of \mathcal{V} , [5]. The space of controls \mathcal{U} is a finite dimensional Euclidean space and \mathcal{Y} is a finite dimensional Euclidean space of measurements having the same dimension as \mathcal{U} , and thus we have square systems. Using the above, we have $A \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, $B_2 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and $C_2 \in \mathcal{L}(\mathcal{H}, \mathcal{U})$.

The well-posedness, stability, optimization and performance of the N networked systems will be examined collectively and thus one must consider the associated aggregate system. Define the space $\{\mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{H}}\}$ as the Hilbert space with inner product

$$\langle \Phi, \Psi \rangle_{\mathbb{H}} = \langle \phi_1, \psi_1 \rangle_{\mathcal{H}} + \langle \phi_2, \psi_2 \rangle_{\mathcal{H}} + \dots + \langle \phi_N, \psi_N \rangle_{\mathcal{H}},$$

with $\Phi = \{\phi_1, \phi_2, \dots, \phi_N\}$, $\Psi = \{\psi_1, \psi_2, \dots, \psi_N\} \in \mathbb{H}$. With this, we define the aggregate state space $\mathbb{H} = (\mathcal{H})^N$. Similarly, we define the spaces \mathbb{V} and \mathbb{V}^* via $\mathbb{V} = (\mathcal{V})^N$ and $\mathbb{V}^* = (\mathcal{V}^*)^N$ with $\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}^*$. Finally, the aggregate input and output spaces are given by $\mathbb{U} = (\mathcal{U})^N$ and $\mathbb{Y} = (\mathcal{Y})^N$ respectively.

To quantify the two goals of model following and synchronization, we first define the reference model

$$\dot{x}_m(t) = A_m x_m(t) + B_2 r(t), \quad (2)$$

where the operator $A_m \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, is the generator of an exponentially stable C_0 semigroup on \mathcal{H} , the reference signal $r \in L_2(0, \infty; \mathcal{U})$ and $x_m(0) \in D(A_m)$. With the above conditions, the reference model (2) is well-posed [6].

The model following problem becomes that of choosing the control signals $u_i(t)$, $i = 1, \dots, N$ so that the errors between the states of each of the networked systems (1) and the state of the reference model (2) converge asymptotically to zero in norm

$$\lim_{t \rightarrow \infty} |x_i(t) - x_m(t)|_{\mathcal{H}} = 0, \quad \forall i = 1, \dots, N. \quad (3)$$

Similarly, the synchronization objective is to choose the coupling between the networked systems (1) so that all pairwise state differences asymptotically converge to zero

$$\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)|_{\mathcal{H}} = 0, \quad \forall i, j = 1, \dots, N. \quad (4)$$

The interaction (information exchange) between the N PDE systems (1) can be described by the appropriate communication topology. An undirected connected graph $G = (V, \mathcal{E})$ is used to describe this communication topology. The nodes $V = \{1, 2, \dots, N\}$ represent the agents (PDE systems in (1)) and the edges $\mathcal{E} \subset V \times V$ represent the communication links between the networked distributed parameter systems in (1). The set of neighbors of the i th system is denoted by $N_i = \{j : (i, j) \in \mathcal{E}\}$, [7]. We denote by L_0 the nominal graph Laplacian matrix associated with the graph G and given by

$L_0 = \mathbf{D} - \mathbf{A}$, where \mathbf{D} is the degree matrix and \mathbf{A} is the adjacency matrix.

III. CONTROLLER ARCHITECTURE AND SYSTEM-THEORETIC CONDITIONS

To address the two objectives (3), (4) we assume that the individual controllers admit the following additive structure

$$u_i(t) = u_{im}(t) + u_{is}(t), \quad i = 1, \dots, N, \quad (5)$$

where the signals u_{im} address the model following objective and u_{is} address the synchronization objective. Since only the outputs $y_i(t)$ can be used by each node to design its own model following controller, then one must assume certain matching conditions that enable each system (1) to follow (2) in the sense of (3). This matching condition requires the existence of a (matrix) gain $\Gamma : \mathcal{U} \rightarrow \mathcal{U}$ such that

