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Numerical Methods for Singular and Impulse Controls of Regime-Switching Insurance Models

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Abstract— **This paper focuses on management of surplus and optimization the stream of dividend payment in a continuous-time financial market. The surplus process is modeled by a Markov regime-switching jump diffusion process subject to mixed regular and singular controls. The regular controls including reinsurance strategies and investment allocations are considered simultaneously to maximize the payoff function. Capital injections, considered as impulse controls, are mandatory to avoid solvency when cash reserve is insufficient. Our goal is to maximize the present value of the difference between the cumulative dividend payment and the possible capital injections. Using dynamic programming principle, the value function is a solution of coupled system of nonlinear integro-differential quasi-variational inequalities. Due to the complexity of the regime-switching process and randomness of capital injections, it is virtually impossible to obtain analytical solutions. Thus numerical schemes are a viable and in many cases only alternative. We construct a two component discrete-time controlled Markov chain to approximate the jump diffusion process and the hidden Markov chain. Convergence of the approximation algorithms is demonstrated; examples are presented to illustrate the applicability of the numerical methods.**

I. INTRODUCTION

Managing the surplus and designing dividend payment policies have long been important research issues in actuarial science and finance literature. The dividend paying decision is crucial. Not only does it represent an important signal about a firm's future growth opportunities and profitability, but also may influence the investment and financing decisions of firms and the wealth of the shareholders. Because of the nature of their product, insurers tend to accumulate relatively large amount of cash, cash equivalents, and investments in order to pay future claims and avoid financial ruin. The payment of dividends to shareholders may reduce insurers' ability to survive adverse investment and underwriting experience.

In the Australian insurance markets, due to the wave of demutualization and global developments in financial markets in the last two decades, some largest insurers including Australian Mutual Provident (Australia's oldest and largest life insurance mutual) have faced significant difficulty on the balance sheets and undergone pressure of distributing the surplus. Hence, the task of optimizing the stream of dividend payments and management of surplus is of high priority. Moreover, people have realized that traditional surplus models fail to capture discrete movements (such as random environment, market trends, interest rates, business cycles, etc.). To reflect the reality, one of the recent trends is to use regime-switching models, where continuous dynamics and discrete events coexist in the systems. Whilst traditional models rely on either ordinary or stochastic differential equations in the continuous setting alone, one of the advantages of regime-switching models is that the models contain discrete events, which describe the economical movements and impacts that cannot be modeled as either ordinary or stochastic differential equations. The switching process between regimes is modeled by a finite state Markov chain. The formulation of regime-switching models is a more general and versatile framework to describe the complicated financial markets and their inherent uncertainty and randomness. Due to the complexity of the construction, the regime-switching diffusion systems can only be solved in very special cases. In general, it is virtually impossible to obtain close-form solutions. Thus numerical schemes are a viable and in many cases only alternative. This paper aims to develop feasible numerical schemes to approximate the optimal strategies in two insurance and financial models.

In this work, we present a survey on some recent progress on numerical methods for regime-switching insurance model with singular and impulse controls. We review and extend the results obtained in [8] and [9]. Because of the page limitation, the verbatim proofs are omitted. Nevertheless, new numerical examples different from that in the aforementioned papers are provided for demonstration. To deal with the numerical algorithm of Markov chain method for solving the system of QVIs, we will construct a two-component discrete-time controlled Markov chain to approximate the jump diffusion process and the Markov switching term. With the piecewise interpolation and upwind discretization, the corresponding dynamic programming equation and transition probabilities will be obtained. To guarantee that the approximating Markov chain is in good alignment with the original jump diffusion process, a definition of local consistency will be introduced and verified. Under simple conditions, the convergence of the approximation sequence to the diffusion process and the convergence of the approximation to the value function will be confirmed. In the actual computation, we simply use the well-known policy iteration (or policy improvement) or value iteration method for our approximation schemes. In the verification of the convergence of approximation sequence, we need to show that

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the approximating sequence is tight and then appropriately characterize the subsequential weak limit, which does not hold in the case of unrestrict dividend payment process. To overcome the difficulty, the basic idea is to suitably re-scale the time so that the processes involved in the convergence analysis are tight in the new time scale; carry out weak convergence analysis with the rescaled processes; and revert back to the original time scale to obtain the convergence of approximating sequence to the original value function. Although the study is devoted to risk management and applications in actuarial sciences, the numerical methods and the techniques used provide insight for the study in systems involving singular and impulsive controls as well.

The rest of the paper is organized as follows. Section II designs a numerical approximation scheme to maximize the payoff function of the total discounted dividend paid out until the lifetime of ruin. Convergence of the approximating sequence is verified. For the singular control part, a technique of time rescaling is used. Using similar techniques, the optimal investment strategy, dividend policies and capital injections are presented in Section III.

II. OPTIMAL RISK CONTROL AND DIVIDEND PAYMENT POLICIES

A. Motivation

To design optimal risk controls and dividend payout strategies for a financial corporation has drawn increasing attention since the introduction of the classical collective risk model in [12], where the probability of ruin was considered as a measure of risk. Realizing that the surplus reaching arbitrarily high and exceeding any finite level are not realistic in practice, [5] proposed a dividend optimization problem. Instead of considering the safety aspect (ruin probability), aiming at maximizing the expected discounted total dividends until lifetime ruin by assuming the surplus process follows a simple random walk, he showed that the optimal dividend strategy is a barrier strategy. Since then, many researchers have analyzed this problem under more realistic assumptions and extended its range of applications. Some recent work can be found in [2], [3], [6] and references therein. To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risk. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Proportional reinsurance is one type of reinsurance policy. Within this scheme, the reinsurance company covers a fixed percentage of losses. The other type of reinsurance policy is nonproportional reinsurance. The most common nonproportional reinsurance policy is the so-called excess-of-loss reinsurance, within which the cedent (primary insurance carrier) will pay all of the claims up to a pre-given level of amount (termed retention level). The comparison of these two types of reinsurance can be found in [1]. In this section, we consider both of these reinsurance policies and provide the numerical solutions of the corresponding Markovian regime-switching models.

