

# Quantized Output Feedback Stabilization of Switched Linear Systems

Masashi Wakaiki and Yutaka Yamamoto

**Abstract**—This paper studies the problem of stabilizing a continuous-time switched linear system by quantized output feedback. We assume that the quantized output and the switching signal are available to the controller at all time. We develop an encoding strategy by using multiple Lyapunov functions and an average dwell-time property. The encoding strategy is based on the results in the case of a single mode. An additional adjustment of the “zoom” parameter is required at every switching time.

**Key words**— Switched systems, quantization, Lyapunov stability, Output feedback stabilization.

**AMS subject classifications**— 93C30, 93D05, 93D15

## I. INTRODUCTION

This paper studies the quantized control problem for switched systems. For linear time-invariant systems, various approaches to quantized control have been developed: Lyapunov-based methods [1]–[3], optimization with  $\ell^\infty$  norm [4], etc. In contrast, few results of quantized control are generalized to switched systems in spite of a wide range of their applications. Recently, based on the results in [5], Liberzon [6] has extended the quantized state feedback strategy from the case of non-switched systems in [5] to that of switched systems. This strategy achieves the global asymptotic stability of a switched system with a sampler and a quantizer. Also in [7], stabilization of sampled-data switched linear systems with memoryless quantizers is discussed. However, stabilization of switched systems by quantized *output* feedback has not yet explored.

Here we consider a continuous-time switching linear system. In [6], [7], the quantized measurement and the switched signal are available only at *each sampling time*. In contrast, in the present paper, they are transmitted to the controller at *all times*, which leads to a simple strategy of encoding.

The objective of this paper is to extend the encoding method of [2], [3] from the non-switched case to the switched case. The key point of the earlier studies is that certain level sets of a Lyapunov function are invariant regions. The difficulty of switched systems is that such level sets are dependent on the modes of the plant and hence change at every switching time. At the “zooming-in” stage, non-switched systems require only periodic reduction of the “zoom” parameter of quantizers. On the other hand, for switched systems, we need to adjust the parameter after each switch. The average dwell time [8] of the switching signal is assumed to be large enough. We develop an output en-

coding for global asymptotic stabilization by using multiple Lyapunov functions.

This paper is organized as follows. In Section II, we give the main result, Theorem 2.4, after explaining the components of the closed-loop system one by one. Section III is devoted to the proof of the main result. We present a numerical example in Section IV and finally conclude this paper in Section V.

*Notation:* Let  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  denote the smallest and the largest eigenvalue of  $P \in \mathbb{R}^{n \times n}$ . Let  $M^T$  denote the transpose of  $M \in \mathbb{R}^{m \times n}$ .

The Euclidean norm of  $v \in \mathbb{R}^n$  is denoted by  $|v| = (v^*v)^{1/2}$ . The Euclidean induced norm of  $M \in \mathbb{R}^{m \times n}$  is defined by  $\|M\| = \sup\{|Mv| : v \in \mathbb{R}^n, |v| = 1\}$ , which equals the largest singular value of  $M$ .

For a piecewise continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , its left-sided limit at  $t_0 \in \mathbb{R}$  is denoted by  $\lim_{t \nearrow t_0} f(t)$ . For  $\alpha \in \mathbb{R}$ ,  $[\alpha]$  is the largest integer not greater than  $\alpha$ .

## II. QUANTIZED OUTPUT FEEDBACK STABILIZATION OF SWITCHED SYSTEMS

### A. Switched linear systems

Consider the switched linear system

$$\dot{x} = A_\sigma x + B_\sigma u, \quad y = C_\sigma x, \quad (\text{II.1})$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the control input, and  $y(t) \in \mathbb{R}^p$  is the output. For a finite index set  $\mathcal{P}$ ,  $\sigma : [0, \infty) \rightarrow \mathcal{P}$  is right-continuous and piecewise constant. We call  $\sigma$  *switching signal* and the discontinuities of  $\sigma$  *switching times*. Let us denote by  $N_\sigma(t, s)$  the number of discontinuities of  $\sigma$  on the interval  $(s, t]$ .

Assumptions on the switched system (II.1) are as follows.

*Assumption 2.1:* For every  $p \in \mathcal{P}$ ,  $(A_p, B_p)$  is stabilizable and  $(C_p, A_p)$  is observable. We choose  $K_p \in \mathbb{R}^{m \times n}$  and  $L_p \in \mathbb{R}^{n \times p}$  so that  $A_p + B_p K_p$  and  $A_p + L_p C_p$  are Hurwitz.

Furthermore, the switching signal  $\sigma$  has an average dwell time [8], i.e., there exist  $\tau_a > 0$  and  $N_0 \geq 1$  such that

$$N_\sigma(t, s) \leq N_0 + \frac{t-s}{\tau_a} \quad (t > s \geq 0). \quad (\text{II.2})$$

### B. Quantizer

In this paper, we use the following class of quantizers proposed in [3].

Let  $\mathcal{Q}$  be a finite subset of  $\mathbb{R}^p$ . A quantizer is a piecewise constant function  $q : \mathbb{R}^p \rightarrow \mathcal{Q}$ . This geometrically implies that  $\mathbb{R}^p$  is divided into a finite number of the quantization regions  $\{y \in \mathbb{R}^p : q(y) = y_i\}$  ( $y_i \in \mathcal{Q}$ ). For the quantizer

M. Wakaiki and Y. Yamamoto are with the Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan (e-mail: wakaiki@acs.i.kyoto-u.ac.jp; yy@i.kyoto-u.ac.jp).