$$A_m = A - B_2 \Gamma C_2, \quad (6)$$

with A_m the generator of an exponentially stable C_0 semigroup on \mathcal{H} . With this condition, then the signals u_{im} can immediately be defined by

$$u_{im}(t) = -\Gamma y_i(t) + r(t), \quad \forall i = 1, \dots, N. \quad (7)$$

The well-posedness of the closed-loop systems with the controllers given by (7) can easily be established. Using the matching condition (6) and the control law (7) in each of the systems in (1) the resulting error system with $e_i(t) = x_i(t) - x_m(t)$ is governed by

$$\dot{e}_i(t) = A_m e_i(t), \quad e_i(0) \in D(A_m), \quad i = 1, \dots, N.$$

The norm convergence of the state errors e_i to zero is attained because A_m generates an exponentially stable semigroup on \mathcal{H} . Therefore, the controller (7) alone achieves model following. In fact, synchronization is also achieved since convergence of the state errors implies the convergence of the pairwise difference in (4). However, one would like to address the synchronization objective by the design of the additional control signal $u_{is}(t)$ in (5) so that the convergence of the pairwise differences $x_i(t) - x_j(t)$ to zero is "faster" than the convergence of $e_i(t)$ and $e_j(t)$ to zero. In fact, for certain class of infinite dimensional systems, one can obtain explicit bounds on the faster convergence of $x_i(t) - x_j(t)$ to zero over the convergence of $e_i(t)$ to zero, [8].

It now remains to define the synchronization signals $u_{is}(t)$. Since each agent can access the output signals $y_i(t)$ of its neighbor, as dictated by the communication topology, then the synchronization signals have the form of an output measurement consensus protocol and given by

$$\begin{aligned} u_{is}(t) &= - \sum_{j \in N_i} a_{ij} (y_i(t) - y_j(t)) \\ &= - \sum_{j=1}^N L_{ij} y_j(t) \end{aligned} \quad i = 1, \dots, N, \quad (8)$$

where a_{ij} denote the edge-dependent scalar *synchronization gains*. The alternative expression for the synchronization controllers involve the parameterized graph matrix L . These gains need to be chosen so that synchronization is enhanced.

To enable the controllers (7) and (8), certain system-theoretic properties of the networked systems (1) need to be made. We thus make the following assumption:

Assumption 1: Consider the networked PDE systems in (1). Assume that the following hold

- 1) The output signal $y_i(t)$ of each system is available to the i th system and to all other networked systems that is linked to, as dictated by the communication topology described by the undirected connected graph G .
- 2) The operator A generates a C_0 semigroup on \mathcal{H} and for any $u_i \in L_2(0, \infty; \mathcal{U})$, the systems (1) are well-posed for any $x_i(0) \in D(A)$.
- 3) The pair (A, B_2) is approximately controllable [9] and the triple (A, B_2, C_2) is statically stabilizable [10], [11]. One then requires the existence of a symmetric positive definite constant gain matrix G such that the closed loop operator $A_m = A - B_2 \Gamma C_2$ generates an exponentially stable C_0 semigroup, with the property

$$PA_m + A_m^* P = -R \text{ in } D(A_m), \quad (9)$$

where R is a coercive bounded self-adjoint operator and P a self-adjoint invertible solution to (9).

To facilitate the analysis of the controllers (5), (7), (8), we consider the model reference error systems governed by

$$\dot{e}_i(t) = A_m e_i(t) - B_2 \sum_{j \in N_i} a_{ij} (y_i(t) - y_j(t)), \quad i = 1, \dots, N.$$

Using the fact that $x_i - x_j = e_i - e_j$, the above simplifies to

$$\dot{e}_i(t) = A_m e_i(t) - B_2 \sum_{j \in N_i} a_{ij} C_2 (e_i(t) - e_j(t)), \quad i = 1, \dots, N.$$

Expressed in terms of the parameterized graph matrix L it becomes

$$\dot{e}_i(t) = A_m e_i(t) - B_2 \sum_{j=1}^N L_{ij} C_2 e_j(t), \quad i = 1, \dots, N. \quad (10)$$

Prior to the optimization of the synchronization gains a_{ij} or equivalently, the parameterized graph matrix L , with respect to an appropriate synchronization measure, we must define the appropriate parameter space needed in the optimization of the synchronization gains.