B. Formulation

In this section, we introduce a dynamic system to describe the surplus processes with reinsurance and dividend payout strategies with Markov regime switching. Let $X(t)$ denote the controlled surplus of an insurance company at time $t \geq 0$. Denote by $u(t)$ and $Z(t)$ the dynamic reinsurance policy at time t and the total dividend paid out up to time t, respectively. Assume the evolution of $X(t)$, subject to reinsurance and dividend payments, follows a one-dimensional temporal homogeneous controlled regime-switching diffusion on an unbounded domain $G = (0, \infty)$:

$$\begin{cases} dX(t) = b(X(t), \alpha(t), u(t))dt \\ \qquad\qquad +\sigma(X(t), \alpha(t), u(t))dW(t) - dZ(t), \\ X(0^-) = x \in G, \quad \alpha(0^-) = \ell \in \mathcal{M}, \end{cases} \tag{1}$$

where u is the regular control and Z is the singular control. Throughout the paper we use the convention that $Z(0^-) = 0$. The jump size of Z at time $t \geq 0$ is denoted by $\Delta Z(t) := Z(t) - Z(t^-)$, and $Z^c(t) := Z(t) - \sum_{0 \leq s \leq t} \Delta Z(s)$ denotes the continuous part of Z. Also note that $\Delta X(t) := X(t) - X(t^-) = -\Delta Z(t)$ for any $t \geq 0$.

A common choice of the payoff is to maximize the total expected discounted value of all dividends until lifetime ruin. Let

$$\tau := \inf\{t > 0 : X(t) \notin G\} \tag{2}$$

be the ruin time, where $G = (0, \infty)$ is the domain of the surplus. Denote by $r > 0$ the discounting factor. For suitable functions f and c and an arbitrary admissible pair $\pi = (u, Z)$, the expected discounted payoff is

$$\begin{aligned} J(x, \ell, \pi) = E_{x,\ell}\Big(\int_0^\tau e^{-rt}[f(X(t), \alpha(t), u(t))dt \\ +c(X(t^-), \alpha(t^-))dZ(t)]\Big). \end{aligned} \tag{3}$$

The pair $\pi = (u, Z)$ is said to be *admissible* if u and Z satisfy

(i) $u(t)$ and $Z(t)$ are nonnegative for any $t \geq 0$,
(ii) Z is càdlàg and nondecreasing,
(iii) $X(t) \geq 0$, for any $t < \tau$, where τ is the ruin time defined in (2),
(iv) both u and Z are adapted to $\mathcal{F}_t := \sigma\{W(s), \alpha(s), 0 \leq s \leq t\}$ augmented by the P-null sets, and
(iv) $J(x, \ell, \pi) < \infty$ for any $(x, \ell) \in G \times \mathcal{M}$ and admissible pair $\pi = (u, Z)$, where J is the functional defined in (3).

Suppose that $\mathcal{A}$ is the collection of all admissible pairs, and U is the collection of possible retention levels $u(t)$. Throughout the paper, we assume that U is a given compact set, and that for each $\ell \in \mathcal{M}$, $c(x, \ell) \geq c(y, \ell)$ for all $0 \leq x \leq y$. That is, the utility function for the dividend is nondecreasing; see examples in [7]. In addition, $c(X(t), \ell) = f(X(t), \ell, u) = 0$ when $t > \tau$. Define the value function as

$$V(x, \ell) := \sup_{\pi \in \mathcal{A}} J(x, \ell, \pi). \tag{4}$$

If the value function V defined in (4) is sufficiently smooth, by applying the dynamic programming principle, we conclude formerly that V satisfies the following coupled system of quasi-variational inequalities (QVIs):

$$\begin{aligned}\max\{H(x,\ell,V'(x,\ell),V''(x,\ell))+Q(x)V(x,\cdot)(\ell)\\ -rV(x,\ell),c(x,\ell)-V'(x,\ell)\}=0,\end{aligned} \tag{5}$$

for all $(t,x,\ell)\in[0,\tau)\times G\times\mathcal{M}$, with boundary condition

$$V(0,\ell)=0,\quad \forall \ell\in\mathcal{M}, \tag{6}$$

where for any $(x,\ell,p,A)\in\mathcal{R}\times\mathcal{M}\times\mathcal{R}\times\mathcal{R}$,

$$\begin{aligned}H(x,\ell,p,A)&:=\sup_{u\in U}\Big\{f(x,\ell,u)+p\cdot b(x,\ell,u)\\ &\qquad+\frac{1}{2}\sigma^2(x,\ell,u)A\Big\},\\ Q(x)V(x,\cdot)(\ell)&:=\sum_{\iota\in\mathcal{M}}q_{\ell\iota}(x)[V(x,\iota)-V(x,\ell)],\end{aligned}$$

and V' and V'' denote the first and the second partial derivatives of V with respect to x.

C. Numerical Algorithm

In this section we construct a locally consistent Markov chain approximation for the mixed regular-singular control model with regime-switching. The discrete-time and finite-state controlled Markov chain is so defined that it is locally consistent with (1). Note that the state of the process has two components x and α. Hence in order to use the methodology in [11], our approximating Markov chain has two components: one component delineates the diffusive behavior whereas the other keeps track of the regimes. Let $h>0$ be a discretization parameter. Define $L_h=\{x: x=kh,k=0,\pm1,\pm2,\dots\}$ and $S_h=L_h\cap G_h$, where $G_h=(0,B+h)$ and B is an upper bound introduced for numerical computation purpose. Moreover, assume without loss of generality that the boundary point B is an integer multiple of h. Let $\{(\xi_n^h,\alpha_n^h),n<\infty\}$ be a controlled discrete-time Markov chain on $S_h\times\mathcal{M}$ and denote by $p^h((x,\ell),(y,\iota)|\pi^h)$ the transition probability from a state (x,ℓ) to another state (y,ι) under the control π^h. We need to define p^h so that the chain's evolution well approximates the local behavior of the controlled regime-switching diffusion (1). At any discrete time n, we can either exercise a regular control, a singular control or a reflection step. That is, if we put $\Delta\xi_n^h=\xi_{n+1}^h-\xi_n^h$, then

$$\begin{aligned}\Delta\xi_n^h&=\Delta\xi_n^h I_{\{\text{singular control at } n\}}\\ &+\Delta\xi_n^h I_{\{\text{regular control at } n\}}+\Delta\xi_n^h I_{\{\text{reflection at } n\}}.\end{aligned} \tag{7}$$

The chain and the control will be chosen so that there is exactly one term in (7) is nonzero. Denote by $\{I_n^h: n=0,1,\dots\}$ a sequence of control actions, where $I_n^h=0,1$ or 2, if we exercise a singular control, regular control, or reflection at time n, respectively.