$q$ , there exist positive numbers  $M$  and  $\Delta$  with  $M > \Delta$  such that

$$|y| \leq M \Rightarrow |q(y) - y| \leq \Delta \quad (\text{II.3})$$

$$|y| > M \Rightarrow |q(y)| > M - \Delta. \quad (\text{II.4})$$

The former condition (II.3) gives an upper bound of the quantization error when the quantizer is not saturated. The latter (II.4) is used for the detection of the saturation.

We make the following assumption on the behavior of the quantizer  $q$  near the origin:

*Assumption 2.2* ([3], [9]): *There exists  $\Delta_0 > 0$  such that  $q(y) = 0$  for every  $y \in \mathbb{R}^p$  with  $|y| \leq \Delta_0$ .*

We use this assumption for the Lyapunov stability of the closed-loop system.

We give the above quantizers with the following adjustable parameter  $\mu > 0$ :

$$q_\mu(y) = \mu q\left(\frac{y}{\mu}\right). \quad (\text{II.5})$$

In (II.5),  $\mu$  is regarded as a ‘‘zoom’’ variable, and  $q_{\mu(t)}(y(t))$  is the data on  $y(t)$  transmitted to the controller. We need to change  $\mu$  to obtain accurate information of  $y$ . The reader can refer to [3], [9], [10] for further discussion.

*Remark 2.3:* The quantized output  $q_\mu(y)$  may chatter on boundaries among the quantization regions. Hence if we generate  $u$  by  $q_\mu(y)$ , the solutions of (II.1) must be interpreted in the sense of Filippov [11]. However this generalization does not affect our Lyapunov-based analysis in this work, because we will work with a single quadratic Lyapunov function between switching times as in [2], [3].

### C. Controller

Similarly to [2], [3], we construct the following dynamic output feedback law based on the standard Luenberger observers:

$$\dot{\xi} = (A_\sigma + L_\sigma C_\sigma)\xi + B_\sigma u - L_\sigma q_\mu(y), \quad u = K_\sigma \xi, \quad (\text{II.6})$$

where  $\xi \in \mathbb{R}^n$  is the estimated state. Then the closed-loop system is given by

$$\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma K_\sigma \xi \\ \dot{\xi} &= (A_\sigma + L_\sigma C_\sigma)\xi + B_\sigma K_\sigma \xi - L_\sigma q_\mu(y). \end{aligned} \quad (\text{II.7})$$

If we define  $z$  and  $F_\sigma$  by

$$z := \begin{bmatrix} x \\ x - \xi \end{bmatrix}, \quad F_\sigma := \begin{bmatrix} A_\sigma + B_\sigma K_\sigma & -B_\sigma K_\sigma \\ 0 & A_\sigma + L_\sigma C_\sigma \end{bmatrix},$$

then we rewrite (II.7) in the form

$$\dot{z} = F_\sigma z + \begin{bmatrix} 0 \\ L_\sigma \end{bmatrix} (q_\mu(y) - y). \quad (\text{II.8})$$

We see from Assumption 2.1 that  $F_p$  is Hurwitz for  $p \in \mathcal{P}$ . For every positive-definite matrix  $Q_p \in \mathbb{R}^{2n \times 2n}$ , there exist a positive-definite matrix  $P_p \in \mathbb{R}^{2n \times 2n}$  such that

$$F_p^\top P_p + P_p F_p = -Q_p \quad (p \in \mathcal{P}). \quad (\text{II.9})$$

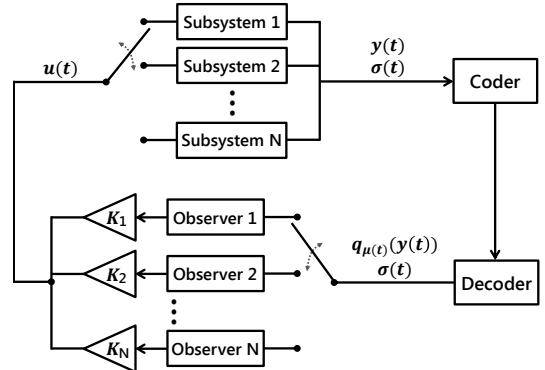


Fig. 1: Continuous-time switched system with quantized output feedback.

We define  $\bar{\lambda}_P$ ,  $\lambda_P$ ,  $\lambda_Q$ , and  $C_{\max}$  by

$$\begin{aligned} \bar{\lambda}_P &:= \max_{p \in \mathcal{P}} \lambda_{\max}(P_p), & \lambda_P &:= \min_{p \in \mathcal{P}} \lambda_{\min}(P_p) \\ \lambda_Q &:= \min_{p \in \mathcal{P}} \lambda_{\min}(Q_p), & C_{\max} &:= \max_{p \in \mathcal{P}} \|C_p\|. \end{aligned} \quad (\text{II.10})$$

### D. Main result

By adjusting the ‘‘zoom’’ parameter  $\mu$  as in the non-switched case [2], [3], we can achieve the global asymptotic stability of the closed-loop system (II.8) in Fig. 1.

*Theorem 2.4:* *Define  $\Theta$  by*

$$\Theta := \frac{2 \max_{p \in \mathcal{P}} \|P_p \hat{L}_p\|}{\lambda_Q}, \quad \text{where } \hat{L}_p := \begin{bmatrix} 0 \\ L_p \end{bmatrix}. \quad (\text{II.11})$$

and let  $M$  be large enough to satisfy

$$M > \max \left\{ 2\Delta, \sqrt{\frac{\bar{\lambda}_P}{\lambda_P}} \Theta \Delta C_{\max} \right\}. \quad (\text{II.12})$$

If the average dwell time  $\tau_a$  is larger than a certain value, then there exists a right-continuous, piecewise-constant function  $\mu$  such that the closed-loop system (II.8) has the following two properties for every  $x(0) \in \mathbb{R}^n$  and every  $\sigma(0) \in \mathcal{P}$ :

*Convergence to the origin:*  $\lim_{t \rightarrow \infty} z(t) = 0$ .