The resulting $N \times N$ gain matrix whose entries represent the synchronization gains will need to be multiplied (Hadamard product, [12]) by the nominal graph Laplacian matrix. To avoid overparameterization of the gain matrices, these matrices are essentially defined as matrices with nonzero entries that match the location of the nonzero entries of the nominal graph Laplacian matrix L_0 , but the zero row sum condition that graph Laplacians satisfy is not imposed. The space $\Theta \in \mathbb{R}^{N \times N}$ is defined as the parameter space of $N \times N$ symmetric positive semidefinite matrices L such that $L = (\mathbf{I}_N + \mathbf{A}) \circ M$ where M is any N dimensional matrix and \circ denotes the Hadamard matrix product. We have

$$\Theta(G) = \{0 \leq L = L^T \in \mathbb{R}^{N \times N} : L = (\mathbf{I}_N + \mathbf{A}) \circ M, M \in \mathbb{R}^{N \times N}\}.$$

Please note that the nominal graph Laplacian matrix L_0 is the matrix associated with the prescribed graph and whose

entries are fixed. Any matrix in $\Theta(G)$ will have nonzero entries at the same locations as L_0 and will keep track on the information exchange amongst the networked systems.

While (4) quantifies the synchronization objective, a measure of synchronization is the *deviation from the mean*

$$z_i(t) = x_i(t) - \frac{1}{N} \sum_{j=1}^N x_j(t), \quad i = 1, \dots, N,$$

and it measures the disagreement of state $x_i(t)$ from the average state of all networked systems. Using the fact that $e_i = x_i - x_m$, then the deviation from the mean can be expressed in terms of the tracking errors

$$z_i(t) = e_i(t) - \frac{1}{N} \sum_{j=1}^N e_j(t), \quad i = 1, \dots, N.$$

It can now be written in terms of the aggregate state vector and the aggregate deviation from the mean as

$$Z(t) = C_1 E(t), \quad C_1 = \mathbf{I}_N - \frac{1}{N} \mathbf{1}_N \cdot \mathbf{1}_N^T, \quad (11)$$

where $E(t) = [e_1(t) \dots e_N(t)]^T$, $Z(t) = [z_1(t) \dots z_N(t)]^T$. It should be noted that C_1 is not an $N \times N$ matrix, but an operator matrix, whose entries are the identity operators on \mathcal{H} . The matrix \mathbf{I}_N should be interpreted as a matrix whose diagonal entries are the identity operators on \mathcal{H} and thus $\mathbf{I}_N \Phi = \Phi$, $\Phi \in \mathbb{H}$. Similarly, $\mathbf{1}_N^T$ denotes the N -dimensional row vector whose entries are the identity operator on \mathcal{H} and given by

$$\mathbf{1}_N^T = [I_{\mathcal{H}} \quad \dots \quad I_{\mathcal{H}}].$$

IV. OPTIMIZATION SCHEMES FOR THE SYNCHRONIZATION GAINS

The closed loop systems representing the tracking errors (10) can be examined collectively when placed in an aggregate form with

$$\mathcal{A}_m \triangleq \mathbf{I}_N \otimes A_m, \quad \mathcal{B}_2 \triangleq \mathbf{I}_N \otimes B_2, \quad \mathcal{C}_2^* \triangleq \mathbf{I}_N \otimes C_2^*.$$

Then one arrives at the aggregate error dynamics

$$\dot{E}(t) = (\mathcal{A}_m - \mathcal{B}_2 L C_2) E(t), \quad E(0) \in D(\mathcal{A}_m), \quad L \in \Theta(G). \quad (12)$$

To argue the well-posedness of (12), one uses the fact that the operator A_m generates an exponentially stable C_0 semigroup on \mathcal{H} . It can easily be argued that \mathcal{A}_m generates a C_0 semigroup on \mathbb{H} . The appropriate choice of L would then allow one to consider $\mathcal{B}_2 L C_2$ as the bounded perturbation of an exponentially stable C_0 semigroup on \mathbb{H} [13].