If $I_n^h=1$, we denote by $u_n^h\subset U$ the random variable that is the regular control action for the chain at time n. Let $\tilde{\Delta}t^h(\cdot,\cdot,\cdot)>0$ be the *interpolation interval* on $S_h\times\mathcal{M}\times U$. Assume $\inf_{x,\ell,u}\tilde{\Delta}t^h(x,\ell,u)>0$ for each $h>0$ and $\lim_{h\to0}\sup_{x,\ell,u}\tilde{\Delta}t^h(x,\ell,u)\to0$. Let $E_{x,\ell,n}^{u,h,1}$, $\mathrm{Var}_{x,\ell,n}^{u,h,1}$ and $P_{x,\ell,n}^{u,h,1}$ denote the conditional expectation, variance, and marginal probability given $\{\xi_k^h,\alpha_k^h,u_k^h,I_k^h,k\le n,\xi_n^h=x,\alpha_n^h=\ell,I_n^h=1,u_n^h=u\}$, respectively. If $I_n^h=0$, then we denote by Δz_n^h the random variable that is the singular control action for the chain at time n if $\xi_n^h\in[0,B]$. Note that $\Delta\xi_n^h=-\Delta z_n^h=-h$. If $I_n^h=2$, or $\xi_n^h=B+h$, a reflection step is exerted definitely. Dividend is paid out to lower the surplus level. Moreover, we require reflection takes the state from $B+h$ to B. That is, if we denote by Δg_n^h the random variable that is the reflection action for the chain at time n, then $\Delta\xi_n^h=-\Delta g_n^h=-h$. Also we require the singular control and reflection to be "impulsive" or "instantaneous." In other words, the interpolation interval on $S_h\times\mathcal{M}\times U\times\{0,1,2\}$ is

$$\begin{aligned}&\Delta t^h(x,\ell,u,i)=\tilde{\Delta}t^h(x,\ell,u)I_{\{i=1\}},\\ &\quad\text{for any } (x,\ell,u,i)\in S_h\times\mathcal{M}\times U\times\{0,1,2\}.\end{aligned} \tag{8}$$

Denote by $\pi^h:=\{\pi_n^h,n\ge0\}$ the sequence of control actions, where

$$\pi_n^h:=\Delta z_n^h I_{\{I_n^h=0\}}+u_n^h I_{\{I_n^h=1\}}+\Delta g_n^h I_{\{I_n^h=2\}}.$$

The sequence π^h is said to be *admissible* if π_n^h is $\sigma\left\{(\xi_0^h,\alpha_0^h),\dots,(\xi_n^h,\alpha_n^h),\pi_0^h,\dots,\pi_{n-1}^h\right\}$-adapted. Put

$$\begin{aligned}&t_0^h:=0,\ t_n^h:=\sum_{k=0}^{n-1}\Delta t^h(\xi_k^h,\alpha_k^h,u_k^h,I_k^h),\\ &\text{and } n^h(t):=\max\left\{n: t_n^h\le t\right\}.\end{aligned}$$

Then the piecewise constant interpolations, denoted by $(\xi^h(\cdot),\alpha^h(\cdot))$, $u^h(\cdot)$, $g^h(\cdot)$, and $z^h(\cdot)$, are naturally defined as

$$\begin{aligned}\xi^h(t)=\xi_n^h,\ \alpha^h(t)=\alpha_n^h,\ g^h(t)&=\sum_{k\le n^h(t)}\Delta g_k^h I_{\{I_k^h=2\}},\\ u^h(t)=u_n^h,\ z^h(t)&=\sum_{k\le n^h(t)}\Delta z_k^h I_{\{I_k^h=0\}},\end{aligned} \tag{9}$$

for $t\in[t_n^h,t_{n+1}^h)$. Let $\eta_h:=\inf\left\{n:\xi_n^h\in\partial G\right\}$. Then the first exit time of ξ^h from G is $\tau^h=t_{\eta_h}^h$. Let $(\xi_0^h,\alpha_0^h)=(x,\ell)\in S_h\times\mathcal{M}$ and π^h be an admissible control. The cost function for the controlled Markov chain is defined as

$$\begin{aligned}J_B^h(x,\ell,\pi^h)=E\sum_{k=1}^{\eta_h-1}e^{-rt_k^h}[f(\xi_k^h,\alpha_k^h,u_k^h)\Delta t_k^h\\ +c(\xi_k^h,\alpha_k^h)\Delta z_k^h],\end{aligned} \tag{10}$$

which is analogous to (3) thanks to the definition of interpolation intervals in (8). The value function of the controlled Markov chain is

$$V_B^h(x,\ell)=\sup_{\pi^h}J_B^h(x,\ell,\pi^h). \tag{11}$$

Practically, we compute $V_B^h(x,\ell)$ by solving the following dynamic programming equation using either value iteration or policy iteration.

$$V_B^h(x,\ell) = \begin{cases} \max_{u\in U}\Big\{\sum_{(y,\iota)} e^{-r\Delta t^h(x,\ell,u,1)} p^h((x,\ell),(y,\iota)|\pi) \\ \times V_B^h(y,\iota) + f(x,\ell,u)\Delta t^h(x,\ell,u,1), \\ \big[\sum_{(y,\iota)} p^h((x,\ell),(y,\iota)|\pi) V_B^h(y,\iota) \\ +c(x,\ell)h\big]\Big\}, & \text{for } x\in S_h, \\ 0, & \text{for } x = 0. \end{cases} \tag{12}$$

For simplicity of notation, we use $V^h(x,\ell)$ for $V_B^h(x,\ell)$ henceforth.

D. Interpolation and Rescaling

Based on the approximating Markov chain constructed above, the piecewise constant interpolation is obtained and the appropriate interpolation interval level is chosen. Recalling (9), the continuous-time interpolations $(\xi^h(\cdot),\alpha^h(\cdot))$, $u^h(\cdot)$, $g^h(\cdot)$, and $z^h(\cdot)$ are defined. In addition, let $\mathcal{U}^h$ denote the collection of controls, which are determined by a sequence of measurable functions $F_n^h(\cdot)$ such that

$$u_n^h = F_n^h(\xi_k^h,\alpha_k^h, k\le n; u_k^h, k\le n). \tag{13}$$

Define $\mathcal{D}_t^h$ as the smallest σ-algebra generated by $\{\xi^h(s),\alpha^h(s),u^h(s),g^h(s),z^h(s),\ s\le t\}$. Note that $\mathcal{U}^h$ is the collection of all piecewise constant admissible controls with respect to $\mathcal{D}_t^h$.