*Lyapunov stability:* To every  $\varepsilon > 0$ , there corresponds  $\delta > 0$  such that

$$|x(0)| < \delta \Rightarrow |z(t)| < \varepsilon \quad (t \geq 0).$$

In the next section, we shall prove Theorem 2.4 with the concrete construction of  $\mu$ . The sufficient condition on  $\tau_a$  is given by (III.25) below.

## III. THE PROOF OF THEOREM 2.4

Let us first consider the fixed ‘‘zoom’’ parameter  $\mu$ . Lemma 3.1 below shows that if switching does not occur, then the state trajectories enter certain level sets in finite time independent of the mode. This lemma is a trivial extension of Lemma 5 in [3] from the case of a single mode to that of multiple modes.

*Lemma 3.1:* Fix  $p \in \mathcal{P}$ , and consider the non-switched system

$$\dot{z} = F_p z + \hat{L}_p (q_\mu(y) - y). \quad (\text{III.1})$$

Choose  $\kappa > 0$ , and suppose that  $M$  satisfies

$$\sqrt{\lambda_P}M > \sqrt{\lambda_P}\Theta\Delta(1+\kappa)C_{\max}, \quad (\text{III.2})$$

where  $C_{\max}$  and  $\Theta$  are defined by (II.10) and (II.11), respectively. Then the two ellipsoids

$$\begin{aligned} \mathcal{R}_1(\mu, p) &:= \left\{ z : z^\top P_p z \leq \frac{\lambda_P M^2 \mu^2}{C_{\max}^2} \right\} \\ \mathcal{R}_2(\mu, p) &:= \left\{ z : z^\top P_p z \leq \bar{\lambda}_P (\Theta\Delta(1+\kappa))^2 \mu^2 \right\} \end{aligned}$$

are invariant sets of every trajectory of (III.1). Furthermore, if  $T$  satisfies

$$T > \frac{\lambda_P M^2 - \bar{\lambda}_P (\Theta\Delta(1+\kappa)C_{\max})^2}{\lambda_Q \kappa (1+\kappa) (\Theta\Delta C_{\max})^2}, \quad (\text{III.3})$$

then every trajectory of (III.1) with an initial state  $z(0) \in \mathcal{R}_1(\mu, p)$  satisfies  $z(T) \in \mathcal{R}_2(\mu, p)$ .

*Proof:* For every  $p \in \mathcal{P}$ , the time derivative of  $z^\top P_p z$  along the trajectories of the system (III.1) satisfies

$$\begin{aligned} \frac{d}{dt}(z^\top P_p z) &= -z^\top Q_p z + 2z^\top P_p \hat{L}_p (q_\mu(y) - y) \\ &\leq -\lambda_{\min}(Q_p)|z|^2 + 2\|P_p \hat{L}_p\| \cdot |z| \cdot |q_\mu(y) - y| \\ &\leq -\lambda_Q |z|^2 + 2 \max_{p \in \mathcal{P}} \|P_p \hat{L}_p\| \cdot |z| \cdot |q_\mu(y) - y| \\ &= -\lambda_Q |z| (|z| - \Theta |q_\mu(y) - y|). \end{aligned} \quad (\text{III.4})$$

On the other hand, since  $|y| = |C_p x| \leq C_{\max}|z|$ , it follows from (II.3) that,

$$C_{\max}|z| \leq M\mu \quad \Rightarrow \quad |q_\mu(y) - y| \leq \Delta\mu.$$

Hence (III.4) shows that if  $M$  satisfies (III.2) for some  $\kappa > 0$ , then

$$\begin{aligned} \Theta\Delta(1+\kappa)\mu &\leq |z| \leq \frac{M\mu}{C_{\max}} \\ \Rightarrow \frac{d}{dt}(z^\top P_p z) &\leq -\lambda_Q \kappa (1+\kappa) (\Theta\Delta\mu)^2. \end{aligned} \quad (\text{III.5})$$

If we define the balls  $\mathcal{B}_1(\mu)$  and  $\mathcal{B}_2(\mu)$  by

$$\begin{aligned} \mathcal{B}_1(\mu) &:= \left\{ z : |z| \leq \frac{M\mu}{C_{\max}} \right\} \\ \mathcal{B}_2(\mu) &:= \left\{ z : |z| \leq \Theta\Delta(1+\kappa)\mu \right\}, \end{aligned}$$

then it follows from (II.10) and (III.2) that

$$\mathcal{B}_2(\mu) \subset \mathcal{R}_2(\mu, p) \subset \mathcal{R}_1(\mu, p) \subset \mathcal{B}_1(\mu)$$

for  $p \in \mathcal{P}$ . Thus (III.5) implies that  $\mathcal{R}_1(\mu, p)$  and  $\mathcal{R}_2(\mu, p)$  are invariant sets of the trajectories of (III.1).

Moreover, the upper bound in (III.5) shows that if  $x(0) \in \mathcal{R}_1(\mu, p)$ , then  $x(T) \in \mathcal{R}_2(\mu, p)$  for  $T$  satisfying (III.3). ■

We use the next result on average dwell time for finite-time estimation of the state at the “zooming-out” stage. Such estimation is needed for Lyapunov stability of the closed-loop system.