To find the optimal choice of the parameterized graph matrix $L \in \Theta(G)$, an appropriate measure of performance must be considered. While the tracking gain Γ can be a priori defined via (6) and thus not be required to undergo any optimization, the choice of the synchronization gains, through the choice of the parameterized graph matrix L can still affect both tracking and synchronization. Thus a combined measure of model following and synchronization

must be considered for the optimization of the parameterized graph matrix $L \in \Theta(G)$. We thus consider the cost

$$J(L) = \int_0^\infty \langle E(\tau), QE(\tau) \rangle_{\mathbb{H}} + \langle Z(\tau), Z(\tau) \rangle_{\mathbb{H}} d\tau,$$

and which includes the term $\langle E(\tau), QE(\tau) \rangle_{\mathbb{H}}$ that explicitly penalizes regulation of the aggregate error to zero and the term $\langle Z(\tau), Z(\tau) \rangle_{\mathbb{H}}$ which represents the norm of the aggregate deviation from the mean. The operator Q is a positive operator used as the weight of the state penalty.

Using (11), the above cost can be written in terms of the aggregate error $E(t)$

$$J(L) = \int_0^\infty \langle E(\tau), (Q + C_1^* C_1) E(\tau) \rangle_{\mathbb{H}} d\tau. \quad (13)$$

Following (11), the $N \times N$ symmetric matrix $C_1 \geq 0$ corresponds to the graph Laplacian with full connectivity, and therefore convergence of $E(t)$ to zero (regulation) implies convergence of $Z(t)$ to zero (synchronization) but not the reverse. The operator matrix Q (i.e. a block diagonal $N \times N$ matrix whose entries are operators in \mathcal{H}) denotes the weighting on the error $E(t)$. Using $\mathbf{M} = Q + C_1^* C_1$ to denote the $N \times N$ operator matrix, the cost (13) can be written as

$$J(L) = \int_0^\infty \langle E(\tau), \mathbf{M}E(\tau) \rangle_{\mathbb{H}} d\tau. \quad (14)$$

For a prescribed $L \in \Theta(G)$, the optimal value of (14) is

$$J^{opt}(L) = \langle E(0), \Pi E(0) \rangle_{\mathbb{H}} \quad (15)$$

where Π solves the operator Lyapunov equation

$$(\mathcal{A}_m - \mathcal{B}_2 L C_2)^* \Pi + \Pi (\mathcal{A}_m - \mathcal{B}_2 L C_2) + \mathbf{M} = 0 \text{ on } D(\mathcal{A}_m).$$

The optimal cost (15) depends on the initial condition of the aggregate error $E(0)$ and the solution to the operator Lyapunov equation. To simplify this optimal value in order to perform the optimization with respect to the graph matrix L , one assumes that the initial condition $E(0)$ is a Gaussian random vector in \mathbb{H} with zero mean and unit covariance. With this assumption, the optimal value of the cost (14) may be written as

$$J^{opt}(L) = \text{tr}(\Pi).$$

When now the graph matrix is constrained in the set $\Theta(G)$, then the optimal value of L is found via

$$L^{opt} = \arg \min_{L_\alpha \in \Theta(G)} \text{tr}(\Pi_\alpha), \quad L_\alpha \in \Theta(G)$$

where Π_α denotes the positive solution to the L_α -parameterized operator Lyapunov equation

$$(\mathcal{A}_m - \mathcal{B}_2 L_\alpha C_2)^* \Pi_\alpha + \Pi_\alpha (\mathcal{A}_m - \mathcal{B}_2 L_\alpha C_2) + \mathbf{M} = 0, \quad (16)$$

for $L_\alpha \in \Theta(G)$.