Using the representations of regular control, singular control, reflection step and the interpolations defined above, (7) yields

$$\xi^h(t) = x + B^h(t) + M^h(t) - z^h(t) - g^h(t) + \varepsilon^h(t), \tag{14}$$

where

$$B^h(t) = \sum_{k=0}^{n-1} b(\xi_k^h,\alpha_k^h,u_k^h)\Delta t^h(\xi_k^h,\alpha_k^h,u_k^h),$$

$$M^h(t) = \sum_{k=0}^{n-1}(\Delta\xi_k^h - E_k^h\Delta\xi_k^h),$$

and $\varepsilon^h(t)$ is a negligible error satisfying

$$\lim_{h\to\infty}\sup_{0\le t\le T} E|\varepsilon^h(t)|^2 \to 0 \quad \text{for any} \quad 0<T<\infty. \tag{15}$$

Also, $M^h(t)$ is a martingale with respect to $\mathcal{D}_t^h$, and its *discontinuity goes to zero* as $h\to 0$. We attempt to represent $M^h(t)$ in a form similar to the diffusion term in (1). Assume $\sigma(\xi^h(s),\alpha^h(s),u^h(s))\neq 0$. Define $w^h(\cdot)$ as

$$w^h(t) = \int_0^t \sigma^{-1}(\xi^h(s),\alpha^h(s),u^h(s))dM^h(s). \tag{16}$$

We can now rewrite (14) as

$$\begin{aligned}\xi^h(t) &= x + \int_0^t b(\xi^h(s),\alpha^h(s),u^h(s))ds \\ &\quad + \int_0^t \sigma(\xi^h(s),\alpha^h(s),u^h(s))dw^h(s) \\ &\quad - z^h(t) - g^h(t) + \varepsilon^h(t).\end{aligned} \tag{17}$$

Next we introduce the rescaled process. The basic idea of rescaling time is to "stretch out" the control and state processes so that they are "smoother" and therefore the tightness of $g^h(\cdot)$ and $z^h(\cdot)$ can be proved. Define $\Delta\hat{t}_n^h$ by

$$\Delta\hat{t}_n^h = \begin{cases} \Delta t^h & \text{for a diffusion on step } n, \\ |\Delta z_n^h| = h & \text{for a singular control on step } n, \\ |\Delta g_n^h| = h & \text{for a reflection on step } n, \end{cases} \tag{18}$$

Define $\hat{t}_n^h = \sum_{i=0}^{n-1}\Delta\hat{t}_i^h$ and $\hat{T}^h(t) = \sum_{i=0}^{n-1}\Delta t_i^h = t_n^h$ for $t\in[\hat{t}_n^h,\hat{t}_{n+1}^h]$. Thus, $\hat{T}^h(\cdot)$ will increase with the slope of unity if and only if a regular control is exerted. In addition, define the rescaled and interpolated process $\hat{\xi}^h(t) = \xi^h(\hat{T}^h(t))$, likewise define $\hat{\alpha}^h(t)$, $\hat{u}^h(t)$, $\hat{g}^h(t)$ similarly. The time scale is stretched out by h at the reflection and singular control steps. We can now write

$$\begin{aligned}\hat{\xi}^h(t) = x &\ + \int_0^t b(\hat{\xi}^h(s),\hat{\alpha}^h(s),\hat{u}^h(s))ds \\ &\ + \int_0^t \sigma(\hat{\xi}^h(s),\hat{\alpha}^h(s),\hat{u}^h(s))dw^h(s) \\ &\ - \hat{z}^h(t) - \hat{g}^h(t) + \varepsilon^h(t).\end{aligned} \tag{19}$$

E. Convergence of Approximating Sequence

To begin with, the technique of time rescaling and the interpolation of the approximation sequences are introduced in Section II-D. We will verify the weak convergence of a sequence of rescaled process $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),\hat{w}^h(\cdot),\hat{z}^h(\cdot),\hat{g}^h(\cdot),\hat{T}^h(\cdot)\}$, where $\hat{m}^h(\cdot)$ is the rescaled relaxed control $m^h(\cdot)$. and establish the convergence of the value function.

Theorem 2.1: Let the approximating chain $\{\xi_n^h,\alpha_n^h,n<\infty\}$ constructed with corresponding transition probabilities be locally consistent with (1), $m^h(\cdot)$ be the relaxed control representation of $\{u_n^h, n<\infty\}$, $(\xi^h(\cdot),\alpha^h(\cdot))$ be the continuous-time interpolation defined in (9), and $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),\hat{w}^h(\cdot),\hat{z}^h(\cdot),\hat{g}^h(\cdot),\hat{T}^h(\cdot)\}$ be the corresponding rescaled processes. Then $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),$ $\hat{w}^h(\cdot),\hat{z}^h(\cdot),\hat{g}^h(\cdot),\hat{T}^h(\cdot)\}$ is tight.

Theorem 2.2: Let $\{\hat{x}(\cdot),\hat{\alpha}(\cdot),\hat{m}(\cdot),\hat{w}(\cdot),\hat{z}(\cdot),\hat{g}(\cdot),\hat{T}(\cdot)\}$ be the limit of weakly convergent subsequence of $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),\hat{w}^h(\cdot),\hat{z}^h(\cdot),\hat{g}^h(\cdot),\hat{T}^h(\cdot)\}$. $w(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process, and $m(\cdot)$ is admissible. Let $\hat{\mathcal{F}}_t$ the σ-algebra generated by $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),\hat{w}^h(\cdot),$ $\hat{z}^h(\cdot),\hat{g}^h(\cdot),\hat{T}^h(\cdot)\}$. Then $\hat{w}(t) = w(\hat{T}(t))$ is an $\hat{\mathcal{F}}_t$-martingale with quadratic variation $\hat{T}(t)$. The limit processes satisfy

$$\begin{aligned}\hat{x}(t) &= x + \int_0^t\int_{\mathcal{U}} b(\hat{x}(s),\hat{\alpha}(s),\phi)\hat{m}^h_{\hat{T}(s)}(d\phi)d\hat{T}(s) \\ &\quad + \int_0^t\int_{\mathcal{U}} \sigma(\hat{x}(s),\hat{\alpha}(s),\phi)\hat{M}_{\hat{T}(s)}(d\phi)d\hat{T}(s) \\ &\quad - \hat{z}(t) - \hat{g}(t).\end{aligned} \tag{20}$$

Theorem 2.3: For $t<\infty$, define the inverse $R(t) = \inf\{s : \hat{T}(s) > t\}$. Then $R(t)$ is right continuous and $R(t)\to\infty$ as $t\to\infty$ w.p.1. For any process $\hat{\varphi}(\cdot)$, define the rescaled process $\varphi(\cdot)$ by $\varphi(t) = \hat{\varphi}(R(t))$. Then, $w(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process and (1) holds.