*Lemma 3.2:* Fix an initial time  $\tau_0 \geq 0$ . Suppose that  $\sigma$  satisfies (II.2). Let  $\tau \in (0, \tau_a)$ , and choose an integer  $N$  so that

$$N > \frac{\tau_a}{\tau_a - \tau} \left( N_0 - \frac{\tau}{\tau_a} \right). \quad (\text{III.6})$$

Then there exists a nonnegative real number  $T \leq (N-1)\tau$  such that  $N_\sigma(\tau_0 + T + \tau, \tau_0 + T) = 0$ .

*Proof:* Let us denote the switching times by  $t_1, t_2, \dots$ , and fix an integer  $N \geq 1$ . Suppose that

$$N_\sigma(\tau_0 + T + \tau, \tau_0 + T) > 0 \quad (\text{III.7})$$

for  $T \leq (N-1)\tau$ . Then we have  $t_k - t_{k-1} \leq \tau$  for  $k = 1, \dots, N$ , where  $t_0 := \tau_0$ . Indeed, if

$$t_k - t_{k-1} > \tau \quad (\text{III.8})$$

for some  $k$  and if we let  $\bar{k}$  be the smallest integer  $k \leq N$  satisfying (III.8), then we obtain  $t_{\bar{k}-1} - t_0 \leq (\bar{k}-1)\tau$  and  $N_\sigma(t_{\bar{k}-1} + \tau, t_{\bar{k}-1}) = 0$ , which contradicts (III.7). Hence for  $0 < \epsilon < t_1$ ,

$$t_N - (t_1 - \epsilon) = \sum_{k=2}^N (t_k - t_{k-1}) + \epsilon \leq (N-1)\tau + \epsilon$$

It follows from (II.2) that

$$N = N_\sigma(t_N, t_1 - \epsilon) \leq N_0 + \frac{(N-1)\tau + \epsilon}{\tau_a}.$$

Therefore  $N$  satisfies the following inequality:

$$N \leq \frac{\tau_a}{\tau_a - \tau} \left( N_0 - \frac{\tau - \epsilon}{\tau_a} \right). \quad (\text{III.9})$$

Since  $\epsilon \in (0, t_1)$  was arbitrary, (III.9) is equivalent to

$$N \leq \frac{\tau_a}{\tau_a - \tau} \left( N_0 - \frac{\tau}{\tau_a} \right). \quad (\text{III.10})$$

Thus we have shown that if (III.7) holds for all  $T \leq (N-1)\tau$ , then  $N$  satisfies (III.10). The contraposition of this statement gives a desired result. ■

#### A. The proof for convergence to the origin

Define  $\Gamma$  by

$$\Gamma := \max_{p \in \mathcal{P}} \|A_p\|.$$

We split the proof into two stages: the “zooming-out” and “zooming-in” stages.

1) *The “Zooming-out” stage:* Set the control input  $u = 0$ , and fix  $\bar{\tau} > 0$  and  $\chi > 0$ . Then increase  $\mu$  in the following way:  $\mu(t) = 1$  for  $t \in [0, \bar{\tau})$ ,  $\mu(t) = e^{(1+\chi)k\Gamma\bar{\tau}}$  for  $t \in [k\bar{\tau}, (k+1)\bar{\tau})$  and  $k = 1, 2, \dots$

Choose  $\tau \in (0, \tau_a)$ , and suppose that we observe

$$|q_{\mu(t)}(y(t))| \leq M\mu(t) - \Delta\mu(t) \quad (\text{III.11})$$

$$\sigma(t) = \sigma(t_0) =: p \quad (\text{III.12})$$

for  $t \in [t_0, t_0 + \tau)$ . First we shall describe how to determine  $\mu(t_0 + \tau)$  after this observation, and next we shall prove the existence of such  $t_0 \geq 0$ .

Define the observability Gramian  $W_p(\tau)$  by

$$W_p(\tau) := \int_0^\tau e^{A_p^\top t} C_p^\top C_p e^{A_p t} dt$$

and the estimated state  $\xi(t_0)$  by

$$\xi(t_0) := W_p(\tau)^{-1} \int_0^\tau e^{A_p^\top t} C_p^\top q_{\mu(t_0+t)}(y(t_0+t)) dt \quad (\text{III.13})$$

Since  $u(t) = 0$ , we also have

$$x(t_0) = W_p(\tau)^{-1} \int_0^\tau e^{A_p^\top t} C_p^\top y(t_0 + t) dt. \quad (\text{III.14})$$

Moreover, if (III.11) holds, then (II.4) gives

$$|y(t)| \leq M\mu(t) \quad (t_0 \leq t < t_0 + \tau),$$

and hence

$$|q_{\mu(t)}(y(t)) - y(t)| \leq \Delta\mu(t) \quad (t_0 \leq t < t_0 + \tau).$$

Therefore (III.13) and (III.14) show that

$$|x(t_0) - \xi(t_0)| \leq \|W_p(\tau)^{-1}\| \tau \Lambda_p^C(\tau) \Delta\mu^-(t_0 + \tau),$$

where

$$\Lambda_p^C(\tau) := \max_{0 \leq t \leq \tau} \|C_p e^{A_p t}\|, \quad \mu^-(t_0 + \tau) := \lim_{t \nearrow t_0 + \tau} \mu(t).$$