The proposed optimization design is summarized below:

$$\left\{ \begin{array}{l} \text{Find } L \in \Theta(G) \text{ to minimize} \\ J(L_\alpha) = \int_0^\infty \langle E_\alpha(\tau), \mathbf{M}E_\alpha(\tau) \rangle_{\mathbb{H}} d\tau. \\ \text{subject to} \\ \dot{E}_\alpha(t) = (\mathcal{A}_m - \mathcal{B}_2 L_\alpha C_2) E_\alpha(t), \quad E_\alpha(0) \in D(\mathcal{A}), \quad L_\alpha \in \Theta(G). \end{array} \right.$$

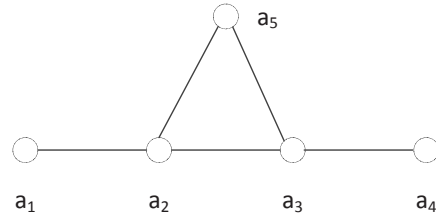


Fig. 1. Undirected connected graph on 5 vertices.

Solution:

$$L^{opt} = \arg \min_{L_\alpha \in \Theta(G)} \text{tr}(\Pi_\alpha)$$

where Π_α solves the L_α parameterized operator Lyapunov equation (16).

V. NUMERICAL STUDIES

Following [8], we consider $N = 5$ systems, each modelled by the following PDE

$$\begin{aligned} \frac{\partial x_i}{\partial t}(t, \xi) &= \alpha \frac{\partial^2 x_i}{\partial \xi^2}(t, \xi) + b(\xi) u_i(t), \quad x_i(0, \xi) = x_{i0}(\xi), \quad \alpha > 0, \\ \frac{\partial x_i}{\partial \xi}(t, 0) &= \frac{\partial x_i}{\partial \xi}(t, 1) = 0, \quad i = 1, \dots, 5, \end{aligned}$$

on the state space $X = L_2(0, 1)$. The spatial distribution $b(\xi)$ of the input operator B is given by the boxcar function

$$b(\xi) = \frac{1}{2\beta} 1_{[\xi_0 - \beta, \xi_0 + \beta]}(\xi) = \begin{cases} \frac{1}{2\beta} & \text{for } \xi_0 - \beta \leq \xi \leq \xi_0 + \beta \\ 0 & \text{elsewhere} \end{cases}$$

where $\xi_0 = 0.5$ and $\beta = 0.025$. The thermal diffusivity is taken to be $\alpha = 0.5$, and the output is taken to be

$$y_i(t) = \int_0^1 x_i(t, \xi) d\xi$$

The feedback gain in (6) is taken as $\Gamma = 1.125$ and the consensus gains in (8) are taken to be uniform

$$u_{is}(t) = -\theta \sum_{j \in N_i} (y_i(t) - y_j(t)), \quad i = 1, \dots, 5, \quad (17)$$

leading to $L_\alpha = \theta L_0$. The connectivity of the 5 systems is represented by the undirected graph of Figure 1 with the corresponding graph Laplacian given by

$$L_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{bmatrix}.$$

To find the optimal value of the parameter θ in (17), the Lyapunov equations (16) were solved for the range of $\theta \in [0, 10]$. Using \mathbf{M} in (14) given by $\mathbf{M} = I_{\mathbb{H}} + 100 C_1^* C_1$, the optimal value minimizing $\text{tr}(\Pi_\alpha)$ was $\theta^{opt} = 4.65$ and thus the optimal synchronization controller (8) was given by

$$\begin{aligned} U_s(t) &= [u_{1s}(t) \quad \dots \quad u_{Ns}(t)]^T \\ &= -\theta^{opt} L_0 C_2 X(t), \end{aligned}$$