Theorem 2.4: $V^h(x,\ell)$ and $V(x,\ell)$ are value functions defined in (12) and (4), respectively. Then $V^h(x,\ell) \to V(x,\ell)$ as $h \to 0$.

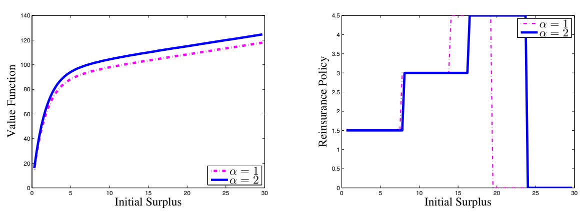
F. Numerical Examples

This section is devoted to a couple of examples. For simplicity, we consider the case that the discrete event has two states. Proportional reinsurance and nonproportional reinsurance are considered, respectively.

Proportional reinsurance. The generator of the Markov chain $\alpha(t)$ is $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$, and $\mathcal{M} = \{1,2\}$. The claim rate depends on the discrete state with $\beta(1) = 1$ and $\beta(2) = 5$. Assume the claim size distribution to be exponential with parameter 1. Then $E[Y] = 1$ and $E[Y^2] = 2$. The (1) follows

$$dX(t) = \beta(\alpha(t))u(t)dt + \sqrt{2\beta(\alpha(t))}u(t)dw(t) - dZ(t),$$
$$X(0^-) = x$$

where the retention level $u(t)$ is the regular control parameter representing the fraction of the claim covered by the cedent and $u(t) \in [0,1]$. Taking the discount rate $r = 0.05$, we compare the cost function in the case of the total expected discounted value of all dividends until lifetime ruin $J(x,\ell,\pi) = E_{x,\ell}\int_{[0,\tau]} e^{-rt}dZ(t)$. We obtained Figure II.1 for this case.



II.1.1 Total expected discounted value of all dividends II.1.2 Optimal reinsurance policy

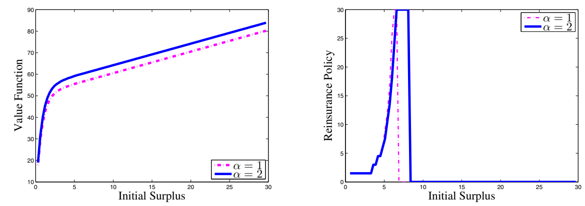
Fig. II.1. Proportional reinsurance with two regimes

Excess-of-loss reinsurance. Assume the claim size distribution to be exponential with parameter 1 and Q, $\beta(1), \beta(2)$ and payoff functions to be the same. Intuitively, the retention level cannot be arbitrarily large, then we restrict the risk control set U to be $[0,1]$. Then the dynamic systems satisfy

$$\begin{cases} dX(t) = \beta(\alpha(t))[1-e^{-u(t)}]dt \\ \quad +\sqrt{2\beta(\alpha(t))[1-e^{-u(t)}(1+u(t))]}dw(t) - dZ(t), \\ X(0^-) = x; \end{cases}$$

see Figure II.2 for this case.

It shows that the value function is concave and the dividend payout strategy is a barrier strategy. It is clear that if the surplus is higher than some barrier level, the extra surplus will be paid as the dividend, with the same time the value functions increase with unity slope. Both the proportional reinsurance and excess-of-loss reinsurance increase at first, maintain the highest rate in an interval, and decrease sharply to zero at a threshold to maximize the total expected discounted value of all dividends. In both regimes, there exists a free boundary (barrier) that separates two regions where the regular control or singular control is dominant. Also, the barrier levels are different in different regimes due to the Markov switching.



II.2.1 Total expected discounted value of all dividends II.2.2 Optimal reinsurance policy

Fig. II.2. Excess-of-loss reinsurance with two regimes

III. DIVIDEND PAYMENT AND INVESTMENT OPTIMIZATION WITH CAPITAL INJECTIONS

A. Motivation

To maximize the expected total discounted dividend payments, the company will bankrupt almost surely if the dividend payment is paid out as a barrier strategy. In practice, [4] suggested that capital injections can be taken into account to avoid insolvency when capital reserve is insufficient. Furthermore, transaction cost will be considered; see also [13], [10], and [14]. Whenever the company is on the verge of financial ruin, the company has the opportunity to raise sufficient funds to survive. A natural payoff function is maximizing the difference between the expected total discounted dividend payment and the capital injections with costs until bankruptcy under the optimal controls.

We aim to obtain the optimal dividend payment and investment strategies using the collective risk model under the Makovian regime-switching setting with capital injections. We allow the investment of surplus in a continuous-time financial market and the management of the dividend payment policy. In our model, borrowing money to do risky investment is not allowed. The insurers cannot put too much money in risky assets for the sake of risk management. That is, there is a natural constraint on the portfolio so that the total weight of the risky assets should be no more than 1. Another constraint on the investment is that short selling risky asset is prohibited. Hence, the proportion of capital invested in the risky asset is denoted as a regular control $u \in [0,1]$. In addition, a dividend process is not necessarily absolutely continuous. In fact, dividends are usually paid out at random discrete times, where insurance companies may distribute dividends at unrestricted payment rate. In such a scenario, the surplus level changes drastically on a dividend payday. Thus, abrupt or discontinuous changes occur due to "singular" dividend distribution policy. Moreover, the capital injections, modeled by impulse controls are exerted when surplus hits not only 0 but also a sufficiently low threshold. To maximize the performance, the impulse controls of capital injections depend on the surplus processes and can be very large, which results in a free boundary of capital injection region and adds more difficulty to analyze the optimal policies. Together with the Markov switching and the incurred claims, this gives rise to a regime-switching jump diffusion stochastic control problem with singular and impulse controls.