Since  $x(t_0 + \tau) = e^{A_p \tau} x(t_0)$ , if we set

$$\xi(t_0 + \tau) = e^{A_p \tau} \xi(t_0), \quad (\text{III.15})$$

then

$$\begin{aligned} & |x(t_0 + \tau) - \xi(t_0 + \tau)| \\ & \leq \|W_p(\tau)^{-1}\| \tau \Lambda_p^C(\tau) \|e^{A_p \tau}\| \Delta\mu^-(t_0 + \tau) =: e(t_0 + \tau). \end{aligned}$$

It follows that

$$\begin{aligned} |z(t_0 + \tau)| & \leq |x(t_0 + \tau)| + |x(t_0 + \tau) - \xi(t_0 + \tau)| \\ & \leq |e(t_0 + \tau)| + 2e(t_0 + \tau) =: E(t_0 + \tau). \end{aligned}$$

Thus if we choose  $\mu(t_0 + \tau)$  so that

$$\mu(t_0 + \tau) \geq \sqrt{\frac{\bar{\lambda}_P}{\underline{\lambda}_P} \frac{C_{\max}}{M}} E(t_0 + \tau), \quad (\text{III.16})$$

then  $z(t_0 + \tau) \in \mathcal{R}_1(\mu(t_0 + \tau), \sigma(t_0 + \tau))$ .

It remains to prove the existence of  $t_0 \geq 0$  satisfying (III.11) and (III.12) for  $t \in [t_0, t_0 + \tau)$ . By the definition of  $\mu$  and (II.12), there is  $\tau_0 \geq 0$  such that

$$|y(t)| \leq M\mu(t) - 2\Delta\mu(t) \quad (t \geq \tau_0).$$

In conjunction with (II.3), this implies that (III.11) holds for  $t \geq \tau_0$ . Let  $N$  be an integer satisfying (III.6). Then Lemma 3.2 guarantees the existence of  $t_0 \in [\tau_0, \tau_0 + (N-1)\tau]$  such that (III.12) holds for  $t \in [t_0, t_0 + \tau)$ .

2) The ‘‘Zooming-in’’ stage: Since we make  $\mu$  increased after each switch during this stage, the term ‘‘zooming-in stage’’ may be misleading. However  $\mu$  decreases overall so we take the name from [3].

Choose  $\kappa$  so that (III.2) holds, and define  $T_0 := t_0 + \tau$ . We consider (II.7) with  $\xi(T_0)$  calculated by (III.13) and (III.15). The discussion above ensures  $z(T_0) \in \mathcal{R}_1(\mu(T_0), \sigma(T_0))$ . Fix  $T$  so that (III.3) is satisfied.

Let us first investigate the case without switching on the interval  $(T_0, T_0 + T)$ . In this case, if we let  $\mu(t) = \mu(T_0)$  for  $t \in [T_0, T_0 + T)$ , then Lemma 3.1 shows that  $z(T_0 + T) \in \mathcal{R}_2(\mu(T_0), \sigma(T_0))$ . Define  $\Omega$  by

$$\Omega := \sqrt{\frac{\bar{\lambda}_P}{\underline{\lambda}_P} \frac{\Theta\Delta(1+\kappa)C_{\max}}{M}}, \quad (\text{III.17})$$

and set  $\mu(T_0 + T) = \Omega\mu(T_0)$ . Then we obtain  $z(T_0 + T) \in \mathcal{R}_1(\mu(T_0 + T), \sigma(T_0 + T))$ . Note that  $\Omega < 1$  by (II.12). As regards after  $T_0 + T$ , if switching does not occur on the interval  $(T_0 + mT, T_0 + (m+1)T]$  for  $m = 1, 2, \dots$ , then we update  $\mu$  in the same way.

We now study the switched case. Let  $T_1, T_2, \dots, T_n$  be switching times on the interval  $(T_0, T_0 + T]$ . We sometimes write  $T_{n+1}$  rather than  $T_0 + T$  for simplicity of notation. For every  $p_1, p_2 \in \mathcal{P}$  with  $p_1 \neq p_2$ , let  $c_{p_2, p_1} > 0$  satisfy

$$z^\top P_{p_1} z \leq c_{p_2, p_1} \cdot z^\top P_{p_2} z. \quad (\text{III.18})$$

for all  $z \in \mathbb{R}^{2n}$ . We can compute  $c_{p_2, p_1}$  not only by linear matrix inequalities but also by an explicit formula in Lemma 13 of [12].

We adjust  $\mu$  at every switching time in the following way:

$$\mu(t) = \sqrt{\prod_{\ell=0}^{k-1} c_{\sigma(T_{\ell+1}), \sigma(T_\ell)}} \cdot \mu(T_0) \quad (T_k \leq t < T_{k+1})$$

for  $k = 0, \dots, n$ . Lemma 3.1 suggests that  $\mathcal{R}_i(\mu(T_k), \sigma(T_k))$  ( $i = 1, 2$ ) are invariant sets for  $t \in [T_k, T_{k+1})$ ,  $k = 0, \dots, n$ . Moreover, by (III.18), if  $z(t) \in \mathcal{R}_i(\mu_0, p_1)$  for some  $\mu_0$ , then  $z(t) \in \mathcal{R}_i(\sqrt{c_{p_2, p_1}}\mu_0, p_2)$  ( $i = 1, 2$ ) for  $p_1, p_2 \in \mathcal{P}$  with  $p_1 \neq p_2$ . Hence it follows that  $z(t) \in \mathcal{R}_1(\mu(t), \sigma(t))$  for  $t \in [T_0, T_{n+1})$ . Also, if there is  $t_1 \in [T_0, T_{n+1})$  such that  $z(t_1) \in \mathcal{R}_2(\mu(t_1), \sigma(t_1))$ , then  $z(t) \in \mathcal{R}_2(\mu(t), \sigma(t))$  for all  $t \in [t_1, T_{n+1})$ . To see the existence of such  $t_1$ , suppose for a contradiction that

$$z(t) \notin \mathcal{R}_2(\mu(t), \sigma(t)), \quad (T_0 \leq t < T_{n+1}). \quad (\text{III.19})$$