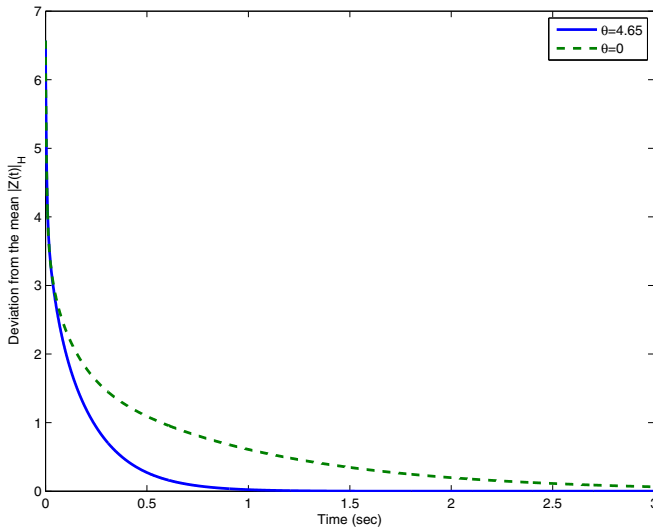


Fig. 2. Evolution of deviation from the mean, $Z(t)$.

where $X(t)$ denotes the aggregate state vector, given by $X(t) = [z_1(t) \dots x_N(t)]^T$.

The system was simulated in the time interval $[0, 3]$ using the Matlab[®] fourth-fifth order Runge Kutta ode solver with $n = 40$ modes. The initial conditions of the 5 states are the same ones used in [8].

To quantify the agreement amongst the 5 states (i.e. synchronization), we used the deviation from the mean $Z(t)$ defined in (11). The evolution of the L_2 norm of $Z(t)$ is depicted in Figure 2 for the case of the optimal value of $\theta = 4.65$ and the value of $\theta = 0$. The last value represents the case of no synchronization control ($u_{is} = 0, i = 1, \dots, N$).

It can be observed that when the synchronization controller (synchronization component of controller (5)) is implemented, the agreement of the networked states converges to zero faster than the case with $u_{is} = 0, i = 1, \dots, N$.

The individual deviations from the mean, defined by the components $z_i(t)$ of $Z(t)$ in (11), and given by

$$z_i(t, \xi) = x_i(t, \xi) - \frac{1}{N} \sum_{i=1}^N x_i(t, \xi), \quad i = 1, \dots, 5,$$

are plotted at the final time $t_f = 3$ versus the spatial variable. Figure 3 depicts the first four $z_i(t_f, \xi)$ for the case of $\theta^{opt} = 4.65$ (solid) and $\theta = 0$ (dotted). When synchronization is implemented, the pointwise convergence of the deviation from the mean is significantly improved.

VI. CONCLUSIONS

The synchronization of networked systems whose identical dynamics are governed by parabolic PDEs, can be improved when the control signals include a coupling term based on a consensus protocol, and which penalize the weighted pairwise differences of the systems' outputs. The weights in the consensus protocol can affect the convergence of an appropriate measure of agreement. The proposed work provided an optimization scheme based on minimization of

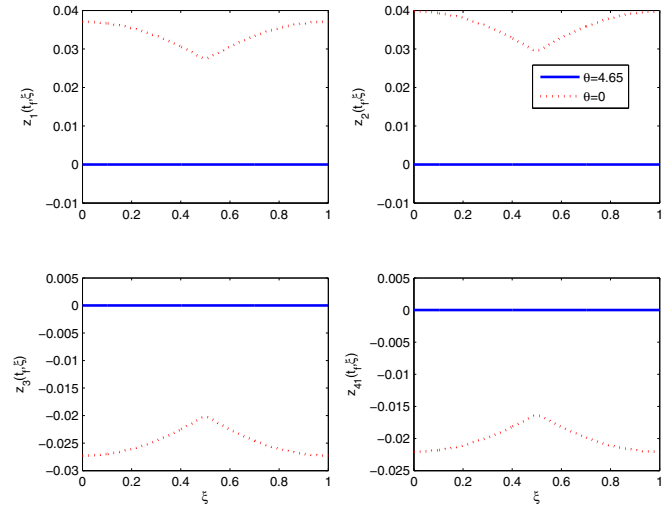


Fig. 3. Deviations from the mean, $z_i(t, \xi)$ at final time .

the energy of the closed loop aggregate system. This optimization was reduced to the optimization of a parameterized operator Lyapunov equation associated with the total energy of the aggregate systems. Extensive simulations studies on a network of 5 diffusion PDEs demonstrated the enhanced synchronization of the networked states when the additional synchronization controller is included.

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