B. Formulation

The surplus process $X(t)$ under consideration is a jump diffusion process with regime-switching under singular and impulse control. To delineate the random environment and other random factors, we use a continuous-time Markov chain $\alpha(t)$ taking values in the finite space $\mathcal{M} = \{1, \dots, n_0\}$. For each $i \in \mathcal{M}$, the premium rate is $c(i) > 0$. Let φ_n be the inter-arrival time of the nth claim, $\nu_n = \sum_{j=1}^{n} \varphi_n$. For a slightly more generality, we consider a Poisson measure in lieu of the traditionally used Poisson process. Suppose $\Gamma \subset \mathcal{R}_+$ is a compact set and the function $q(X, i, \rho)$ is the magnitude of the claim sizes, where ρ has distribution $\Pi(\cdot)$. $N(t, H) =$ number of claims on $[0, t]$ with claim size taking values in $H \in \Gamma$. Note that our formulation is general, the claim sizes are assumed to depend on the switching regime. Then the Poisson measure $N(\cdot)$ has intensity $\lambda dt \times \Pi(d\rho)$ where $\Pi(d\rho) = f(\rho)d\rho$. Assume that $q(\cdot, i, \rho)$ is continuous for each ρ and each $i \in \mathcal{M}$. At different regimes, the values of $q(\cdot, i, \rho)$ could be much different, which takes into consideration of random environment. Then the surplus process in the absence of dividend payment and investment is a regime-switching jump process given by

$$\begin{aligned} d\widetilde{X}(t) &= \sum_{i\in\mathcal{M}} I_{\{\alpha(t)=i\}}(c(i)dt - dR(t)) \\ &= c(\alpha(t))dt - \int_\Gamma q(X(t^-), \alpha(t), \rho)N(dt, d\rho). \end{aligned} \tag{21}$$

We consider the financial market with a risk free asset $Y(t)$ and a risky asset $S(t)$ with prices satisfying

$$\left\{ \begin{aligned} &\frac{dY(t)}{Y(t)} = g(\alpha(t))dt, \\ &\frac{dS(t)}{S(t)} = b(\alpha(t))dt + \sigma(\alpha(t))dW(t), \end{aligned} \right. \tag{22}$$

where for each $i \in \mathcal{M}$, $g(i)$ and $b(i)$ are the return rates of the risk free and risky asset, respectively. $\sigma(\alpha(t))$ is the corresponding volatility and $W(t)$ is a standard Brownian motion. The investment behavior of the insurer is modelled as a portfolio process $u(t)$, where proportional surplus $u(t) \in [0, 1]$ was invested in the risky asset $S(t)$. The dividend strategy is denoted by $Z(\cdot)$ as in II.

The capital injection process $l(t) = \sum_{n=1}^{\infty} I_{\{\tau_n \le t\}}\zeta_n$ is described by a sequence of increasing stopping times $\{\tau_n, n = 1, 2, \dots\}$ and a sequence of random variables $\{\zeta_n, n = 1, 2, \dots\}$, which represent the times and the sizes of capital injections. A control policy π is described by $\pi = \{u, z; l\} = \{u, z; \tau_1, \dots, \tau_n, \dots; \zeta_1, \dots, \zeta_n, \dots\}$. Assume the evolution of $X(t)$, subject to capital injections and dividend payments, follows a one-dimensional process on an unbounded domain $G' = (0, \infty)$. The surplus process considers dividend payment, capital injection and investment satisfy the following stochastic differential equation

$$\left\{ \begin{aligned} X(t) &= x + \int \Big[[g(\alpha(t))(1 - u(t)) + u(t)b(\alpha(t))]X(t) \\ &\quad + c(\alpha(t)) \Big] dt + \int u(t)\sigma(\alpha(t))X(t)dW(t) - R(t) \\ &\quad - Z(t) + \sum_{n=1}^{\infty} I_{\{\tau_n \le t\}}\zeta_n, \\ X(0) &= x. \end{aligned} \right. \tag{23}$$

for all $t < \tau$ and we impose $X(t) = 0$ for all $t > \tau$, where $\tau = \inf\{t \ge 0 : X(t) \le 0\}$ represents the time of ruin.

We assume that the shareholders can get the proportion of β_1 for every dividend payment, where $0 < \beta_1 < 1$. We omit the fixed transaction costs in the dividends payout process. Moreover, we assume that the shareholders need to pay $K + \beta_2\zeta, \beta_2 > 1$, to meet the capital injection of ζ. $K > 0$ is the fixed transaction costs, $(\beta_2 - 1)\zeta$ is the proportional transaction costs. Denote by $r > 0$ the discount factor. For an arbitrary admissible pair $\pi = (u, Z, l)$, the performance function is

$$\begin{aligned} J(x, i, \pi) &= E_{x,i}\Big[\int_0^\infty e^{-rt}\beta_1 dZ - \sum_{n=1}^{\infty} e^{-r\tau_n}(K + \beta_2\zeta_n) \\ &\qquad \times I_{\{\tau_n < \infty\}} \Big]. \end{aligned} \tag{24}$$

The pair $\pi = (u, z, l)$ is said to be *admissible* if u, z, and l satisfy

- (i) $u(t)$, $Z(t)$, and $l(t)$ are nonnegative for any $t \ge 0$, and $u \in [0, 1]$,
- (ii) Z is right continuous, has left limits, and is nondecreasing,
- (iii) $X(t) \ge 0$, for any $t \le \tau$,
- (iv) u, Z and l are adapted to $\mathcal{F}_t := \sigma\{\alpha(s), W(s), N(s), 0 \le s \le t\}$ augmented by the P-null sets,
- (v) τ_n is a stopping time w.r.t. $\mathcal{F}_t$, and $0 \le \tau_1 < \cdots < \tau_n < \cdots$, a.s.
- (vi) ζ_n is measurable w.r.t. $\mathcal{F}_t$,
- (vii) $P(\lim_{n\to\infty} \tau_n < T) = 0, \forall T > 0$, and
- (viii) $J(x, i, \pi) < \infty$ for any $(x, i) \in G \times \mathcal{M}$ and admissible pair $\pi = (u, Z, l)$, where J is the functional defined in (24).

Suppose that $\mathcal{A}$ is the collection of all admissible pairs. Define the value function as

$$V(x, i) := \sup_{\pi\in\mathcal{A}} J(x, i, \pi). \tag{25}$$

For an arbitrary $\pi \in \mathcal{A}$, $i = \alpha(t) \in \mathcal{M}$, and $V(\cdot, i) \in$

$C^2(\mathcal{R})$, define an operator $\mathcal{L}^\pi$ by

$$\begin{aligned}\mathcal{L}^\pi V(x,i) &= V_x(x,i)([g(i)(1-u)+ub(i)]x+c(i))\\ &+\frac{1}{2}\sigma(i)^2u^2x^2V_{xx}(x,i)\\ &+\lambda\int_0^\infty [V(x-q(x,i,\rho),i)-V(x,i)]f(\rho)d\rho\\ &+QV(x,\cdot)(i),\end{aligned} \tag{26}$$

where V_x and V_{xx} denote the first and second derivatives with respect to x, and

$$QV(x,\cdot)(i)=\sum_{j\neq i}q_{ij}(V(x,j)-V(x,i)).$$

Define another capital injection operator $\mathcal{H}$ by

$$\mathcal{H}V(x,i)=\sup_{\widetilde{y}\geq 0}\{V(x+\widetilde{y},i)-\beta_2\widetilde{y}-K\} \tag{27}$$