First we examine the case  $T_0 + T = T_{n+1} > T_n$ . Since a Filippov solution is (absolutely) continuous,  $\lim_{t \nearrow T_{n+1}} z(t)^\top P_{\sigma(t)} z(t)$  exists and (III.19) gives

$$\lim_{t \nearrow T_{n+1}} z(t)^\top P_{\sigma(t)} z(t) \geq \bar{\lambda}_P (\Theta\Delta(1+\kappa))^2 \mu(T_n)^2. \quad (\text{III.20})$$

On the other hand, since  $z(t) \in \mathcal{B}_1(\mu(t))$  and  $z(t) \notin \mathcal{B}_2(\mu(t))$  for  $t < T_{n+1}$ , (III.5) shows that

$$\begin{aligned} & \lim_{t \nearrow T_1} z(t)^\top P_{\sigma(t)} z(t) \\ & \leq \left( \frac{\lambda_P M^2}{C_{\max}^2} - (T_1 - T_0) \underline{\lambda}_Q \kappa (1 + \kappa) (\Theta\Delta)^2 \right) \mu(T_0)^2, \end{aligned}$$

and hence we have

$$\begin{aligned} & z(T_1)^\top P_{\sigma(T_1)} z(T_1) \\ & \leq c_{\sigma(T_1), \sigma(T_0)} \cdot \left( \lim_{t \nearrow T_1} z(t)^\top P_{\sigma(t)} z(t) \right) \\ & = \left( \frac{\lambda_P M^2}{C_{\max}^2} - (T_1 - T_0) \underline{\lambda}_Q \kappa (1 + \kappa) (\Theta\Delta)^2 \right) \mu(T_1)^2. \end{aligned}$$

If we repeat this process and use (III.3), then

$$\begin{aligned} & \lim_{t \nearrow T_{n+1}} z(t)^\top P_{\sigma(t)} z(t) \\ & \leq \left( \frac{\lambda_P M^2}{C_{\max}^2} - T \underline{\lambda}_Q \kappa (1 + \kappa) (\Theta\Delta)^2 \right) \mu(T_n)^2 \\ & < \bar{\lambda}_P (\Theta\Delta(1+\kappa))^2 \mu(T_n)^2, \end{aligned} \quad (\text{III.21})$$

which contradicts (III.20). Hence we obtain

$$z(T_{n+1}) = \lim_{t \nearrow T_0+T} z(t) \in \mathcal{R}_2(\mu(T_n), \sigma(T_n)). \quad (\text{III.22})$$

In the case  $T_0 + T = T_{n+1} = T_n$ , (III.20) and (III.21) hold with  $T_{n-1}$  in place of  $T_n$ , and then we have

$$\begin{aligned} z(T_{n+1}) &\in \mathcal{R}_2(\mu(T_{n-1}), \sigma(T_{n-1})) \\ &\subset \mathcal{R}_2(\sqrt{c_{\sigma(T_n), \sigma(T_{n-1})}} \cdot \mu(T_{n-1}), \sigma(T_n)). \end{aligned}$$

Thus if  $n$  switches occur, then we set

$$\mu(T_0 + T) = \Omega \sqrt{\prod_{\ell=0}^{n-1} c_{\sigma(T_{\ell+1}), \sigma(T_\ell)}} \cdot \mu(T_0).$$

The discussion above implies  $z(T_0 + T) \in \mathcal{R}_1(\mu(T_0 + T), \sigma(T_0 + T))$ . We update  $\mu$  in the same way after  $T_0 + T$ .

Finally, define

$$c := \max_{p_1 \neq p_2} c_{p_2, p_1}. \quad (\text{III.23})$$

Then (II.2) gives

$$\begin{aligned} \mu(T_0 + mT) &\leq \Omega^m \sqrt{c^{N_\sigma(T_0+mT, T_0)}} \mu(T_0) \\ &\leq \sqrt{c^{N_0}} \cdot \left( \Omega \sqrt{c^{T/\tau_a}} \right)^m \mu(T_0) \end{aligned} \quad (\text{III.24})$$

for  $m \in \mathbb{N}$ . If  $\Omega \sqrt{\lambda^{T/\tau_a}} < 1$ , that is, if the average dwell time  $\tau_a$  satisfies

$$\tau_a > \frac{\log(c)}{2 \log(1/\Omega)} T, \quad (\text{III.25})$$

then  $\lim_{m \rightarrow \infty} \mu(T_0 + mT) = 0$ . Since  $x(t) \in \mathcal{B}_1(\mu(t))$  for  $t \geq T_0$ , we obtain  $\lim_{t \rightarrow \infty} x(t) = 0$ . ■

*Remark 3.3:* (a) The proposed method of adjusting  $\mu$  is causal but sensitive to the time-delay of the switching signal at the “zooming-in” stage. To allow such a delay, we must examine the bound of an error due to the mismatch of modes between the plant and the controller. However we do not proceed along this line to avoid technical issues.

(b) Here we have changed  $\mu$  at every switching time in the “zooming-in” stage. If we would not, switching might lead to instability of the closed-loop system. Without adjustment of  $\mu$ , the quantizer does not saturate right after the switch because the trajectory belongs to  $\mathcal{B}_1(\mu)$ . However,  $\mathcal{B}_1(\mu)$  is not an invariant set. Hence if we do not change  $\mu$ , the trajectory may leave  $\mathcal{B}_1(\mu)$ . This leads to saturation of the quantizer.