If the value function V defined in (25) is sufficiently smooth, V formally satisfies the following quasi-variational inequalities (QVIs):

$$\begin{aligned}&\max\Big\{\mathcal{L}^\pi V(x,i)-rV(x,i),\beta_1-V_x(x,i),\\ &\mathcal{H}V(x,i)-V(x,i)\Big\}=0, \quad \text{for each } \ i\in\mathcal{M}.\end{aligned} \tag{28}$$

Similar to [14], we divide the set of surplus to three regions

(i) Continuation region:

$$\begin{aligned}\mathcal{C} := \{&\mathcal{L}^\pi V(x,i)-rV(x,i)=0,\beta_1<V_x(x,i),\\ &\mathcal{H}V(x,i)<V(x,i)\}\end{aligned}$$

(ii) Dividend payout region:

$$\begin{aligned}\mathcal{D} := \{&\mathcal{L}^\pi V(x,i)-rV(x,i)<0,\beta_1=V_x(x,i),\\ &\mathcal{H}V(x,i)<V(x,i)\}\end{aligned}$$

(iii) Capital injection region:

$$\begin{aligned}\mathcal{I} := \{&\mathcal{L}^\pi V(x,i)-rV(x,i)<0,\beta_1<V_x(x,i),\\ &\mathcal{H}V(x,i)=V(x,i)\}.\end{aligned}$$

Boundary Conditions. Intuitively, for all $i\in\mathcal{M}$, on the boundary of the capital injection region, the value function obeys

$$V(x,i)=\sup_{\widetilde{y}\geq 0}\{V(x+\widetilde{y},i)-\beta_2\widetilde{y}-K\} \tag{29}$$

To make it computationally feasible, we truncate x at some large value B. When B is large enough, it follows

$$V_x(B,i)=\beta_1. \tag{30}$$

Combining (28), (29) and (30), the system of QVIs with the boundary conditions is given by

$$\begin{cases}\max\Big\{\mathcal{L}^\pi V(x,i)-rV(x,i),\\ \quad\beta_1-V_x(x,i),\mathcal{H}V(x,i)-V(x,i)\Big\}=0,\ i\in\mathcal{M},\\ V_x(B,i)=\beta_1,\\ V(0,i)=\sup\limits_{0\leq\widetilde{y}\leq B}\{V(\widetilde{y},i)-\beta_2\widetilde{y}-K\}.\end{cases} \tag{31}$$

C. Numerical Algorithm and Convergence

By using similar techniques shown in II, we construct an appropriate interpolation sequence $\{\xi^h(\cdot),\alpha^h(\cdot),\ m^h(\cdot), W^h(\cdot),N^h(\cdot),R^h(\cdot),Z^h(\cdot),l^h(\cdot),\hat{T}^h(\cdot)\}$ to approximate the value function. The system of dynamic programming equations follows

$$V^h(x,i) = \begin{cases}\max\limits_{\pi\in\mathcal{A}}\Big[(1-\lambda\Delta t^h(x,i,u,2)+\delta^h(x,i,u,2))\\ \times e^{-r\Delta t^h(x,i,u,2)}\sum\limits_{(y,j)}(p_D^h((x,i),(y,j))|\pi)V^h(y,i)\\ +(\lambda\Delta t^h(x,i,u,2)+\delta^h(x,i,u,2))e^{-r\Delta t^h(x,i,u,2)}\\ \times\int_0^\infty V^h(x-q_h(x,i,\rho),i)\Pi(d\rho),\\ V^h(x-h,i)+\beta_1 h,\\ \sup\limits_{0\leq\widetilde{y}\leq B-x} V^h(x+\widetilde{y},i)-\beta_2\widetilde{y}-K\Big], & \text{for } x\in S_h,\\ \sup\limits_{0\leq\widetilde{y}\leq B} V^h(\widetilde{y},i)-\beta_2\widetilde{y}-K, & \text{for } x=0,\\ V^h(B-h,i)+\beta_1 h, & \text{for } x=B.\end{cases} \tag{32}$$

To analyze the asymptotic properties of the proposed approximating Markov chain proposed we will deal with weak convergence of $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\ \hat{m}^h(\cdot),\ \hat{W}^h(\cdot),\ \hat{N}(\cdot), \hat{R}^h(\cdot),\ \hat{Z}^h(\cdot),\hat{l}^h(\cdot),\hat{T}^h(\cdot)\}$, a sequence of rescaled process. As a result, a sequence of controlled surplus processes converges to a limit surplus process. By using the techniques of inversion, the convergence of the value function will be established. To proceed, we need the following assumptions.

(A) Let $u(\cdot)$ be an admissible ordinary control with respect to $W(\cdot)$, $\alpha(\cdot)$, and $N(\cdot)$, and suppose that $u(\cdot)$ is piecewise constant and takes only a finite number of values. For each initial condition, there exists a solution to relaxed presentation of $\hat{\xi}^h(\cdot)$, where $m(\cdot)$ is the relaxed control representation of $u(\cdot)$ and the solution is unique in the weak sense.

Lemma 3.1: Assume (A). Using appropriate transition probabilitie, the interpolated process of the constructed Markov chain $\{\hat{\alpha}^h(\cdot)\}$ converges weakly to $\hat{\alpha}(\cdot)$, the rescaled Markov chain with generator $Q=(q_{ij})$.

Theorem 3.2: Under the conditions of Lemma 3.1, let the approximating chain $\{\xi_n^h,\alpha_n^h,n<\infty\}$ be constructed with transition probabilities be locally consistent with (23), $m^h(\cdot)$ be the relaxed control representation of $\{u_n^h,n<\infty\}$, $(\xi^h(\cdot),\alpha^h(\cdot))$ be the continuous-time interpolation defined in (9), and $\{\hat{\xi}^h(\cdot),\hat{\alpha}^h(\cdot),\hat{m}^h(\cdot),\hat{W}^h(\cdot),\hat{N}^h(\cdot),\hat{R}^h(\cdot), \hat{Z}^h(\cdot),\hat{l}^h(\cdot),\hat{T}^h(\cdot)\}$ be the corresponding rescaled processes. Then $\{\hat{\xi}^h(\cdot),\ \hat{\alpha}^h(\cdot),\ \hat{m}^h(\cdot),\ \hat{W}^h(\cdot),\ \hat{N}^h(\cdot),\ \hat{R}^h(\cdot),\ \hat{Z}^h(\cdot), \hat{l}^h(\cdot),\ \hat{T}^h(\cdot)\}$ is tight.