(c) We can handle the case where  $\sigma$  are not available for the zooming strategy if we use the following conservative balls  $\mathcal{B}_3$  and  $\mathcal{B}_4$ : Define  $\mathcal{B}_3(\mu)$  by the largest ball contained in all  $\mathcal{R}_1(\mu, p)$  and  $\mathcal{B}_4(\mu)$  by the smallest ball containing all  $\mathcal{R}_2(\mu, p)$ . Assume that  $\mathcal{B}_3 \supset \mathcal{B}_4$ . Clearly,  $\mathcal{B}_3$  and  $\mathcal{B}_4$  are invariant sets for every mode. Moreover, the state goes to  $\mathcal{B}_4$  from  $\mathcal{B}_3$  for a finite period of time. Note that here we still assume that the observer has the exact knowledge of  $\sigma$ . Such a set-up makes sense if  $\mu$  is updated by a high-level supervisor while controllers are implemented locally.

### B. The proof for Lyapunov stability

The proof of Lyapunov stability follows in a line similar to that in Sec. 5.5 of [6].

Let us denote by  $\mathcal{B}_\varepsilon$  the open ball with center at the origin and radius  $\varepsilon$  in  $\mathbb{R}^{2n \times 2n}$ . In what follows, we use the letters in the previous subsection and assume that (III.25) holds.

Let  $\delta > 0$  be small enough to satisfy

$$C_{\max} e^{\Gamma N \tau} \delta < \Delta_0. \quad (\text{III.26})$$

Then  $q_{\mu(t)}(y(t)) = 0$  for  $t \in [0, N\tau]$ . The argument on the existence of  $t_0$  at the “zooming-out” stage implies that the time  $T_0$ , at which the stage changes from “zooming-out” to “zooming-in”, satisfies  $T_0 \leq N\tau$  for every switching signal.

Fix  $\alpha > 0$ . By (III.13),  $\xi(T_0) = 0$ , and hence we see from (III.16) that  $\mu(T_0)$  achieving  $z(T_0) \in \mathcal{R}_1(\mu(T_0), \sigma(T_0))$  can be chosen so that

$$\alpha \leq \mu(T_0) \leq \bar{\mu}, \quad (\text{III.27})$$

where  $\bar{\mu}$  is defined by

$$\begin{aligned} \bar{\mu} := \max \left\{ \alpha, \sqrt{\frac{\bar{\lambda}_P}{\lambda_P}} \frac{\Delta \tau C_{\max} e^{(1+\chi) \lfloor N\tau/\bar{\tau} \rfloor \Gamma \bar{\tau}}}{M} \right. \\ \left. \times \max_{p \in \mathcal{P}} \left( \|W_p(\tau)^{-1}\| \Lambda_p^C(\tau) \|e^{A_p \tau}\| \right) \right\}. \end{aligned}$$

Note that  $\bar{\mu}$  is independent on switching signals.

By (III.24), if  $m$  satisfies

$$m > \frac{\log(\bar{\mu} M \sqrt{c^{N_0}} / (\varepsilon C_{\max}))}{\log(1/(\Omega \sqrt{c^{T/\tau_a}}))}, \quad (\text{III.28})$$

then we have

$$\mathcal{R}_1(\mu(T_0 + mT), \sigma(T_0 + mT)) \subset \mathcal{B}_\varepsilon. \quad (\text{III.29})$$

Let  $\bar{m}$  be the smallest integer satisfying (III.28).

Define  $T_1 := T_0 + \bar{m}T \leq N\tau + \bar{m}T$  and  $\underline{c} := \min\{1, \min_{p_1 \neq p_2} c_{p_2, p_1}\}$ . By (III.27), we have

$$\begin{aligned} \mu(t) &\geq \Omega^{\bar{m}} \sqrt{\underline{c}^{N_0 + \bar{m}T/\tau_a}} \mu(T_0) \\ &\geq \alpha \Omega^{\bar{m}} \sqrt{\underline{c}^{N_0 + \bar{m}T/\tau_a}} =: \eta. \end{aligned} \quad (\text{III.30})$$

for  $t \in [T_0, T_1]$ . Let  $\delta > 0$  satisfy

$$C_{\max} e^{\Gamma(N\tau + \bar{m}T)} \delta < \eta \Delta_0 \quad (\text{III.31})$$

$$e^{\Gamma(N\tau + \bar{m}T)} \delta < \min \left\{ \varepsilon, \sqrt{\frac{\lambda_P}{\bar{\lambda}_P}} \frac{M\eta}{C_{\max}} \right\}. \quad (\text{III.32})$$

By (III.26), (III.30), and (III.31),  $q_{\mu(t)}(y(t)) = 0$  on the interval  $[0, T_1]$ , so  $\xi(t) = 0$  and  $u(t) = 0$  on the same interval. Combining this with (III.32), we obtain  $|x(t)| \leq e^{\Gamma(N\tau + \bar{m}T)} \delta < \varepsilon$  for  $t \leq T_1$ . Thus

$$|z(t)| = |x(t)| < \varepsilon \quad (t \leq T_1). \quad (\text{III.33})$$

On the other hand, (III.30) and (III.32) gives

$$\begin{aligned} z(T_1)^\top P_{\sigma(T_1)} z(T_1) &\leq \lambda_{\max}(P_{\sigma(T_1)}) |z(T_1)|^2 \\ &< \frac{\lambda_P M^2 \eta^2}{C_{\max}^2} \leq \frac{\lambda_P M^2 \mu(T_1)^2}{C_{\max}^2} \end{aligned}$$

for every  $p \in \mathcal{P}$ , and hence  $z(T_1) \in \mathcal{R}_1(\mu(T_1), \sigma(T_1)) \subset \mathcal{B}_\varepsilon$  by (III.29). In addition, since

$$\begin{aligned} \mu(T_1 + kT) &\leq \sqrt{c^{N_0}} \cdot \left(\Omega \sqrt{c^{T/\tau_a}}\right)^{\bar{m}+k} \mu(T_0) \\ &\leq \sqrt{c^{N_0}} \cdot \left(\Omega \sqrt{c^{T/\tau_a}}\right)^{\bar{m}} \bar{\mu} \end{aligned}$$

for all  $k \geq 0$  and since  $\bar{m}$  satisfies (III.28), it follows that that  $\mathcal{R}_1(\mu(T_1 + kT), \sigma(T_1 + kT))$  also lies in  $\mathcal{B}_\varepsilon$ . Recall that  $\mathcal{R}_1(\mu(t), \sigma(t))$  is an invariant set for  $t \geq T_0$ . Thus we have

$$|z(t)| < \varepsilon \quad (t \geq T_1). \quad (\text{III.34})$$

From (III.33) and (III.34), we see that Lyapunov stability can be achieved. ■

*Remark 3.4:* Through Lemma 3.2, we implicitly use the average dwell time property to obtain the upper bound  $\bar{\mu}$  in (III.27).

#### IV. NUMERICAL EXAMPLES

Consider the continuous-time switched system (II.7) with the following two modes:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = [1 \quad 1], \\ A_2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [0 \quad -1]. \end{aligned}$$

As the feedback gain and the observer gain of each mode, we take

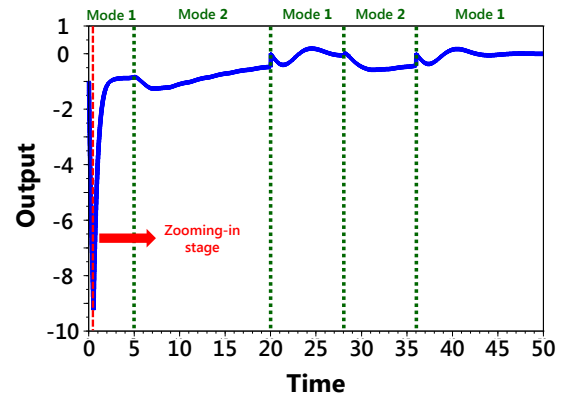
$$K_1 = [-3 \quad 1], \quad L_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad K_2 = [0 \quad -1], \quad L_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let the quantizer  $q$  be uniform-type, and define the parameters  $M$  and  $\Delta$  of the quantizer by  $M = 20$ ,  $\Delta = 0.1$ . Also, define  $Q_1$  and  $Q_2$  in (II.9) and  $\kappa$  in (III.2) by  $Q_1 = \text{diag}(2, 8, 2, 8)$ ,  $Q_2 = \text{diag}(1, 1, 1, 1)$ ,  $\kappa = 2.5$ , where  $\text{diag}(e_1, \dots, e_4)$  means a diagonal matrix whose diagonal elements starting in the upper left corner are  $e_1, \dots, e_4$ . Then we obtain  $T \approx 2.20$  in (III.3),  $\Omega \approx 0.824$  in (III.17),  $c \approx 4.03$  in (III.23), and  $\tau_a \approx 7.90$  in (III.25).

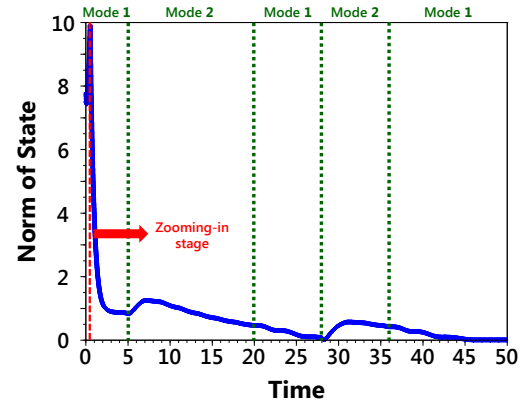
Fig. 2 (a) and (b) show that the output  $y$  and the Euclidean norm of the state  $x$  of the switched system (II.1) with  $x(0) = [-6 \ 5]^\top$  and  $\mu(0) = 1$ . In this example, the “zooming-out” stage finished at  $t = 0.5$ . We see the non-smooth behaviors of  $y$  and  $x$  at the switching times  $t = 5, 20, 28, 36$ . In particular, we observe from the behaviors of  $x$  at  $t = 5, 28$  that, not surprisingly, adjustments of  $\mu$  at some switching times may be conservative.

#### V. CONCLUDING REMARKS

The stabilization of continuous-time switched linear systems by quantized output feedback has been studied. We have assumed that the quantized measurement and the switched signal are transmitted to the controller at all times. We have proposed an output encoding method for globally asymptotic stability. The encoding method is based on the non-switched case, and an additional adjustment of the zoom parameter is needed at every switching time in the zooming-in stage. We have discussed the effect of switching by using multiple Lyapunov functions and an average dwell-time assumption.



(a) The output  $y$ .



(b) The Euclidean norm of the state  $x$ .

Fig. 2: Simulation with  $x(0) = [-6 \ 5]^\top$  and  $\sigma(0) = 1$ .

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