Theorem 3.3: Under the conditions of Theorem 3.2, let $\{\hat{X}(\cdot),\ \hat{\alpha}(\cdot),\ \hat{m}(\cdot),\ \hat{W}(\cdot),\ \hat{N}(\cdot),\ \hat{R}(\cdot),\ \hat{z}(\cdot),\ \hat{l}(\cdot),\ \hat{T}(\cdot)\}$ be the limit of the weakly convergent subsequence of $\{\hat{\xi}^h(\cdot),\ \hat{\alpha}^h(\cdot), \hat{m}^h(\cdot),\hat{W}^h(\cdot),\hat{N}^h(\cdot),\hat{R}^h(\cdot),\hat{Z}^h(\cdot),\hat{l}^h(\cdot),\hat{T}^h(\cdot)\}$, $W(\cdot)$ be a standard $\mathcal{F}_t$-Wiener process, and $m(\cdot)$ be admissible. Let $\hat{\mathcal{F}}_t$ be the σ-algebra generated by $\{\hat{\xi}^h(s),\ \hat{\alpha}^h(s),\ \hat{m}^h(s),\ \hat{W}^h(s), \hat{N}^h(s),\ \hat{R}^h(s),\ \hat{Z}^h(s),\ \hat{l}^h(s),\ \hat{T}^h(s): s\leq t\}$. Then $\hat{W}(t)=$

$W(\hat{T}(t))$ is an $\hat{\mathcal{F}}_t$-martingale with quadratic variation $\hat{T}(t)$. The limit processes satisfy

$$\begin{aligned}\hat{X}(t) &= x + \int_0^t \int_{\mathcal{U}} \Big[[g(\hat{\alpha}(s))(1-\psi) + \psi b(\hat{\alpha}(s))]\hat{X}(s) \\ &\quad + c(\hat{\alpha}(s)) \Big] \hat{m}^h_{\hat{T}(s)}(d\psi) d\hat{T}(s) \\ &\quad + \int_0^t \int_{\mathcal{U}} \psi \sigma(\hat{\alpha}(s)) \hat{X}(s) \hat{M}_{\hat{T}(s)}(d\psi) d\hat{T}(s) \\ &\quad - \hat{R}(t) - \hat{Z}(t) - \hat{l}(t).\end{aligned} \tag{33}$$

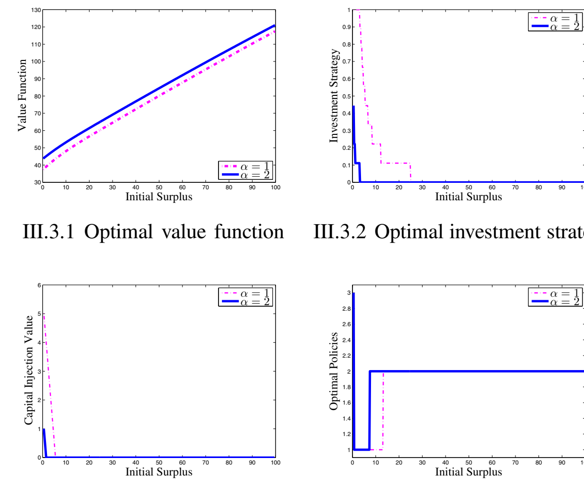
Theorem 3.4: Under the conditions of Theorem 3.3, for $t < \infty$, define the inverse $\overline{T}(t) = \inf\{s : \hat{T}(s) > t\}$. Then $\overline{T}(t)$ is right continuous and $\overline{T}(t) \to \infty$ as $t \to \infty$ w.p.1. For any process $\hat{\chi}(\cdot)$, define the rescaled process $\chi(\cdot)$ by $\chi(t) = \hat{\chi}(\overline{T}(t))$. Then, $W(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process, $N(\cdot)$ is a Poisson measure and (23) holds.

Theorem 3.5: For $V^h(x,i)$ and $V(x,i)$, value functions defined in (32) and (25), respectively, we have $V^h(x,i) \to V(x,i)$ as $h \to 0$.

D. Numerical Examples

We still consider the two-state Markov chain. Exponential claim distribution is assumed. The corresponding capital injection sizes and barriers for regions are also obtained in the numerical examples. The continuous-time Markov chain $\alpha(t)$ representing the discrete event state has the generator $Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$, and takes values in $\mathcal{M} = \{1, 2\}$. The claim severity distribution follows exponential distribution with density function $f(y) = ae^{-ay}$ where $a = 0.1$ and $\lambda = 0.1$. The premium rate depends on the discrete state with $c(1) = 2$ and $c(2) = 10$. The portfolio rate $u(t)$ taking values in $[0, 1]$ is the control. The yield rates of the riskless asset are $g(1) = 0.03$, $g(2) = 0.04$, $b(1) = 0.06$ and $b(2) = 0.08$. The volatility of the financial market $\sigma(\alpha(t))$ is valued as $\sigma(1) = 0.1$ and $\sigma(2) = 0.4$. Let the upper bound of the computation interval $B = 100$, the discount rate $r = 0.05$, the fixed capital injection cost $K = 1$, the parameters for the proportion costs $\beta_1 = 0.95$ and $\beta_2 = 1.05$.

Due to the singular and impulse controls, the value function in each regime is verified to be a concave function and defined separately in three regions. We use "1" to denote the region in the QVIs when regular control is dominant, "2" to denote the region in the QVIs when dividend payment is dominant, and "3" to denote the region in the QVIs when capital injections are dominant. Taking into consideration of capital injections, the capital injections have to be ordered if the surplus violates the capital requirement for running the business. Hence, the impulse controls of capital injections will occur for sure at zero surplus. In addition, the barrier of capital injection region is a free boundary. Thus, the impulse controls of capital injections depend on the surplus process and can be very large. These state-dependent capital injections lead to the formulation of free boundary problem, and the state-dependent "threshold" curve, separates the capital injection region and continuation region. We observe that the dividend payment is dominant when the investment in risky assets becomes zero. It seems the insurer chooses to put money in the riskless asset or pay out the surplus as dividend when it is high enough to avoid the possible risk. Moreover, with the sufficient low surplus, capital injections are optimal and the investment is preferred in risky assets.



III.3.1 Optimal value function

III.3.2 Optimal investment strategy

III.3.3 Optimal capital injections

III.3.4 Three regions

Fig. III.3. Excess-of-loss reinsurance with two regimes

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